Coloring in graphs definable in a pure set

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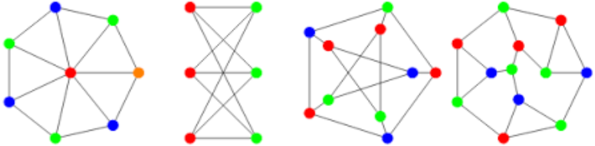
Overview

1. The notions
2. Results
3. Part of the proof
4. Questions

Notation

- A *(vertex) coloring* of a graph $G = (V, E)$ is an assignment of colors to the vertices of G so that no two adjacent vertices receive the same color.



- The *chromatic number* of G is the minimum number $\chi(G)$ of colors needed.
- Class = hereditary class = closed under induced subgraph = closed under removing vertices from graphs
- For subgraph, we are allowed to removed vertices and edges.

χ-BOUNDEDNESS

- The basic questions: What makes a graph class have unbounded chromatic number? Which graphs is it necessary/sufficient to contain?
- We can ask if $\chi(G)$ "controlled" by the size of the largest clique in G (denoted $\omega(G)$).
- If we ask that $\chi(G) = \omega(G)$ for every $G \in \mathcal{C}$, this yields a class of *perfect graphs*.

Definition

A graph class $\mathcal{C}$ is *χ-bounded* if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in \mathcal{C}$.

- Not every class is χ-bounded: there exist classes with $\chi(G)$ unbounded and $\omega(G)$ bounded.

SET-DEFINED CLASSES

Definition

A class $\mathcal{C}$ of graphs is *set-defined (of dimension d)* if it is a subclass of the age of a graph definable (equivalently, interpretable or trace definable) in $(\mathbb{N}, =)$ (in dimension d).

Example

- Disjoint unions of cliques ($d = 2$): $(x, y)E(x', y') \iff x = x'$.
- (2-dim.) Shift graphs ($d = 2$): $(x, y)E(x', y') \iff y = x'$.
- k-dim. shift graph $\mathcal{S}_k$ ($d = k$): $(x_1, \ldots, x_k)E(x'_1, \ldots, x'_k) \iff x_2 = x'_1 \wedge \cdots \wedge x_k = x'_{k-1}$
- Graph classes of bounded degeneracy (iff $\mathcal{C}$ is set-defined and forbids some $K_{t,t}$).

- $\mathcal{S}_2 \supset \mathcal{S}_3 \supset \ldots$; each $\mathcal{S}_i$ is triangle-free, but has unbounded χ.

Questions

Question

When is a set-defined class χ-bounded?

- Not always, since shift graphs are not χ-bounded.

Question

Are classes of shift graphs the only obstruction?

Question

Given a presentation of a set-defined class, can we decide whether it is χ-bounded?

- Presentation: A *complete set-defined class* $\mathcal{C}$ consists of all graphs representable by a fixed adjacency formula. (I.e. $\mathcal{C} = Age(G)$ for some G defined in $(\mathbb{N}, =)$.)

Results

Theorem

Let $\mathcal{C}$ be a complete-set defined class. Then either

1. *$\mathcal{C}$ is (polynomially) χ-bounded; or*
2. *$\mathcal{C}$ contains a subclass of some $\mathcal{S}_k$ that is not χ-bounded.*

Furthermore, which case is decidable given the boolean formula for $\mathcal{C}$.

Corollary

The Gyárfás-Sumner conjecture is true for complete set-defined classes: if $\mathcal{C}$ is not χ-bounded, then it contains every tree.

Theorem

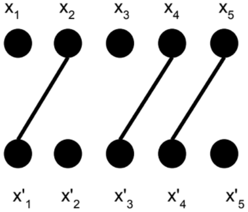
If a set-defined class $\mathcal{C}$ is not χ-bounded, then this is witnessed by some $\mathcal{D} \subset \mathcal{C}$ such that $\mathcal{D}$ has bounded clique number and its homomorphism-closure contains a (sub)class of shift graphs of unbounded χ.

The most interesting point of the proof

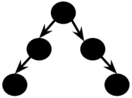
- After several reductions, it suffices to consider subclasses of a particular form.
- If a certain system of (in)equalities can be satisfied, then the subclass contains $\mathcal{S}_k$ for some k.
- These (in)equalities are either of the form "$x_i = x_{i+1} + 1$" or "$x_i \leq \max(x_1, \ldots, x_k)$".
- This is a linear program in the max-plus (aka tropical) algebra $(\mathbb{R}, \oplus, \otimes)$.
- Farkas analogue: If there is no solution, this is witnessed by a winning strategy for one player in a certain (mean payoff) game on an auxiliary graph.
- This winning strategy can be used to show the subclass (or a homomorphic projection) forbids a directed V as a subgraph, and so has bounded chromatic number.

FORBIDDING A V

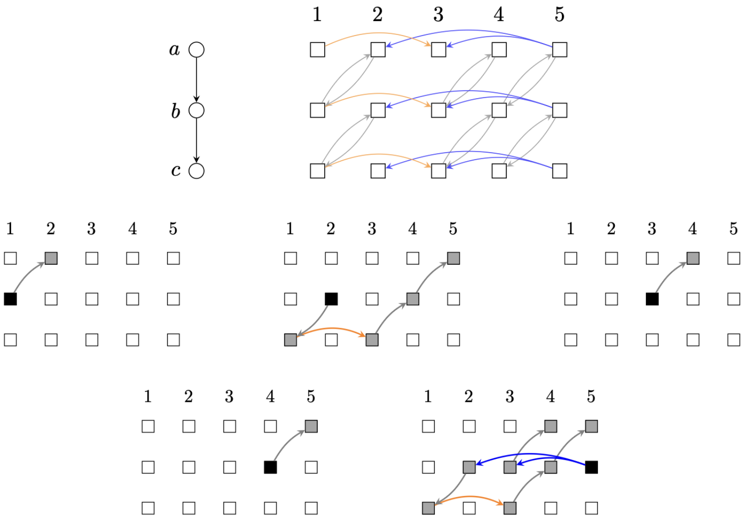
- We reduce to considering subclasses like the following:



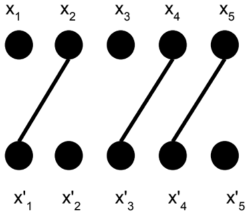
- On 5-tuples where the 3rd coordinate determines the 1st, and the 2nd and 3rd coordinates jointly determine the 5th.
- For this class, we wish to show that in a directed path of length 3, the (coordinates of the) starting vertex uniquely determines the (coordinates of the) second vertex.
- Thus the following is forbidden as a subgraph:



Tracing coordinates in a path



FROM INEQUALITIES TO COORDINATE TRACING



On 5-tuples where the 3rd coordinate determines the 1st, and the 2nd and 3rd coordinates jointly determine the 5th.

$$x_2 = x_1 + 1$$
$$x_4 = x_3 + 1$$
$$x_5 = x_4 + 1$$
$$x_5 \leq \max\{x_2, x_3\}$$
$$x_1 \leq \max\{x_3\},$$

A TOUCH OF STABILITY

- Set-defined classes are edge-stable. Our approach seems entirely unrelated to this.(?)
- But note work of Halevi-Kaplan-Shelah on Taylor's conjecture, showing stable graphs with large (uncountable) chromatic number contain all (finite) cliques or some $\mathcal{S}_k$ as (non-induced) subgraphs.
- The ω-stable and superstable cases reduce to showing that set-defined classes of unbounded chromatic number contain all cliques or some $\mathcal{S}_k$ as (non-induced) subgraphs.
- The strictly stable case seems to reduce to the same for order-defined classes.

Further questions

- Why tropical linear programming? Is it somehow natural/inevitable?
- The Gyárfás-Sumner conjecture for general set-defined classes?
- Other properties besides χ-boundedness?
- Do these results generalize to edge-stable classes? (Are shift graphs there only obstructions to χ-boundedness?)
- How does this generalize to graph classes defined in other NIP homogeneous structures? (New obstructions appear.) In $(\mathbb{Q}, <)$ (equiv. semilinear graph classes)? In tree-defined classes?

Model checking in finite fields and groups

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Overview

Parameterized complexity

Definition

First-order/MSO model checking: Given a structure M and a first-order/MSO sentence ϕ, determine if $M \models \phi$.

- For FO, can naively be solved in time $O(n^{\mathrm{qd}(\phi)})$.

Definition

Let $\mathcal{C}$ be a class of structures. First-order model checking is *fixed-parameter tractable (fpt)* on $\mathcal{C}$ if it can be solved in time $f(|\phi|)|M|^c$ for some computable f.

- Note the degree of the polynomial no longer depends on ϕ.
- [Gro07] posed model checking in algebraic structures.
- [BM15] showed that first-order model checking is fpt on the class of finite abelian groups, and MSO is not for abelian groups with a logarithmic encoding

Finite fields

Theorem (Ax [Ax68])

Let ϕ be a sentence. Then there is a sentence ψ that is a boolean combination of existentials over one variable such that $T_{pf} \models \phi \leftrightarrow \psi$, and there is some $N_\phi \in \mathbb{N}$ such that for every prime power $n \geq N_\phi$, we have $\mathbb{F}_n \models \phi \leftrightarrow \psi$. Furthermore, both ψ and N_ϕ are computable from ϕ.

Corollary

First-order model checking is fpt in the class of finite fields.

Proof.

Given ϕ, compute ψ and N_ϕ. If $n < \max(|\psi|, N_\phi)$, then just brute-force check if $\mathbb{F}_n \models \phi$. Otherwise, we can efficiently check if polynomials are irreducible over $\mathbb{F}_n$, so check if $\mathbb{F}_n \models \psi$. $\square$

Cyclic groups of prime-power order

Theorem

Assume E $\neq$ NE. Then MSO model checking on the class of cyclic groups of prime-power order is not in XP (i.e. there is some ϕ such that model checking for ϕ is not in P).

- If E $\neq$ NE, then there is a unary language L that is in NP$\setminus$P, with a nondeterminstic Turing machine in time $o(n^k)$.
- By the prime number theorem, we may assume L only has strings of prime length.
- To check if $1^p \in L$, consider $C_{p^{2k}}$. Let ϕ define a $p^k \times p^k$ grid using the subgroup of order p^k and its cosets.
- Define the input of 1^p using the subgroup of order p. Then let ϕ simulate a Turing machine for L.

(Pseudo)Finite fields

Theorem

The class of finite fields is finitely axiomatizable in MSO.

- The fields axioms.
- The characteristic is non-zero.
- The field is a simple extension $\mathbb{F}_0(\alpha)$ of its prime subfield $\mathbb{F}_0$, and α has an inverse in the ring $\mathbb{F}_0[\alpha]$.

FO MODEL CHECKING IN GROUPS

Theorem

If $\mathcal{C}$ is a hereditary graph class that is definable in the class of finite groups with only polynomial blow-up, then $\mathcal{C}$ is monadically stable.

Proof.

By [MN87], $\mathcal{G}_n \leq n^{O(\log^2(n))}$, and so $\mathcal{C}_n \leq n^{O(\log^2(n))}$. By [BL26], $\mathcal{C}$ must be monadically NIP. It must also be atomically stable, and therefore monadically stable. $\square$

- First-order model checking is fpt on every monadically stable graph class [DEM+24].

Question

What about for finite simple groups? For alternating groups?

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