k-distality, k-triviality, and k-arity

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k-triviality

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Goode also introduced the stronger notion of *k-total triviality*.

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Say that a tuple $(a_1, ..., a_n)$ is *independent* over a small set D if $a_i \mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}_D a_{<i}$ for all $i \in [n]$.

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Likewise, say that T is *strictly* k-totally trivial if it is k-totally trivial but not $(k-1)$-totally trivial.

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The following example (Goode, 1991) shows that they are not equivalent.

Consider the two-sorted structure $(X_1, X_2, g : X_1 \times X_2 \to X_2)$, whose axioms say that g induces a free action of $F(X_1)$ on X_2 (free: no non-trivial word has a fixed point).

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This construction can be iterated to give an n-sorted structure
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This separates k-triviality from k-total triviality for all $k \in \mathbb{N}^+$.

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Does this collapse hold among stable theories?

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Does this collapse hold among stable theories?

Theorem (T., 2026+)

Yes, even among simple theories.

Connection to k-arity

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A stable $(k+1)$-ary theory is k-totally trivial (and hence k-trivial).

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A simple theory with QE in a *finite* relational language is trivial.

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Theorem (Baldwin–Freitag–Mutchnik, 2024+)

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Remaining gap: simple theories with QE in an infinite relational language of bounded arity.

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— Amador Martin-Pizarro

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2. *(External characterisation)* Let $\mathcal{I}$ and $\mathcal{J}$ be infinite sequences (dense, without endpoints), $a \in \mathcal{U}$, and $B \subseteq \mathcal{U}$. If $\mathcal{I} + \mathcal{J}$ is B-indiscernible and $\mathcal{I} + a + \mathcal{J}$ is indiscernible, then $\mathcal{I} + a + \mathcal{J}$ is B-indiscernible.

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Examples of distal theories/structures

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- (T., 2025) $(\mathbb{Z}, <, +, R)$ for $R = \{2^n : n \in \mathbb{N}\}, \{n! : n \in \mathbb{N}\}, \{\text{Fibonacci}\}, \ldots$

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Have: distality $\Leftrightarrow$ 1-distality $\Leftrightarrow$ strong 1-distality. Obvious that strong k-distality $\Rightarrow$ k-distality for all $k \in \mathbb{N}^+$. (Also, k-distality $\Rightarrow$ NIP_k.)

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To answer this, we need to go back to k-triviality.

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Observe that no examples were given that are NIP and strictly k-distal for some $k \geq 3$. (Goode's examples are strictly *strongly* k-distal for arbitrarily large k.)

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Corollary

If an NIP theory is strictly k-distal for some $k \geq 3$, it must be unstable.

I would like to know if an example exists!

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Since T is binary, $\phi(x, y_1, y_2)$ is equivalent in T to a Boolean combination of some $\psi_1(x, y_1, y_2), \ldots, \psi_l(x, y_1, y_2) \in L(\mathcal{I}_0 \mathcal{J}_0)$, where each ψ_j either has no x-dependence or no y_i-dependence for some $i \in [2]$. Write $\tau(\psi_1, \ldots, \psi_l)$ for this Boolean combination.

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It will be helpful to talk about the precise *arity* of a theory T, $\mathrm{ar}(T)$; define this to be the least $k \in \omega$ such that T is k-ary if this exists, and ω otherwise.

So, for $k \geq 2$, we seek strictly (strongly) k-distal theories T with $\mathrm{ar}(T) > k$.

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Let T_n denote the theory of Goode's n-sorted labelled free pseudoplane. Recall that T_n is strictly 2-distal and strictly strongly $(2^{n-1}+1)$-distal.

Proposition (T., 2026+)

For $n \geq 2$, $\mathrm{ar}(T_n) > 2^n - 1 \geq 2^{n-1} + 1 > 2$.

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Now R is a ternary relation with

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Now R is a ternary relation with

$$\mathbb{1}_R(v_1, v_2, v_3) \equiv \mathbb{1}_E(v_1, v_2) + \mathbb{1}_E(v_2, v_3) + \mathbb{1}_E(v_3, v_1) \mod 2.$$

Firstly, we show that T_2 is not (1-)distal. Indeed, let $v_1, v_2 \in V$ and $\mathcal{I}_0, \mathcal{I}_1, \mathcal{I}_2$ be infinite sequences of elements of V, such that:

1. The sequence $\mathcal{J} := \mathcal{I}_0 + v_1 + \mathcal{I}_1 + v_2 + \mathcal{I}_2$ consists of pairwise distinct elements,
2. $V \models \neg E(v_1, v_2)$;
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Then, for distinct $x_1, x_2, x_3 \in \mathcal{J}$, $V \models R(x_1, x_2, x_3)$ if and only if $\{v_1, v_2\} \not\subseteq \{x_1, x_2, x_3\}$.

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For $k \in \mathbb{N}^+$, the *Johnson graph* $\mathfrak{J}(k)$ is the structure $(\mathbb{N}^{(k)}, E)$, with $E(x, y) :\Leftrightarrow |x \cap y| = k - 1$. Let $T_k := \mathrm{Th}(\mathfrak{J}(k))$.

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$\mathfrak{J}(k)$ is homogeneous in the language $\{E_{ij} : i,j \in \mathbb{N}, 2 \leq i \leq k+1, 0 \leq j \leq k-1\}$, where E_{ij} is an i-ary relation symbol interpreted as

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Also, $\mathrm{ar}(T_2) \geq 3$, so $\mathrm{ar}(T_k) \geq 3$ for all $k \geq 2$. In fact, $\mathrm{ar}(T_k) \xrightarrow{k \to \infty} \infty$.

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So, we obtain strictly (strongly) 2-distal theories with arbitrarily large finite arity. Can we have infinite arity?

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Let $M := \{\{\{a, b\}, \{c, d\}\} : a, b, c, d \in \mathbb{N} \text{ distinct}\}$, and let $\mathrm{Sym}(\mathbb{N})$ act on M (and hence M^k for each $k \in \mathbb{N}^+$) in the obvious way. For each $k \in \mathbb{N}^+$, there are finitely many orbits on M^k.

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The language L is thus relational, with finitely many relation symbols of each arity. Let $T = \mathrm{Th}_L(M)$. [Cherlin–Lachlan, 1986]: $\mathrm{ar}(T) = \omega$.

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The language L is thus relational, with finitely many relation symbols of each arity. Let $T = \mathrm{Th}_L(M)$. [Cherlin–Lachlan, 1986]: $\mathrm{ar}(T) = \omega$. As $\mathrm{ar}(T) = \omega$, our previous proof of (strong) 2-distality does not go through. Nonetheless:

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Observe that M is interpretable in $(\mathbb{N}, =)^{\mathrm{eq}}$, so T is ω-categorical and ω-stable. It thus suffices to prove that T is (1-)totally trivial. As before, it suffices to prove that T is trivial.

As an ω-categorical and ω-stable theory, T is 1-based. Suppose T were not trivial. [De Piro–Kim, 2003]: an infinite vector space is interpretable in every non-trivial, 1-based, ω-categorical theory, and hence in T. [Thomas, 1991]: M is a reduct of a structure M' that is homogeneous in a finite binary relational language, so M' interprets an infinite vector space. This contradicts [Macpherson, 1991], which says that no infinite group is interpretable in a structure that is homogeneous in a finite relational language.

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I do not imagine that every (strongly) k-distal theory is the reduct of a k-ary one, or vice versa, but I do not have counterexamples for either direction.

On the topic of reducts...

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[Goode, 1991]: stable, 1-based, and trivial theories are totally trivial, so in fact T is totally trivial. Thus, T is strongly 2-distal, and hence strongly k-distal for all $k \geq 2$. Now, T' is not trivial, so not k-trivial for any $k \in \mathbb{N}^+$, and hence not k-distal for any $k \geq 2$.

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Theorem (T. 2026++), modulo slight lie

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Let $\phi(x_1, x_2; y) \in L$, $B \subseteq M \models T$ with $|B| \geq 2$ and $a_1, a_2 \in M$. Then there are $\psi(x_1, x_2; z), \psi_1(x_1, y; z_1), \psi_2(x_2, y; z_2) \in L$ such that, for all finite $\bar{B} \subseteq B$, there are $c, c_1, c_2 \in B$ such that, for all $b \in \bar{B}$,

$$(a_1, a_2) \models \psi(x_1, x_2; c) \wedge \psi_1(x_1, b; c_1) \wedge \psi_2(x_2, b; c_2) \vdash \phi(x_1, x_2; b) \leftrightarrow \phi(a_1, a_2; b).$$

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Key ingredient in forthcoming work on hypergraph regularity lemmas for NIP strongly k-distal theories (also in my PhD thesis).