# Generalising Kim's lemma to abstract independence relations

Alberto Miguel Gómez

University of Cambridge

Workshop on Model Theory and Applications (Chania)

# Outline

# What is an independence relation?

Fix a complete theory $T$ and monster $\mathbb{M} \models T$.

For us, an **independence relation** is an $\mathrm{Aut}(\mathbb{M})$-invariant ternary relation on small subsets of $\mathbb{M}$. We denote it by $\mathop{\smile\!\!\!\!\!\!|}\,$, and write $A \mathop{\smile\!\!\!\!\!\!|}_C B$ if $(A, B, C) \in \mathop{\smile\!\!\!\!\!\!|}\,$.

For the rest of this talk, every independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$ will satisfy the following two properties:

- *Full existence*: for any sets $A, B$ and $C$, there is $A' \equiv_C A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B$.
- *Monotonicity*: if $A \mathop{\smile\!\!\!\!\!\!|}_C B$ and $A' \subseteq A$, $B' \subseteq B$, then $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.

# A look at the classification map

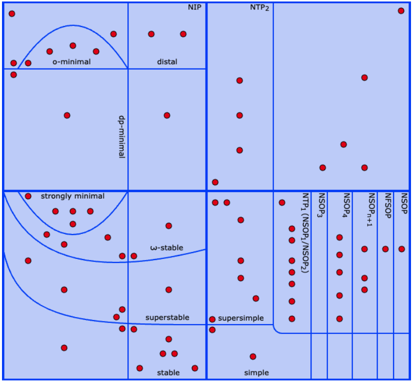


Figure: forkinganddividing.com (Conant)

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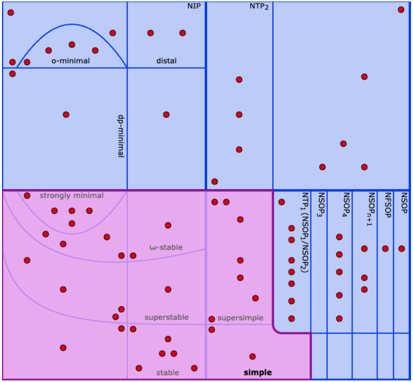


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# Simple theories

**Fact (Kim's Lemma)**

*A theory $T$ is simple iff, whenever $\varphi(x, b)$ divides over $C$, it divides along every Morley sequence over $C$ starting at $b$.*

Kim and Pillay characterised simplicity in terms of the behaviour of non-forking:

**Definition**

$A \mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{f}_{C} B :=$ there is no formula in $\mathrm{tp}(A/BC)$ that forks over $C$.

If $T$ is simple:

- $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{f}$ is symmetric.
- $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{f}$ satisfies local character.
- $\mathop{\smile\!\!\!\!\!\!\mid}\nolimits^{f}$ satisfies the independence theorem over models.

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# Adler's framework

## Definition

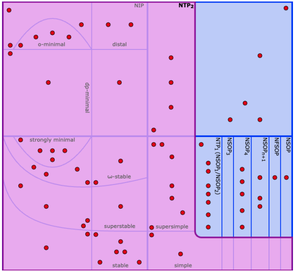
We say $\mathop{\smile\!\!\!\!\!\!|}\,$ is an **Adler independence relation** if it satisfies monotonicity, right base monotonicity, left transitivity, left normality, finite character, local character, and right extension.

Each of these properties is "algebraic".

## Fact (Adler)

*Every Adler independence relation is symmetric.*

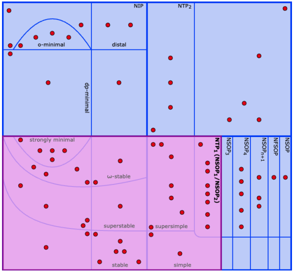
# Beyond simplicity




NTP$_2$ (Chernikov-Kaplan):

- A theory $T$ is NTP$_2$ iff, whenever $\varphi(x, b)$ divides over $C$, it divides along every *strict* Morley sequence over $C$ starting at $b$.
- If $T$ is NTP$_2$, forking and dividing coincide over models.

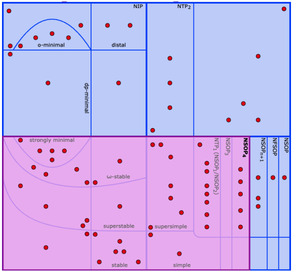
# Beyond simplicity




NSOP$_1$ (Kaplan, Shelah, Ramsey):

- A theory $T$ is NSOP$_1$ iff, whenever $\varphi(x, b)$ *Kim*-divides over $M$, it divides along every $M$*-invariant* Morley sequence starting at $b$.
- If $T$ is NSOP$_1$, $\mathop{\smile\hskip -0.9em ^| \ }^{K}$ satisfies symmetry, *chain* local character, and the independence theorem over models.

# Beyond simplicity




NSOP$_4$ (Mutchnik):

- In many NSOP$_4$ theories, there is a stationary independence relation satisfying the *relative Kim's lemma*.
- If *Conant*-independence is symmetric, then $T$ is NSOP$_4$.

# Main question

How can we understand the recent developments in neostability theory in terms of abstract independence relations?

# Outline

# Generalised Morley sequences

Fix an independence relation $\mathop{\smile\kern-0.9em\mid}$. We look at "special" sequences:

## Definition

A sequence $(a_i)_{i<\omega}$ is $\mathop{\smile\kern-0.9em\mid}$-**Morley over** $C$ if it is $C$-indiscernible and $a_i \mathop{\smile\kern-0.9em\mid}_C a_{<i}$ for all $i < \omega$.

## Examples

- $(a_i)_{i<\omega}$ is $\mathop{\smile\kern-0.9em\mid}^f$-Morley over $C$ iff it is Morley over $C$.
- $(a_i)_{i<\omega}$ is $\mathop{\smile\kern-0.9em\mid}^i$-Morley over $M$ iff it is $M$-invariant Morley.
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- $(a_i)_{i<\omega}$ is $\mathop{\smile\!\!\!\!\!\!\;|}^{\mathrm{ist}}$-Morley over $C$ iff it is strict Morley over $C$.

# The Key Idea

For an independence relation $\mathop{\smile\!\!\!\!\!\!\mid}$:

- $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-**Kim-divides over** $M$ if it divides along *some* $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over $M$.
- $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-**Conant-divides over** $M$ if it divides along *every* $\mathop{\smile\!\!\!\!\!\!\mid}$-Morley sequence over $M$.
- $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-**Kim-forks** (resp., $\mathop{\smile\!\!\!\!\!\!\mid}$-**Conant-forks**) **over** $M$ if $\varphi(x, b) \vdash \bigvee_{i<n} \psi_i(x, c_i)$ and each $\psi_i(x, c_i)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-Kim-divides (resp., $\mathop{\smile\!\!\!\!\!\!\mid}$-Conant-divides) over $M$.

# Freeness

We say $\varphi(x, b)$ is $\mathop{\smile\!\!\!\!\!\!|}\,$-**free over** $C$ if there is some $a \models \varphi(x, b)$ such that $a \mathop{\smile\!\!\!\!\!\!|}_C b$.

## Examples

- $\varphi(x, b)$ is $\mathop{\smile\!\!\!\!\!\!|}^{u}$-free over $C$ iff there is $c \in C$ such that $\models \varphi(c, b)$.
- $\varphi(x, b)$ is $\mathop{\smile\!\!\!\!\!\!|}^{f}$-free over $C$ iff it does not fork over $C$.
- $\varphi(x, b)$ is $\mathop{\smile\!\!\!\!\!\!|}^{a}$-free over $C$ iff it does not *strongly fork* over $C$.

$A \mathop{\smile\!\!\!\!\!\!|}^{u}_{C} B$ iff $\mathrm{tp}(A/BC)$ is a coheir over $C$

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## Some Adlerian operations

We also need some operations on an independence relation $\mathop{\smile\!\!\!\!\!\!|}\,$:

- Define $\mathop{\smile\!\!\!\!\!\!|}^*$ by "forcing right extension": $A \mathop{\smile\!\!\!\!\!\!|}^*_C B$ iff, for every $B' \supseteq B$, there is $A' \equiv_{BC} A$ such that $A' \mathop{\smile\!\!\!\!\!\!|}_C B'$.
- Define $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{sfc}}$ by "forcing strong finite character": $A \mathop{\smile\!\!\!\!\!\!|}^{\mathrm{sfc}}_C B$ iff every formula in $\mathrm{tp}(A/BC)$ is $\mathop{\smile\!\!\!\!\!\!|}$-free over $C$.

# Relativising Kim- and Conant-independence

- Define $A \mathop{\smile\!\!\!\!\!\!\;|}^{+}_{M} B$ iff, for every $\mathop{\smile\!\!\!\!\!\!\;|}$-Morley sequence $I$ over $M$ starting at $B$, there is $A' \equiv_{MB} A$ such that $I$ is $MA'$-indiscernible.

- $\mathop{\smile\!\!\!\!\!\!\;|}$**-Kim-independence** $= ((\mathop{\smile\!\!\!\!\!\!\;|}^{+})^{\mathrm{sfc}})^*$.

- We say $\varphi(x, b)$ **weakly** $\mathop{\smile\!\!\!\!\!\!\;|}$**-Kim-divides over** $M$ if it is *not* $\mathop{\smile\!\!\!\!\!\!\;|}^{+}$-free over $M$.

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# Relativising Kim- and Conant-independence

## Theorem

- $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-*Conant-divides over* $M$ *iff it divides along* every $\mathop{\smile\!\!\!\!\!\!\mid}$-*Morley sequence over* $M$.
- $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-*Kim-forks (resp.,* $\mathop{\smile\!\!\!\!\!\!\mid}$-*Conant-forks) over* $M$ *iff* $\varphi(x, b) \vdash \bigvee_{i<n} \psi_i(x, c_i)$ *and each* $\psi_i(x, c_i)$ $\mathop{\smile\!\!\!\!\!\!\mid}$-*Kim-divides (resp.,* $\mathop{\smile\!\!\!\!\!\!\mid}$-*Conant-divides) over* $M$.
- $A$ *is* $\mathop{\smile\!\!\!\!\!\!\mid}$-*Kim-independent (resp.,* $\mathop{\smile\!\!\!\!\!\!\mid}$-*Conant-independent) from* $B$ *over* $M$ *iff there is no formula in* $\mathrm{tp}(A/BM)$ *that* $\mathop{\smile\!\!\!\!\!\!\mid}$-*Kim-forks (resp.,* $\mathop{\smile\!\!\!\!\!\!\mid}$-*Conant-forks) over* $M$.

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- $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!|}$-*Kim-forks (resp.,* $\mathop{\smile\!\!\!\!\!\!|}$-*Conant-forks) over* $M$ *iff* $\varphi(x, b) \vdash \bigvee_{i<n} \psi_i(x, c_i)$ *and each* $\psi_i(x, c_i)$ $\mathop{\smile\!\!\!\!\!\!|}$-*Kim-divides (resp.,* $\mathop{\smile\!\!\!\!\!\!|}$-*Conant-divides) over* $M$.
- $A$ *is* $\mathop{\smile\!\!\!\!\!\!|}$-*Kim-independent (resp.,* $\mathop{\smile\!\!\!\!\!\!|}$-*Conant-independent) from* $B$ *over* $M$ *iff there is no formula in* $\mathrm{tp}(A/BM)$ *that* $\mathop{\smile\!\!\!\!\!\!|}$-*Kim-forks (resp.,* $\mathop{\smile\!\!\!\!\!\!|}$-*Conant-forks) over* $M$.

# Relativising Kim- and Conant-independence

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# Back to Kim's lemma

We will say $\mathop{\smile\!\!\!\!\!\!\,|}^1$ **has GUWP w.r.t.** $\mathop{\smile\!\!\!\!\!\!\,|}^2$ if, whenever $\varphi(x, b)$ $\mathop{\smile\!\!\!\!\!\!\,|}^2$-Kim-divides over $M$, it also $\mathop{\smile\!\!\!\!\!\!\,|}^1$-Conant-divides over $M$.

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*Suppose $\mathop{\smile\!\!\!\!\!\!\,|}^1$ and $\mathop{\smile\!\!\!\!\!\!\,|}^2$ satisfy full existence and monotonicity. TFAE:*

1. *$\mathop{\smile\!\!\!\!\!\!\,|}^1$ witnesses $\mathop{\smile\!\!\!\!\!\!\,|}^2$.*
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# Back to Kim's lemma

## Examples

- If $T$ is simple, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{f}}$ witnesses $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$.
- If $T$ is NTP$_2$, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{ist}}$ witnesses $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{a}}$.
- If $T$ is NSOP$_1$, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ witnesses $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$.
- If $T$ is NBTP, then $\mathop{\smile\!\!\!\!\!\!|}^{K\mathrm{st}}$ witnesses $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$.
- If $T$ is NCTP, then $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{bi}}$ witnesses $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$.

# Outline

# The Old

**Theorem**

*Suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies full existence and monotonicity. Assume $\mathop{\smile\!\!\!\!\!\!|}$ is "compatible" with $\mathop{\smile\!\!\!\!\!\!|}$-Kim-independence, and $\mathop{\smile\!\!\!\!\!\!|}$-Kim-independence satisfies left transitivity. TFAE:*

1. *$\mathop{\smile\!\!\!\!\!\!|}$-Kim-independence is symmetric.*
2. *$\mathop{\smile\!\!\!\!\!\!|}$-Kim-independence witnesses $\mathop{\smile\!\!\!\!\!\!|}$.*

*In this case, $\mathop{\smile\!\!\!\!\!\!|}$-Kim-independence also satisfies chain local character.*

**Examples**

1. This recovers symmetry and local character for $\mathop{\smile\!\!\!\!\!\!|}^{f}$ in simple theories.
2. Assuming symmetry, we recover "witnessing" and chain local character for Kim-independence in $NSOP_1$ theories.

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# The Old

**Theorem**

*Suppose $\mathop{\smile\!\!\!\!\!\!|}\,' \implies \mathop{\smile\!\!\!\!\!\!|}$ satisfy full existence, monotonicity and right extension, and additionally $\mathop{\smile\!\!\!\!\!\!|}\,'$ satisfies left extension and $\mathop{\smile\!\!\!\!\!\!|}$ "weak transitivity". Assume $\mathop{\smile\!\!\!\!\!\!|} \implies \mathop{\smile\!\!\!\!\!\!|}$-Kim-independence. If $\mathop{\smile\!\!\!\!\!\!|}$-Kim-independence satisfies left transitivity and symmetry, then it also satisfies the $\mathop{\smile\!\!\!\!\!\!|}\,'$-independence theorem over models.*

**Examples**

1. Independence theorem over models in simple theories.
2. Assuming symmetry, "weak independence theorem" in $\mathrm{NSOP}_1$ theories.
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# The New

**Definition (repeated)**

We say $\mathop{\smile\!\!\!\!\!\!|}$ is an **Adler independence relation** if it satisfies monotonicity, right base monotonicity, left transitivity, left normality, finite character, local character, and right extension.

**Theorem**

*For any Adler independence relation $\mathop{\smile\!\!\!\!\!\!|}$:*

1. $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}^{\times}$. *In particular, $\varphi(x, b)$ is $\mathop{\smile\!\!\!\!\!\!|}$-free over $M$ iff it does not $\mathop{\smile\!\!\!\!\!\!|}$-Conant-fork over $M$.*
2. $\mathop{\smile\!\!\!\!\!\!|}$ *satisfies strong finite character iff* $\mathop{\smile\!\!\!\!\!\!|} = \mathop{\smile\!\!\!\!\!\!|}$*-Conant-independence.*

# The New

**Theorem**

*Suppose $\mathop{\smile\!\!\!\!\!\!|}$ satisfies full existence, monotonicity, and quasi-strong finite character*[1]*. If $\mathop{\smile\!\!\!\!\!\!|}^{\mathrm{i}}$ witnesses $\mathop{\smile\!\!\!\!\!\!|}$, then $T$ is either* $\mathrm{NSOP}_1$ *or* BTP*.*

As a consequence, this shows that most strictly $\mathrm{NSOP}_4$ theories have BTP.

**Conjecture**

If $T$ is $\mathrm{NSOP}_4$, then it is either $\mathrm{NSOP}_1$ or BTP.

[1] That is, for types $p(x), q(y) \in S(M)$, "$p(x) \cup q(y) \cup x \mathop{\smile\!\!\!\!\!\!|}_M y$" is type-definable over $M$.

## The Old and New

**Theorem**

*Let $\mathop{\smile\!\!\!\!\!\!|}^1 \implies \mathop{\smile\!\!\!\!\!\!|}^2$ be independence relations satisfying full existence and monotonicity such that $\mathop{\smile\!\!\!\!\!\!|}^1$ also satisfies left and right extension. If $\mathop{\smile\!\!\!\!\!\!|}^2$-Conant-independence satisfies the $\mathop{\smile\!\!\!\!\!\!|}^2$-independence theorem over models, then $\mathop{\smile\!\!\!\!\!\!|}^1$ witnesses $\mathop{\smile\!\!\!\!\!\!|}^2$.*

This recovers, without using trees, the fact that $T$ is NSOP$_1$ iff Conant-independence satisfies the independence theorem over models (Chernikov, Kaplan, Ramsey).

# The Old and New

**Theorem (Mutchnik)**

*Let $\downfree$ be an independence relation satisfying full existence, monotonicity, stationarity, and the generalised freedom axiom. If $\downfree$-Kim-independence is symmetric, then $\downfree^{\mathrm{i}}$ witnesses $\downfree$.*

**Proof.**

Using the properties of $\downfree$, it can be shown that $\downfree$-Kim-independence satisfies left transitivity. Thus, by our Theorem, as $\downfree$-Kim-independence is symmetric, it follows that $\downfree$-Kim-independence witnesses $\downfree$. But $\downfree^{\mathrm{i}} \implies \downfree$-Kim-independence. Thus, $\downfree^{\mathrm{i}}$ witnesses $\downfree$. $\square$

# The End

Thank you!