Revisiting the Conjectures of Shapiro and Schanuel

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Exponential Polynomials

Define the ring of *exponential sums*

$$\mathcal{E}_s := \{\lambda_1 \exp(\mu_1 z) + \cdots + \lambda_n \exp(\mu_n z) : \lambda_i, \mu_i \in \mathbb{C}\},$$

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$$\mathcal{E}_p^\times = \{\lambda \exp(\mu z) : \lambda \in \mathbb{C}^\times, \mu \in \mathbb{C}\}.$$

An element of $\mathcal{E}_p$ is *irreducible* if it is not the product of two non-units.

Shapiro's conjecture

Conjecture (H. S. Shapiro, 1958)

If $f, g \in \mathcal{E}_s$ have infinitely many common roots, then there is $h \in \mathcal{E}_s$ such that h divides both f and g in $\mathcal{E}_s$, and h has infinitely many zeros.

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H. L. Montgomery: although these are problems about analytic functions, it seems analytic methods on their own are doomed to fail.

Factorization theory of exponential polynomials

Definition

Given $f(z) = \sum_{i=1}^{n} \lambda_i(z) \exp(\mu_i z) \in \mathcal{E}_p$ we define the *support* of f, denoted $\mathrm{supp}(f)$, as

$$\mathrm{supp}(f) := \mathrm{Span}_{\mathbb{Q}}(\mu_1, \ldots, \mu_n).$$

We say that f is *simple* if $\mathrm{l.dim}_{\mathbb{Q}}(\mathrm{supp}(f)) = 1$.

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Remark

$$\forall \lambda \in \mathbb{C}^{\times}, \mu \in \mathbb{C} \left[\lambda - \exp(\mu z) = \left(\sqrt{\lambda} - \exp\left(\frac{\mu}{2} z \right) \right) \left(\sqrt{\lambda} + \exp\left(\frac{\mu}{2} z \right) \right) \right]$$

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J. F. Ritt (1927–1929): every $f \in \mathcal{E}_p$ is a finite product of irreducible exponential polynomials and simple ones.

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This statement is often accompanied by the comment that all known proofs use p-adic methods.

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Proposition (A. J. van der Poorten–R. Tijdeman, 1975)

Skolem–Mahler–Lech is equivalent to the following statement: if an exponential polynomial and a simple exponential sum have infinitely many common zeros, then they have a non-trivial common divisor in $\mathcal{E}_p$.

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Corollary

Let $f(z) = \sum_{j=1}^{n} \lambda_j \exp(\mu_j z)$ be an exponential sum, and let $M = \{z \in \mathbb{Z} : f(z) = 0\}$. Then M is a finite union of arithmetic progressions.

A connection with Schanuel's conjecture

Conjecture (S. Schanuel, 1960s)

If $z_1, \ldots, z_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent, then

$$\mathrm{tr.deg.}_{\mathbb{Q}} \mathbb{Q}\left(\mathbf{z}, \exp(\mathbf{z})\right) \geq n.$$

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Theorem (S. Fischler–T. Rivoal, 2025)

Schanuel's conjecture solves Shapiro's question when $f, g \in \mathcal{E}_p$ are defined over $\overline{\mathbb{Q}}$.

Main Result

Theorem (P. D'Aquino–S.E., in progress)

Suppose $f, g \in \mathcal{E}_p$ have infinitely many common zeros. Then Schanuel's conjecture implies that there is $h \in \mathcal{E}_p$ such that h vanishes precisely at all the common zeros of f and g.

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$$d = \mathrm{l.dim}_{\mathbb{Q}}(\mathrm{supp}(f) \cup \mathrm{supp}(g)),$$

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For every $s \in S'$ there exists a unique $\mathbf{q}_s \in \mathbb{Q}^\ell$ so that $s = \mathbf{q}_s \cdot \mathbf{t}$.
Thus, the system of equations $f(\mathbf{z} \cdot \mathbf{t}) = g(\mathbf{z} \cdot \mathbf{t}) = 0$ has infinitely many solutions $\mathbf{z} \in \mathbb{Q}^\ell$.

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Schanuel's conjecture, weak Zilber–Pink and Ramsey's theorem together imply that there is an infinite subset $S' \subseteq S$ with $\mathrm{l.dim}_{\mathbb{Q}}(S') < \infty$. Let $\mathbf{t} = \{t_1, \ldots, t_\ell\} \subset S'$ be a basis.
For every $s \in S'$ there exists a unique $\mathbf{q}_s \in \mathbb{Q}^\ell$ so that $s = \mathbf{q}_s \cdot \mathbf{t}$.
Thus, the system of equations $f(\mathbf{z} \cdot \mathbf{t}) = g(\mathbf{z} \cdot \mathbf{t}) = 0$ has infinitely many solutions $\mathbf{z} \in \mathbb{Q}^\ell$.
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Afterthoughts

Theorem (F. Gallinaro, 2023)

Choose $\mu_1, \ldots, \mu_n, \lambda_1, \ldots, \lambda_n \in \mathbb{C}^\times$ *such that* $\mu_1, \ldots, \mu_n$ *are* $\mathbb{Q}$*-linearly independent. Then the exponential sum*

$$f(z) = \lambda_1 \exp(\mu_1 z) + \cdots + \lambda_n \exp(\mu_n z)$$

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This can rephrased as an "exponential algebraic closedness" problem (EAC), in that from f we build varieties $L \subseteq \mathbb{C}^d$ and $W \subseteq (\mathbb{C}^\times)^d$, where L is defined by linear equations, so that $L \times W$ is *additively free* and *rotund*.

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Further afterthoughts

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Conjecture (Uniform version of Schanuel's conjecture)

For any positive integer n and for any algebraic variety $V \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ defined over $\overline{\mathbb{Q}}$ satisfying $\dim V < n$, there is a finite set $\mathscr{L}$ of proper linear subspaces of $\mathbb{G}_a^n$ defined over $\mathbb{Q}$ and a finite set $\mathcal{H}$ of proper torsion cosets of $\mathbb{G}_m^n(\mathbb{C})$, with each $H \in \mathcal{H}$ of codimension at least 2, such that for any $\mathbf{z} \in \mathbb{G}_a^n(\mathbb{C})$ satisfying $(\mathbf{z}, \exp(\mathbf{z})) \in V$ we have either $\mathbf{z} \in L$, for some $L \in \mathscr{L}$, or $\exp(\mathbf{z}) \in H$ for some $H \in \mathcal{H}$.

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Schanuel's conjecture and the Zilber–Pink conjecture together imply the uniform version of Schanuel's conjecture (J. Pila, 2025).

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Question

Does the uniform Schanuel conjecture imply Skolem–Mahler–Lech?