How I convinced myself that trace definability is really interesting

Erik Walsberg

June 4, 2026

Everything is first order, all theories are consistent & complete.

"Definable" means "first order definable, with parameters".

We all like linearity vs field structure dichotomies.

Field structure: Interprets an infinite field.

Linearity: def. sets are "linear objects", one-basedness, local modularity, etc ...

Thm (Peterzil-Starchenko)

$\mathcal{R}$ is an o-minimal expansion of $(\mathbb{R}; +, <)$. TFAE

- $\mathcal{R}$ is **not** a reduct of $\mathbb{R}_{\mathrm{Vec}} := (\mathbb{R}; +, <, (t \mapsto \lambda t)_{\lambda \in \mathbb{R}})$.
- $\mathcal{R}$ defines a copy of $(\mathbb{R}; +, \times)$.
- $\mathcal{R}$ interprets an infinite field.

Of course a version of this holds for o-minimal expansions of ordered groups.

Original goal: Generalize this to dp-minimal expansions of archimedean oags.

Simon: An expansion of $(\mathbb{R}; +, <)$ is o-minimal iff it is dp-minimal.

An expansion of $(\mathbb{Q}; +, <)$

$\mathcal{M}$ a structure, $A \subseteq M^m$ arbitrary.

$X \subseteq A^n$ is a **trace** of a def. set if $X = Y \cap A^n$ for $\mathcal{M}$-def. $Y \subseteq M^{mn}$.

induced structure on A is the structure generated by all traces of $\mathcal{M}$-def. sets.

Fix $t \in \mathbb{R}$, $t > 1$. $P = \{t^q : q \in \mathbb{Q}\}$, $\mathcal{P}$ = structure induced on P by $(\mathbb{R}; +, \times)$.

Identify $\mathcal{P}$ with an expansion of $(\mathbb{Q}; +, <)$ under $(\mathbb{Q}; +, <) \to (P; \times, <)$, $q \mapsto t^q$.

---

van den Dries and Günaydin: $\mathcal{P}$-def. sets = traces of semialgebraic sets.
Hence $\mathcal{P}$ is weakly o-minimal, hence dp-min.

Question: Is $\mathcal{P}$ "linear" ?????

On one hand: acl in $\mathrm{Th}(\mathcal{P})$ = acl in the underlying group over P

Pantelis: No model of $\mathrm{Th}(\mathcal{P})$ interprets an infinite field.

On the other:

The Shelah completion

$\mathcal{M}$ is NIP

Shelah completion: The structure $\mathcal{M}^{\mathrm{Sh}}$ induced on $\mathcal{M}$ by any big $\mathcal{M} \prec \mathcal{N}$.

Shelah: $\mathcal{M}^{\mathrm{Sh}}$-def. sets $=$ traces of $\mathcal{N}$-def. sets.

If $\mathcal{M}$ expands a linear order then all convex subsets of M are $\mathcal{M}^{\mathrm{Sh}}$-def.

NIP-theoretic properties transfer from $\mathcal{M}$ to $\mathcal{M}^{\mathrm{Sh}}$.

A good NIP-theoretic notion of linearity should transfer from $\mathcal{M}$ to $\mathcal{M}^{\mathrm{Sh}}$.

Let $\mathcal{P} \prec \mathcal{Q}$ be big.

$\mathcal{Q}$ doesn't interpret an infinite field but $\mathcal{Q}^{\mathrm{Sh}}$ interprets $(\mathbb{R}; +, \times)$!!!!

Fin = finite elements, Inf = infinitesimal elements, both $\mathcal{Q}^{\mathrm{Sh}}$-def. as convex

Induced structure on $\mathrm{Fin}/\mathrm{Inf}$ is a copy of $(\mathbb{R}; +, \times)$.

Similar examples

Let R be a big real closed field

$A \subseteq R$ dense alg. independent, $\mathcal{A}$ is the induced structure on A.

Dolich-Miller-Steinhorn: $\mathcal{A}$-def. $=$ trace of R-def.

Pantelis, Berenstein-Vassiliev: models of $\mathrm{Th}(\mathcal{A})$ can't interpret infinite groups.

Me: $\mathcal{A}^{\mathrm{Sh}}$ interprets $(\mathbb{R}; +, \times)$.

One possible response: I don't care about these structures.

Let $\mathbb{K}$ be a big p-adically closed field.

$\mathcal{B}$ the induced structure on the set of balls in $\mathbb{K}$, so interpretable in $\mathbb{K}$.

Thm: $\mathcal{B}$ does not interpret an infinite field but $\mathcal{B}^{\mathrm{Sh}}$ interprets $\mathbb{Q}_p$.

Surely $\mathcal{B}$ is a natural structure.

I propose: Use a weaker notion of interpretability to define "field structure".

A weaker notion of interpretability

$\mathcal{N}$ **trace defines** $\mathcal{M}$ if the following equiv. conditions hold:

- Up to iso. $M \subseteq N^n$ for some n and every $\mathcal{M}$-def. subset of every M^m is a trace of an $\mathcal{N}$-def. subset of N^{nm}.
- $\exists$ injection $\tau\colon M \hookrightarrow N^n$ for some n s.t. every $\mathcal{M}$-def. subset of every M^m is the pullback of an $\mathcal{N}$-def. subset of N^{mn}.

Trace definability is transitive.

$\mathcal{N}$ interprets $\mathcal{M}$ $\quad\Longrightarrow\quad$ $\mathcal{N}$ trace defines $\mathcal{M}$

$\mathcal{M}$ NIP $\quad\Longrightarrow\quad$ $\mathcal{M}^{\mathrm{Sh}}$ trace def. in any big $\mathcal{M} \prec \mathcal{N}$.

$(\mathbb{R}; +, \times)$ trace defs $\mathcal{P}$, $\quad$ $\mathcal{N}$ trace defs $(\mathbb{R}; +, \times)$ for big $\mathcal{P} \prec \mathcal{N}$

$\mathbb{Q}_p \prec \mathbb{K}$ interprets $\mathcal{B}$, $\quad$ $\mathcal{B}$ trace defs $\mathbb{Q}_p$.

T trace defines $\mathcal{M}$ if some model of T does.

T trace defines T^* if T trace defs every (equiv: some) T^*-model.

T, T^* are **trace equivalent** if each trace defines the other.

Two structures are trace equivalent ($\equiv_{\mathrm{tr}}$) when their theories are.

A weaker notion of interpretability

$\mathcal{N}$ **trace defines** $\mathcal{M}$ if the following equiv. conditions hold:

- Up to iso. $M \subseteq N^n$ for some n and every $\mathcal{M}$-def. subset of every M^m is a trace of an $\mathcal{N}$-def. subset of N^{nm}.
- $\exists$ injection $\tau\colon M \hookrightarrow N^n$ for some n s.t. every $\mathcal{M}$-def. subset of every M^m is the pullback of an $\mathcal{N}$-def. subset of N^{mn}.

Trace definability is transitive.

$\mathcal{N}$ interprets $\mathcal{M} \quad\Longrightarrow\quad \mathcal{N}$ trace defines $\mathcal{M}$

$\mathcal{M}$ NIP $\quad\Longrightarrow\quad \mathcal{M}^{\mathrm{Sh}}$ trace def. in any big $\mathcal{M} \prec \mathcal{N}$.

$(\mathbb{R}; +, \times)$ trace defs $\mathcal{P}$, $\mathcal{N}$ trace defs $(\mathbb{R}; +, \times)$ for big $\mathcal{P} \prec \mathcal{N}$ $\quad \mathcal{P} \equiv_{\mathrm{tr}} (\mathbb{R}; +, \times)$

$\mathbb{Q}_p \prec \mathbb{K}$ interprets $\mathcal{B}$, $\mathcal{B}$ trace defs $\mathbb{Q}_p$. $\quad \mathcal{B} \equiv_{\mathrm{tr}} \mathbb{Q}_p$

T trace defines $\mathcal{M}$ if some model of T does.

T trace defines T^* if T trace defs every (equiv: some) T^*-model.

T, T^* are **trace equivalent** if each trace defines the other.

Two structures are trace equivalent ($\equiv_{\mathrm{tr}}$) when their theories are.

Side note

$\mathcal{N}$ **trace defines** $\mathcal{M}$ if the following equiv. conditions hold:

- Up to iso. $M \subseteq N^n$ for some n and every $\mathcal{M}$-def. subset of every M^m is a trace of an $\mathcal{N}$-def. subset of N^{nm}.
- $\exists$ injection $\tau \colon M \hookrightarrow N^n$ for some n s.t. every $\mathcal{M}$-def. subset of every M^m is the pullback of an $\mathcal{N}$-def. subset of N^{mn}.

$\mathcal{N}$ **interprets** $\mathcal{M}$ if

- $\exists$ surjection $\pi \colon N^n \to M$ for some n s.t. every $\mathcal{M}$-def. subset of every M^m pulls back to an $\mathcal{N}$-def. subset of N^{mn}.

Trace definability

$\mathcal{N}$ **trace defines** $\mathcal{M}$ if:

- Up to iso. $M \subseteq N^n$ for some n and every $\mathcal{M}$-def. subset of every M^m is a trace of an $\mathcal{N}$-def. subset of N^{nm}.

T trace defines $\mathcal{M}$ if some model of T does.

T trace defines T^* if T trace defs some (equiv: every) T^*-model.

T, T^* are **trace equivalent** if each trace defines the other.

Two structures are trace equivalent when their theories are.

T is unstable	$\iff$	T trace defines $(\mathbb{Q}; <)$
T is IP	$\iff$	T trace defines the Erdős-Rado graph.

Properties preserved under trace definability:

Stability k-NIP, total transcendence, superstability, strong dependence, finiteness of Morley rank/U-rank/dp-rank/op-dimension.

All but one can be characterized in terms of trace definability.

Back to dp-minimal expansions of archimedean oags

Simon: Expansion of $(\mathbb{R}; +, <)$ is dp-minimal iff it is o-min.

Me: An exp. of a divisible archimedean oag is dp-min. iff it is weakly o-min.

ADHMS-MV: No proper dp-minimal expansion of $(\mathbb{Z}; +, <)$.

$(S; +, <)$ is a substructure of $(\mathbb{R}; +, <)$. $\mathcal{S}$ is a dp-min. expansion of $(S; +, <)$.
$\mathcal{S}^\square$ is the o-minimal completion of $\mathcal{S}$, an o-minimal expansion of $(\mathbb{R}; +, <)$.
S dense, $\mathcal{S}^\square$ is the exp. of $(\mathbb{R}; +, <)$ by all closures in $\mathbb{R}^n$ of $\mathcal{S}$-def. subsets of S^n.
S discrete, take $\mathcal{S}^\square = (\mathbb{R}; +, <)$.

Thm: Every $\mathcal{S}$-def. set is a boolean combination of $(S; +)$-def. sets and traces of $\mathcal{S}^\square$-def. sets.

Black box: $\mathcal{S} \equiv_{\mathrm{tr}} \mathcal{S}^\square$.

So should have: $\mathcal{S}$ "linear" $\iff$ $\mathcal{S}^\square$ "linear" $\iff$ $\mathcal{S}^\square$ a reduct of $\mathbb{R}_{\mathrm{Vec}}$.

But: How do we know that $\mathbb{R}_{\mathrm{Vec}}$ does not trace define an infinite field???????

How do we know that anything doesn't trace define an infinite field?????

Near linear Zarankiewicz bounds

$E \subseteq V \times W$ has **NLZB** if $\forall \varepsilon > 0, k$ there is $\lambda > 0$ such that:
if $V_0 \subseteq V, W_0 \subseteq W$ finite and E is $K_{k,k}$-free on V_0, W_0, then

$$|E \cap (V_0 \times W_0)| \leq \lambda(|V_0| + |W_0|)^{1+\varepsilon}$$

$\mathcal{M}$ has NLZB when every def. relation does, T has NLZB when its models do.

Infinite fields do not have NLZB (point-line incidence relation).

Preserved under trace definability.

Basit-Chernikov-Starchenko-Tao-Tran: Ordered vector spaces have NLZB.

So TFAE for $\mathcal{S}$ a dp-minimal expansion of an archimedean oag:

- $\mathcal{S}^\square$ is a reduct of $\mathbb{R}_{\mathrm{Vec}}$.
- $\mathcal{S}$ has NLZB.
- $\mathcal{S}$-def. sets are boolean comb. of $(S;+)$-def. sets and traces of $\mathbb{R}_{\mathrm{Vec}}$-def. sets.
- $\mathrm{Th}(\mathcal{S})$ does not trace define an infinite field.
- $\mathrm{Th}(\mathcal{S})$ does not trace define $(\mathbb{R};+,\times)$.

Breaking news

Gou-Mittal-Tran-Yang: One-based structures have NLZB.

So do $(\mathbb{Q}_p; +, <_p)$ and $(\mathbb{C}_p; +, <_p)$.

Hence these structures cannot trace define an infinite field.

…Now let's go inside the black box.

I claimed: $\mathcal{S}$ is trace equivalent to $\mathcal{S}^{\square}$. Let's sketch the proof for S dense.

$\mathrm{Th}(\mathcal{S})$ trace defines $\mathcal{S}^{\square}$:

$\mathcal{S} \prec \mathcal{N}$ big.

$\mathrm{Fin} = \{a \in N : |a| \leq n \text{ for some } n\}$, $\mathrm{Inf} = \{a \in N : |a| \leq \frac{1}{n} \text{ for all } n \geq 1\}$

Both convex, hence $\mathcal{N}^{\mathrm{Sh}}$-def.

Structure induced on $\mathrm{Fin}/\mathrm{Inf}$ by $\mathcal{N}^{\mathrm{Sh}}$ is $\mathcal{S}^{\square}$.

$\mathrm{Th}(\mathcal{S}^{\square})$ trace defines $\mathcal{S}$:

Every $\mathcal{S}$-def. $X \subseteq S^n$ is a bool. comb. of $(S;+)$-def. sets & traces of $\mathcal{S}^{\square}$-def. sets.

$\Longrightarrow$ Suffices to show that $\mathrm{Th}(\mathcal{S}^{\square})$ trace defines $(S;+)$.

dp-minimality $\Longrightarrow$ S is non-singular, i.e. $|S/nS| < \infty$ for all $n \geq 1$.

Another black box: G is a non-divisible non-singular torsion free abelian group.

A theory T trace defs G iff T trace defs $(\mathbb{R};+)$ and is not totally trans.

So G is the simplest structure above $(\mathbb{R};+)$ which is not totally trans.

Hence $\mathrm{Th}(\mathbb{R};+,<)$ trace defines $(S;+)$.

Along the same lines, $(\mathbb{Z};+,<) \equiv_{\mathrm{tr}} (\mathbb{R};+,<)$

Things to prove

- One-basedness + finite U-rank is preserved under trace definability.
- Forking triviality + finite U-rank is preserved under trace definability.
- An ordered abelian group cannot trace define an infinite field.
- ACF_p cannot trace define ACF_0.
- There are $2^{\aleph_0}$ countable theories modulo trace equivalence.
- A theory with qe in a bounded arity relational language cannot trace define an infinite group.
- A structure with qe in a finite binary rel. language is trace equivalent to either the trivial structure, $(\mathbb{Q}; <)$, or the Erdős-Rado graph.
- Define something like NLZB that rules out trace definability of infinite groups.
- Give a dichotomy between field structure and linearity for the induced structure on a definable set of imaginaries in a p-adically closed field.

$\vdots$

Thank you.