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Bialgebraic varieties for the Γ function

Joint with Sebastian Eterović and Roy Zhao

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Question

Let $W, V \subset \mathbb{C}^n$ be algebraic varieties with $0 < \dim W, \dim V < n$.

Suppose f is a transcendental meromorphic function.

If $f(W) \subseteq V$, what can we conclude about V and W?

Notation: $f(W) = \{(f(w_1), \ldots, f(w_n)) : (w_1, \ldots, w_n) \in W\}$

The Ax–Lindemann–Weierstrass Theorem

If $\exp(W) \subseteq V$, what can we say about W and V?

Theorem (Ax)

If W is a maximal irreducible component of $\exp^{-1}(V)$, then W is a translate of a rational subspace of $\mathbb{C}^n$.

Example:
$W = \{(w_1, w_2) \in \mathbb{C}^2 : 2w_1 + 3w_2 = \sqrt{5}\}$
$V = \{(v_1, v_2) \in \mathbb{C}^2 : (v_1)^2(v_2)^3 = e^{\sqrt{5}}\}$

Analogs of (a strengthening of) this result are known for other differentially algebraic functions.

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The Lindemann–Weierstrass Theorem

Theorem (Ax)

If W is a maximal irreducible component of $\exp^{-1}(V)$, then W is a translate of a rational subspace of $\mathbb{C}^n$.

Theorem (Lindemann–Weierstrass)

If $a_1, \ldots, a_n \in \overline{\mathbb{Q}}$ and $\exp(a_1), \ldots, \exp(a_n)$ are algebraically dependent, then $a_1, \ldots, a_n$ are $\mathbb{Q}$-linearly dependent.

Schanuel's Conjecture: If $\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(z_1, \ldots, z_n, \exp(z_1), \ldots, \exp(z_n)) < n$ for $z_1, \ldots, z_n \in \mathbb{C}$, then $z_1, \ldots, z_n$ are $\mathbb{Q}$-linearly dependent.

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The Γ function

Γ extends the factorial function on positive integers holomorphically to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$.

Γ satisfies three kinds of functional equations:

1. $\Gamma(z+1) = z\Gamma(z)$
2. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$
3. $\prod_{k=0}^{m-1} \Gamma\left(z + \frac{k}{m}\right) = (2\pi)^{\frac{1}{2}(m-1)} m^{\frac{1}{2}-mz}\Gamma(mz)$

Stirling's formula: $\Gamma(z) = \sqrt{2\pi} e^{(z-\frac{1}{2})\log z - z + \varphi(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

What kinds of varieties W and V satisfy $\Gamma(W) \subseteq V$?

1. $\Gamma(z+2) = (z+1)\Gamma(z+1) = \left(\frac{\Gamma(z+1)}{\Gamma(z)} + 1\right)\Gamma(z+2)$

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So $\Gamma(W) \subseteq V$ for

$$W = \{(w_1, w_2, w_3) : w_1 + 1 = w_2, w_1 + 2 = w_3\}$$

$$V = \{(v_1, v_2, v_3) : v_3 = \left(\frac{v_2}{v_1} + 1\right) v_2\}$$

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What kinds of varieties W and V satisfy $\Gamma(W) \subseteq V$?

2. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ gives

$$\frac{\pi^2}{\Gamma(z)^2\Gamma(1-z)^2} + \frac{\pi^2}{\Gamma(z+\frac{1}{2})^2\Gamma(\frac{1}{2}-z)^2} = 1$$

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3. $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = \sqrt{\pi}2^{1-2z}\Gamma(2z)$ gives, for any $a \in \mathbb{C}$,

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \cdot \sqrt{\pi}2^{1-2(z+a)}\Gamma(2z+2a)$$
$$= \sqrt{\pi}2^{1-2z}\Gamma(2z) \cdot \Gamma(z+a)\Gamma\left(z+a+\frac{1}{2}\right)$$

What kinds of varieties W and V satisfy $\Gamma(W) \subseteq V$?

1. $\Gamma(az+1) = a\frac{\Gamma(z+1)}{\Gamma(z)}\Gamma(az)$
2. $\frac{\pi^2}{\Gamma(z)^2\Gamma(1-z)^2} + \frac{\pi^2}{\Gamma(z+\frac{1}{2})^2\Gamma(\frac{1}{2}-z)^2} = 1$
3. $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \cdot 2^{-2a}\Gamma(2z+2a) = \Gamma(2z) \cdot \Gamma(z+a)\Gamma\left(z+a+\frac{1}{2}\right)$

Observations: In all three examples

- $\dim W = 1 < \dim V$
- Some coordinates of W are related by an affine rational transformation $w_i = qw_j + c$ for $q \in \mathbb{Q}$ and $c \in \mathbb{C}$.

Can any instance of $\Gamma(W) \subseteq V$ be explained by an affine rational relationship between some coordinates of W?

Conjecture (Rohrlich–Lang)

If $a_1, \dots, a_n \in \mathbb{Q}$ and $\Gamma(a_1), \dots, \Gamma(a_n)$ are algebraically dependent, then the dependency comes from the functional equations Γ is known to satisfy:

1. $\Gamma(z+1) = z\Gamma(z)$
2. $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$
3. $\prod_{k=0}^{n-1} \Gamma\left(z + \frac{k}{n}\right) = (2\pi)^{\frac{1}{2}(n-1)} n^{\frac{1}{2} - nz} \Gamma(nz)$

In particular, each a_i is related to some a_j by a rational linear transformation.

Two functional transcendence results for Γ

Theorem (1)

Let $q_1, \ldots, q_n \in \mathbb{Q}$ and $b_1, \ldots, b_n \in \mathbb{C}$. If $\Gamma(q_1 z + b), \ldots, \Gamma(q_n z + b_n)$ are algebraically dependent, then the dependency comes from the functional equations Γ is known to satisfy.

Theorem (2)

If $W, V \subset \mathbb{C}^n$ are algebraic varieties with $\dim W = \dim V$ and $\Gamma(W) \subseteq V$, then W and V are defined by equations of the form $x_i = x_j$ and $x_i = a \in \mathbb{C}$.

Strategies for proving functional transcendence results

- O-minimality and point counting

Theorem (P., Speissegger)

Γ *is definable on certain unbounded complex domains in* $\mathbb{R}_{\mathcal{G},\exp}$.

But there are serious obstacles to adapting the standard approach to Γ

- Differential algebra
  But Hölder proved Γ is differentially transcendental
  - Difference algebra?

A Lang-Rohrlich theorem for linear functions

Theorem (1)

Let $q_1, \ldots, q_n \in \mathbb{Q}$ *and* $b_1, \ldots, b_n \in \mathbb{C}$. *If* $\Gamma(q_1 z + b), \ldots, \Gamma(q_n z + b_n)$ *are algebraically dependent, then the dependency comes from the functional equations* Γ *is known to satisfy:*

1. $\Gamma(z+1) = z\Gamma(z)$
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We may assume $q_1, \ldots, q_n$ are even integers by rescaling z.

Sketch of Theorem (1)

Theorem (1)

*Let $q_1, \ldots, q_n$ be **even** integers and $b_1, \ldots, b_n \in \mathbb{C}$. If $\Gamma(q_1 z + b_1), \ldots, \Gamma(q_n z + b_n)$ are algebraically dependent, then the dependency comes from the functional equations Γ is known to satisfy.*

$K =$ field of 1-periodic meromorphic functions. $\sin(q_j z + b) \in K$.

Use $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)}$ to make $q_1, \ldots, q_n$ positive even integers and $0 \leq \Re(b_1), \ldots, \Re(b_n) < 1$.

We can do this at the cost of introducing z and elements of K into the algebraic dependency.

Sketch of Theorem (1)

Call $S = \{\Gamma(q_1 z + b_1), \ldots, \Gamma(q_n z + b_n)\}$ a *normalized* set if $q_1, \ldots, q_n$ are positive even integers and $0 \leq \Re b_j < 1$ for $j = 1, \ldots, n$.

Theorem (1)

Let $S = \{\Gamma(q_1 z + b_1), \ldots, \Gamma(q_n z + b_n)\}$ be a normalized set. If S is algebraically dependent ***over*** *$K(z)$, then the dependency comes from the functional equations Γ is known to satisfy.*

We show if S is a normalized set, then it is multiplicatively dependent via instances of the multiplication equation.

Sketch of Theorem (1)

Theorem (Eterović, P., Zhao)

Let $S = \{\Gamma(q_1 z + b_1), \ldots, \Gamma(q_n z + b_n)\}$ *be a normalized set.* S *is algebraically dependent over* $K(z)$ *if and only if there are integers* $d_1, \ldots, d_n$ *not all zero such that*

$$\prod_{i=1}^{n} \Gamma(q_i z + b_i)^{d_i} = b,$$

where $b = \prod_{i=1}^{n} \left((2\pi)^{(1-q_i)/2} q_i^{b_i - 1/2} \right)^{d_i}$*, and* $\sum_{i=1}^{n} d_i q_i \log(q_i) = 0$.

A differential Lang-Rohrlich theorem for linear functions

Theorem (Eterović, P., Zhao)

*Let $S = \{\Gamma(q_1 z + b_1), \ldots, \Gamma(q_n z + b_n)\}$ be a normalized set. S is **differentially** algebraically dependent over $K(z)$ if and only if there are integers $d_1, \ldots, d_n$ not all zero such that*

$$\prod_{i=1}^{n} \Gamma(q_i z + b_i)^{d_i} = b \exp(qz),$$

where $q = \sum_{i=1}^{n} d_i q_i \log(q_i)$ and $b = \prod_{i=1}^{n} \left((2\pi)^{(1-q_i)/2} q_i^{b_i - 1/2} \right)^{d_i}$.

Strategies for proving functional transcendence results

- Difference algebra, but unclear how to proceed for nonlinear W
- Valuation theory, asymptotic expansions

Let $f = \exp$ or Γ and assume $f(W) \subseteq V \subset \mathbb{C}^n$. For simplicity, assume $n = 2$ and $\dim W = \dim V = 1$, no constant coordinates.

Let $\alpha(z)$ and $\beta(z)$ be algebraic functions such that $(x, \alpha(x)) \in W$ and $(x, \beta(x)) \in V$ for all large enough real x.

Then $f(\alpha(x)) = \beta(f(x))$ for all large enough x.
As $x \to \infty$, $f(\alpha(x))$ and $\beta(f(x))$ have different grow rates **unless** α and β are of particular forms.

Formalize this idea with transseries

The field of logarithmic-exponential transseries $\mathbb{T}$ is an ordered valued differential field of formal sums of transmonomials (built iteratively from $\mathbb{R}((x^{\mathbb{R}}))$ using $\exp$, $\log$, compositions $\ldots$) with coefficients in $\mathbb{R}$.

Example

- real Puiseux series at $+\infty$
- $\exp(x) = e^x$
- $\log\Gamma(x) = x\log x - x - \frac{1}{2}\log x + \sum_{l=1}^{\infty} \frac{B_{2l}}{2l(2l-1)x^{2l-1}}$

Idea: Expand $f(\alpha(x))$ and $\beta(f(x))$ as transseries. Show the transseries expansions are different **unless** α and β are of particular forms.

Formalize this idea with transseries

When can we expand $f(\alpha(x))$ and $\beta(f(x))$ as transseries?

Fact

The germs at $+\infty$ of functions definable in an o-minimal expansion of the real field form an ordered differential field (called a Hardy field).

Theorem (van den Dries, Macintyre, Marker)

The field of germs at $+\infty$ of one-variable functions definable in $\mathbb{R}_{\mathrm{an,exp}}$ embeds into the field $\mathbb{T}$ of transseries as a differential field.

Theorem (van den Dries, Speissegger)

The field of germs at $+\infty$ of one-variable functions definable in $\mathbb{R}_{\mathcal{G},\mathrm{exp}}$ embeds into the field $\mathbb{T}$ of transseries as a differential field.

Sketch of Theorem (2)

Theorem (2)

If $W, V \subset \mathbb{C}^n$ are algebraic varieties with $\dim W = \dim V$ and $\Gamma(W) \subseteq V$, then W and V are defined by equations of the form $x_i = x_j$ and $x_i = a \in \mathbb{C}$.

- Suppose $\Gamma(\alpha(x)) = \beta(\Gamma(x))$ and α, β are nonconstant. This is actually two equations: $\Re\Gamma(\alpha(x)) = \Re\beta(\Gamma(x))$ and $\Im\Gamma(\alpha(x)) = \Im\beta(\Gamma(x))$.
- P. Speissegger and I characterized the domains where Γ is definable in $\mathbb{R}_{\mathcal{G},\exp}$. Show $\Gamma(\alpha(x)) = \beta(\Gamma(x))$ holds definably in $\mathbb{R}_{\mathcal{G},\exp}$ (picture)
- Using the embedding of definable functions into transseries, get two equations of transseries (real and imaginary parts).
- Do transseries computations to show $\alpha(x) = \beta(x) = x$.

Remarks

Theorem (3)

Suppose $W \subset \mathbb{C}^n$ and $V \subseteq \mathbb{C}^{2n}$ are irreducible algebraic varieties satisfying that for all $(z_1, \ldots, z_n) \in W$ we have $(z_1, \ldots, z_n, \Gamma(z_1), \ldots, \Gamma(z_n)) \in V$. If $\dim V = 2\dim W$, then W is completely defined by equations of the forms $X_i = X_j + b_{ij}$, where $b_{ij} \in \mathbb{Z}$, and $X_i = c$, for some $c \in \mathbb{C}$.

- Prove the full statement of Theorem (2) by induction.
- We also have another proof of Theorem (2).
- The valuation theory/transseries strategy adapts to give a new proof of the Ax–Lindemann–Weierstrass theorem.

Next steps

If $\Gamma(W) \subseteq V$, what can we say about W and V?

- Theorem (2) gives the answer if $\dim W = \dim V$.
- Theorem (1) gives the answer if W is a translate of a $\mathbb{Q}$-line.
- Di Vizio and Pellarin: algebraic dependencies between functions of the form $\Gamma(z - f(z))$ where $f(z)$ is algebraic over K.
- In general, when is $S = \{\Gamma(\alpha_1(z)), \ldots, \Gamma(\alpha_n(z))\}$ algebraically dependent? Do the dependencies come from the known functional equations for Γ?
- What about other o-minimal functions with transseries expansions?