On the elementary theory of the real exponential field

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(joint with A. Berarducci,
building on unpublished work
of A. Berarducci and M. Mamino)

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It is a "lower bound" for transcendence of exp: below a certain threshold, all algebraic relations between numbers and their exponentials must be explained by $\mathbb{Q}$-linear dependence and the fact that exp is a group homomorphism.

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Clearly this axiom scheme is recursive, so we also get a new proof of the Macintyre–Wilkie conditional decidability result (using that every complete, recursively axiomatized theory is decidable.)

Proof strategy

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Theorem (Ressayre 1993)

Let $(R, \exp)$ be an exponential field. If $\exp$ satisfies some growth axioms (essentially, if it grows faster than every polynomial) and $(R, \exp_{|(-1,1)})$ is elementarily equivalent to $\mathbb{R}_{\exp_{|(-1,1)}}$, then $(R, \exp)$ is elementarily equivalent to $\mathbb{R}_{\exp}$.

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Assume $SC_{\mathbb{R}}$. Then the prime model of the complete theory of $\mathbb{R}_{\exp_{|(-1,1)}}$ embeds into any definably complete restricted exponential field.

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Theorems 1 and 2 imply that under $\mathrm{SC}_{\mathbb{R}}$ the theory of definably complete restricted exponential fields is complete. With Ressayre, we obtain the main theorem.

Comments

- Ressayre's growth axioms follow from definable completeness and the differential equation, so we do not need to worry about them. Under definable completeness, Ressayre's theorem becomes just that if $(R, \exp_{|(-1,1)})$ is elementarily equivalent to $\mathbb{R}_{\exp_{|(-1,1)}}$ then $(R, \exp)$ is elementarily equivalent to $\mathbb{R}_{\exp}$.

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- L.S. Krapp has proved independently the unrestricted version of Theorem 2.

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- An (ℓ, n)-Khovanskii point over R is a solution of a system of the form $F_{p_1}(x_1, \ldots, x_n) = \cdots = F_{p_n}(x_1, \ldots, x_n) = 0$ and $\mathrm{Jac}(F_{p_1}, \ldots, F_{p_n})(x_1, \ldots, x_n) \neq 0$

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Proposition (Servi)

Let $M \subseteq N$ be definably complete restricted exponential fields.
If $F_{p_1}, \dots, F_{p_k}$ are $\exp_|$-polynomials over M of complexity $\leq \ell$ and they have a common zero, then the zero set of $F_{p_1}, \dots, F_{p_k}$ contains a projection of a Khovanskii point over M of complexity $\leq \ell$.

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Applying the Proposition and some tricks, we reduce model completeness to: for all $M \subseteq N$ definably complete restricted exponential fields, every Khovanskii point in N over M has coordinates in M.

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We parametrize C by $(\phi_1(t), \dots, \phi_n(t))$. Our key argument is that there is $m \in \mathbb{N}$ such that $|\phi_n(t) - \exp_|(\phi_\ell(t))| > t^{-m}$ for every sufficiently large t.

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Proving the embedding theorem, I

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Assume $SC_{\mathbb{R}}$. Then the prime model of the theory of $\mathbb{R}_{\exp_{|(-1,1)}}$ embeds into any definably complete restricted exponential field.

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However, this tells us that a lift of a Khovanskii point exists, but what if there are two different ones? Then their difference is infinitesimal, and the field they generate cannot embed in $\mathbb{R}$...

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Note that if $SC_{\mathbb{R}}$ holds, then the prime model embeds into N and it is contained in every convex subring of N. So in this case the hypothesis of the theorem is always true.

More recent and future work

Following comments from Wilkie, we observed that we can prove something similar to our main theorem using what Macintyre–Wilkie call *Weak Schanuel's Conjecture* (which is a statement equivalent to the decidability of the theory of $\mathbb{R}_{\exp}$).

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Is it true unconditionally that every definably complete restricted exponential field is polynomially bounded?

This would be a key step in showing that the theory of definably complete exponential fields is model complete.