On Generic Derivations and Endomorphisms

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*joint work with Antongiulio Fornasiero

Motivations

- Robinson: The theory of fields of characteristic 0 with one derivation has a model completion, the theory DCF_0 of differentially closed fields of characteristic 0.
- Blum: Axioms for DCF_0, i.e. K is a field of characteristic 0, and, whenever p and q are differential polynomials in one variable, with q not constant and of order strictly less than the order of p, then there exists a in K such that $p(a) = 0$ and $q(a) \neq 0$.
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Basic Definition

T is a model complete theory if not replace it with its Morleyzation.

Definition

Let T be a theory expanding the theory of fields of characteristic 0. T admits generic derivations if T^{δ} , i.e. T plus

$$\delta(x+y)=\delta x+\delta y,\ \delta(x\cdot y)=x\delta y+y\delta x.$$

has a model completion, T_g^{δ}.

Definition

Let T and T^* be theories in the same language L.
T^* is a model completion of T if the following hold:

1. If $A \models T^*$, then $A \models T$;
2. If $A \models T$, then there exists a $B \supset A$ such that $B \models T^*$;
3. If $A \models T$, $A \subset B$, $A \subset C$, where $B, C \models T^*$, then B is elementary equivalent to C over A.

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GOALS

Main results

- If T is algebraically bounded, then T admits generic derivations.
- T_g^δ inherits model theoretic properties from T: stability, NIP, simplicity.
- Conditions to have elimination of imaginaries of T_g^δ.
- If T is linearly bounded, then T admits generic endomorphism.
- Theories that do not admit a generic derivation.

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Algebraically bounded field

We fix an L-structure $\mathbb{K}$ expanding a field of characteristic 0.

Definition (Van den Dries)

$\mathbb{K}$ is **algebraically bounded** if, for any formula $\phi(\overline{x}, y)$, there exist finitely many polynomials $p_1, \ldots, p_m \in K[\overline{x}, y]$ such that for any $\overline{a}$, if $\phi(\overline{a}, \mathbb{K})$ is finite, then $\phi(\overline{a}, K)$ is contained in the zero set of $p_i(\overline{a}, y)$ for some i such that $p_i(\overline{a}, y)$ doesn't vanish.

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Algebraically bounded field

Fact

The following are equivalent:

1. The model theoretic algebraic closure coincide with the field theoretic algebraic closure in every elementarily extension of $\mathbb{K}$;
2. $\mathbb{K}$ is algebraically bounded;

Theories with generic derivations

- ACF_0 [Robinson '59, Blum '68]
- RCF [Singer '78]
- Henselian valued fields [Point et al.]
- Model complete "large" fields [Tressl '05]

Algebraically bounded fields

- Algebraically closed valued fields;
- p-adics;
- pseudo-finite fields.
- The expansion of an algebraically bounded structure by a generic predicate is still algebraically bounded;

Notations

Let L be the language of $\mathbb{K}$ and T be its theory.

Let $\overline{\delta} := \delta_1, \ldots, \delta_k$. We obtain two theories:

- $T^{\overline{\delta}}$ the expansion of T saying that the δ_i are derivations which commute with each other, i. e. for every $i, j \leq k$, $\delta_i \circ \delta_j = \delta_j \circ \delta_i$.
- $T^{\overline{\delta},nc}$ the expansion of T saying that the δ_i are derivations without any further conditions.

For convenience, we use $T_g^{\overline{\delta},?}$ to denote either of the model completions, both for commuting derivations and the non-commuting case.

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We denote by Γ the free commutative (or non commutative) monoid generated by $\overline{\delta}$, we can define the canonical partial order $\preceq$ given by $\beta \preceq \alpha\beta$, for all $\alpha, \beta \in \Gamma$.

Remark

If Γ is the free non commutative monoid generated by $\overline{\delta}$ then $\preceq$ is a well-founded partial order on Γ, but it is not a well-partial-order (i.e., there exist infinite anti-chains).

Remark

If Γ is the free commutative monoid generated by $\overline{\delta}$, with the canonical partial order $\preceq$, it is isomorphic to $\mathbb{N}^k$.

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Two axiomatizations

Definition

Let $X \subseteq \mathbb{K}^n$ be L-definable with parameters. X is large if $\dim(X) = n$.

Two possible axiomatizations for the model completion T_g^δ are given by T^δ and either of the following axiom schemas:

Deep

For every $Z \subseteq \mathbb{K}^{n+1}$ $L(\mathbb{K})$-definable, if $\Pi_n(Z)$ is large, then there exists $c \in \mathbb{K}$ such that $Jet_n(c) = \langle \delta^i c : i \leq n \rangle \in Z$;

Wide

For every $W \subseteq \mathbb{K}^n \times \mathbb{K}^n$ $L(\mathbb{K})$-definable, if $\Pi_n(W)$ is large, then there exists $\overline{c} \in \mathbb{K}^n$ such that $< \overline{c}, \delta\overline{c} > \in W$.

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Proposition

Let T and T^* be theories in the same language L such that $T \subseteq T^*$. The following are equivalent:

1. T^* is the model completion of T and T^* eliminates quantifiers.
2.
    1. For every $A \models T$, for every $\sigma_1, \ldots, \sigma_n \in T^*$, there exists $B \models T$ such that $A \subseteq B$ and $B \models \sigma_1 \wedge \ldots \wedge \sigma_n$;
    2. For every L-structures A, B, C such that $B \models T$, $C \models T^*$, and A is a common substructure, for every quantifier-free $L(A)$-formula $\phi(x)$, and for every $b \in B$ such that $B \models \phi(b)$, there exists $c \in C$ such that $C \models \phi(c)$.

Main result

Notations:

$T^{\delta}_{deep} := T^{\delta} \cup Deep$ and $T^{\delta}_{wide} := T^{\delta} \cup Wide$

Theorem (Fornasiero and T. 2024)

The model completion T^{δ}_{g} of T^{δ} exists, and the theories T^{δ}_{deep} and T^{δ}_{wide} are two possible axiomatizations of T^{δ}_{g}.

Sketch of the proof

- $T^{\delta}_{wide} \vdash T^{\delta}_{deep}$
- We apply Criteria (2) when $T = T^{\delta}$. (1) holds for $T^{*} = T^{\delta}_{wide}$ and (2) holds for $T^{*} = T^{\delta}_{deep}$.

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Several commuting and non commuting derivations

McGrail

Axiomatized $DCF_{0,m}$. The axiomatization is complicate.
$DCF_{0,m}$ is ω-stable, it eliminates imaginaries its models are algebraically closed field.

Moosa and Scanlon 2014

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The theory is stable but not ω-stable.

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Generic derivations

Theorem (Fornasiero, T. 2024)

1. *If T is model complete then $T_g^{\overline{\delta},?}$ is the model completion of $T^{\overline{\delta},?}$.*
2. *If T eliminates quantifiers, then $T_g^{\overline{\delta},?}$ eliminates quantifiers.*
3. *For every $L^{\overline{\delta}}$-formula $\alpha(\overline{x})$ there exists an L-formula $\beta(\overline{x})$ such that*

$$T_g^{\overline{\delta},?} \models \forall \overline{x}(\alpha(\overline{x}) \leftrightarrow \beta(Jet(\overline{x})).$$

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Stability, NIP

Theorem (Fornasiero, T. 2024)

1. If T is stable, then $T_g^{\bar{\delta},?}$ is stable.
2. If T is NIP, then $T_g^{\bar{\delta},?}$ is NIP.

Proof

1. QE + T is stable iff any indiscernible sequence is totally indiscernible.
2. QE + T is dependent iff every indiscernible sequence does not alternate infinitely many times .

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Stability, NIP

Theorem (Fornasiero, T. 2024)

1. If T is stable, then $T_g^{\bar{\delta},?}$ is stable.
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Theorem (Fornasiero, T. 2025)

$T_g^{\overline{\delta}}$ is ω-stable iff $T = ACF_0$ and the derivations commute.

Proof

Theorem (Fornasiero, Kaplan and Matthews)

If T is geometric and stable, then it is strongly minimal.

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Elimination of imaginaries

Definition

T eliminates imaginaries (T has EI) if every imaginary is interdefinable with a real tuple: i.e., for every imaginary e there is a real tuple a such that $dcl^{eq}(e) = dcl^{eq}(a)$.

Conjecture 2024

Let T be algebraically bounded. Then, $T_g^{\bar{\delta},?}$ has elimination of imaginaries modulo T^{eq}.

Knew cases

ACF_0 and RCF.

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Generic endomorphism

Assumption

- $\mathbb{M}$ is a structure expanding vector space.
- $M_0 = acl(\emptyset)$, L is the language of $\mathbb{M}$ expanding L_K and we assume that it contains a constant for each element of M_0.
- T is its L-theory and we assume model complete theory.
- We consider η endomorphism of M_0 and ϕ is some endomorphism of M extending η.
- We denote by T^ϕ the L^ϕ-expansion of T saying that ϕ is an endomorphism extending η;

Generic endomorphism

Theorem (Fornasiero, T. 2026)

- T admits generic endomorphism that we denote by T_g^{ϕ}
- If T has QE then T_g^{ϕ}.
- If T is stable then $T_g^{\phi}$is stable.
- If T is NIP then T_g^{ϕ} is NIP.

No model completion

Theories with no model completion

1. Theories of groups; [Eklof-Sabbagh '71]
2. Theories of modules over certain rings; [Eklof-Sabbagh '71]
3. Theories of exponential fields. [Haykazyan- Kirby 2021]

No generic derivations

Theorem (Fornasiero, T. 2025)

If T expands a non trivial exponential field of characteristic 0, then T does not admit generic derivations.

Corollary

$(\mathbb{C}, exp)$, $(\mathbb{B}, E(x))$ *where* $\mathbb{B}$ *is a Zilber field and* $(\mathbb{R}, exp)$ *do not admit generic derivations.*

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Sketch of the proof

Definition

T^δ has a model companion if the class $\mathcal{E}$ of existentially closed models of T^δ is elementary, and in that case the model companion is the theory of $\mathcal{E}$.

Definition

A model $M \models T$ is existentially closed if, for every quantifier free formula $\phi(\overline{x}, \overline{y})$ and all $\overline{a}$ in M, if there is an extension $M \subset B$ such that $B \models T$ and $B \models \exists \overline{x} \phi(\overline{x}, \overline{a})$ then

$$M \models \exists \overline{x} \phi(\overline{x}, \overline{a}).$$

Lemma

Let $(\mathbb{K}, \delta) \in \mathcal{E}$. Let $X \subset K^2$ be $\mathbb{K}$-definable, and consider $\delta_{\restriction X} : X \to K^2$. We have that $\delta_{\restriction X}$ is a surjective iff X is Zariski dense in K^2.

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Restricted exponential function

Theorem (Fornasiero, T. 2025)

There exists an L-definable family $(\Gamma_c : c \in I)$ of subsets of K^2 with $I = [0,1]$, such that Γ_c is not Zariski dense in K^2 iff $c \in \mathbb{Q}$.

Corollary

Let $(\mathbb{K}, \delta) \in \mathcal{E}$. We have δ_c is surjective iff $c \notin \mathbb{Q}$.

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Restricted sine

Assumption

$L \supseteq \{\sin x, \cos x, \pi\}$ and T expands the theory of ordered fields. Let $\mathbb{K} \models T$, and we denote by $D = [0, 2\pi) \subseteq K$, S the field $K + iK$, S^* the multiplicative group of S.

Definition

We say that $\mathbb{K}$ defines a restricted sine if $\pi > 0$, and the function $F : D \to S^*$, $x \mapsto \sin x + i \cos x$ is a group homomorphism that is not constant. We say that T defines a restricted sine if the above holds in all models of T.

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