Pseudofinite fields with an additive and a multiplicative character

Stefan Ludwig

University of Freiburg

Chania 02.06.2026

Introduction - Pseudofinite fields

- For this talk we write $q = p^e$ where p prime, $e \in \mathbb{N}$.

Introduction - Pseudofinite fields

- For this talk we write $q = p^e$ where p prime, $e \in \mathbb{N}$.
- Work in $\mathcal{L}_{\mathrm{ring}} = \{\cdot, +, -, 1, 0\}$.

Introduction - Pseudofinite fields

- For this talk we write $q = p^e$ where p prime, $e \in \mathbb{N}$.
- Work in $\mathcal{L}_{\text{ring}} = \{\cdot, +, -, 1, 0\}$.
- Consider $\mathrm{Th}((\mathbb{F}_q)_q) = \{\phi \;\; \mathcal{L}_{\text{ring}}\text{-sentence} \mid \text{For all } q, \;\; \mathbb{F}_q \models \phi\}$

Introduction - Pseudofinite fields

- For this talk we write $q = p^e$ where p prime, $e \in \mathbb{N}$.
- Work in $\mathcal{L}_{\text{ring}} = \{\cdot, +, -, 1, 0\}$.
- Consider $\mathrm{Th}((\mathbb{F}_q)_q) = \{\phi \ \ \mathcal{L}_{\text{ring}}\text{-sentence} \mid \text{For all } q, \ \ \mathbb{F}_q \models \phi\}$

Theorem (Ax, 68)

Let F be an infinite field. Then, $F \models \mathrm{Th}((\mathbb{F}_q)_q)$ if and only if

- *F is perfect and $\mathrm{Gal}(F) = \hat{\mathbb{Z}}$*
- *F is PAC: $V(F) \neq \emptyset$ for every abs. irr. variety V defined over F.*

Introduction - Pseudofinite fields

- For this talk we write $q = p^e$ where p prime, $e \in \mathbb{N}$.
- Work in $\mathcal{L}_{\mathrm{ring}} = \{\cdot, +, -, 1, 0\}$.
- Consider $\mathrm{Th}((\mathbb{F}_q)_q) = \{\phi \;\; \mathcal{L}_{\mathrm{ring}}\text{-sentence} \mid \text{For all } q, \;\; \mathbb{F}_q \models \phi\}$

Theorem (Ax, 68)

Let F be an infinite field. Then, $F \models \mathrm{Th}((\mathbb{F}_q)_q)$ if and only if

- *F is perfect and $\mathrm{Gal}(F) = \hat{\mathbb{Z}}$*
- *F is PAC: $V(F) \neq \emptyset$ for every abs. irr. variety V defined over F.*

Lang-Weil bounds: $\forall r \in \mathbb{N} \; \exists K(r) \in \mathbb{N}$, s. th. for V an absolutely irreducible variety of dimension d defined over $\mathbb{F}_q$ with $\mathrm{complexity}(V) \leq r$:

$$\Big| |V(\mathbb{F}_q)| - q^d \Big| \leq K q^{d - \frac{1}{2}}$$

Introduction - Pseudofinite fields

- For this talk we write $q = p^e$ where p prime, $e \in \mathbb{N}$.
- Work in $\mathcal{L}_{\text{ring}} = \{\cdot, +, -, 1, 0\}$.
- Consider $\mathrm{Th}((\mathbb{F}_q)_q) = \{\phi \ \ \mathcal{L}_{\text{ring}}\text{-sentence} \mid \text{For all } q, \ \ \mathbb{F}_q \models \phi\}$

Theorem (Ax, 68)

Let F be an infinite field. Then, $F \models \mathrm{Th}((\mathbb{F}_q)_q)$ if and only if

- *F is perfect and $\mathrm{Gal}(F) = \hat{\mathbb{Z}}$*
- *F is PAC: $V(F) \neq \emptyset$ for every abs. irr. variety V defined over F.*

Lang-Weil bounds: $\forall r \in \mathbb{N}\ \exists K(r) \in \mathbb{N}$, s. th. for V an absolutely irreducible variety of dimension d defined over $\mathbb{F}_q$ with $\mathrm{complexity}(V) \leq r$:

$$\left| |V(\mathbb{F}_q)| - q^d \right| \leq K q^{d - \frac{1}{2}}$$

$\implies \quad \forall r \in \mathbb{N}\ \exists q_0(r)$ s.th. for V as above and $q \geq q_0$, $V(\mathbb{F}_q) \neq \emptyset$.

Introduction

Question: Ax's theorem for enriched finite fields?

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

E. Kowalski (05) obtains bounds on *character sums* over definable sets in finite fields.

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

E. Kowalski (05) obtains bounds on *character sums* over definable sets in finite fields.

Goal: Determine $\mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$.

- For any χ_q, $\ker(\chi_q) = \{x^{\mathrm{ord}(\chi_q)} \mid x \in \mathbb{F}_q\}$

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

E. Kowalski (05) obtains bounds on *character sums* over definable sets in finite fields.

Goal: Determine $\mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$.

Remark

- For any Ψ_q, $\ker(\Psi_q) = \{a(x^p - x) \mid x \in \mathbb{F}_q\}$ for some $a \in \mathbb{F}_q$.

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

E. Kowalski (05) obtains bounds on *character sums* over definable sets in finite fields.

Goal: Determine $\mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$.

Remark

- For any Ψ_q, $\ker(\Psi_q) = \{a(x^p - x) \mid x \in \mathbb{F}_q\}$ for some $a \in \mathbb{F}_q$.
- For any χ_q, $\ker(\chi_q) = \{x^{\mathrm{ord}(\chi_q)} \mid x \in \mathbb{F}_q\}$

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

E. Kowalski (05) obtains bounds on *character sums* over definable sets in finite fields.

Goal: Determine $\mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$.

Remark

- For any Ψ_q, $\ker(\Psi_q) = \{a(x^p - x) \mid x \in \mathbb{F}_q\}$ for some $a \in \mathbb{F}_q$.
- For any χ_q, $\ker(\chi_q) = \{x^{\mathrm{ord}(\chi_q)} \mid x \in \mathbb{F}_q\}$

$\to$ **Interesting case:**

$$\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q) \cup \{\mathrm{ord}(\Psi) = \infty = \mathrm{ord}(\chi)\}$$

Introduction

Question: Ax's theorem for enriched finite fields?

- $\Psi_q : \mathbb{F}_q \to S^1$ additive character
- $\chi_q : \mathbb{F}_q^* \to S^1$ multiplicative character

Remark

Any Ψ_q is of the form $\Psi_q(x) = \exp(2\pi i a \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}(x))$ for some $a \in \mathbb{F}_q$.

E. Kowalski (05) obtains bounds on *character sums* over definable sets in finite fields.

Goal: Determine $\mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q)$.

Remark

- For any Ψ_q, $\ker(\Psi_q) = \{a(x^p - x) \mid x \in \mathbb{F}_q\}$ for some $a \in \mathbb{F}_q$.
- For any χ_q, $\ker(\chi_q) = \{x^{\mathrm{ord}(\chi_q)} \mid x \in \mathbb{F}_q\}$

$\to$ **Interesting case:**

$\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{Th}((\mathbb{F}_q, \Psi_q, \chi_q)_q) \cup \{\mathrm{ord}(\Psi) = \infty = \mathrm{ord}(\chi)\}$
(This implies characteristic 0.)

Generalisations of Ax's theorem

Question: In which language?

Generalisations of Ax's theorem

Question: In which language?

Fact (Hrushovski)

$\mathrm{Th}(\mathbb{F}_p, I_p)$ *decides* $(\mathbb{N}, +, \cdot)$ *for an interval* I_p *with* $|I_p|, |I_p^c|$ *unbounded*

Generalisations of Ax's theorem

Question: In which language?

Fact (Hrushovski)

$\mathrm{Th}(\mathbb{F}_p, I_p)$ *decides* $(\mathbb{N}, +, \cdot)$ *for an interval* I_p *with* $|I_p|, |I_p^c|$ *unbounded*

Generalisations of Ax's theorem

Question: In which language?

Fact (Hrushovski)

$\mathrm{Th}(\mathbb{F}_p, I_p)$ *decides* $(\mathbb{N}, +, \cdot)$ *for an interval* I_p *with* $|I_p|, |I_p^c|$ *unbounded*

Fact (Tran)

The theory of $(\bar{\mathbb{F}}_p, \triangleleft)$ *where* $\triangleleft$ *is a circular order stemming from a multiplicative character is* TP_2.

Generalisations of Ax's theorem

Question: In which language?

Fact (Hrushovski)

$\mathrm{Th}(\mathbb{F}_p, I_p)$ *decides* $(\mathbb{N}, +, \cdot)$ *for an interval* I_p *with* $|I_p|, |I_p^c|$ *unbounded*

Fact (Tran)

The theory of $(\bar{\mathbb{F}}_p, \lhd)$ *where* $\lhd$ *is a circular order stemming from a multiplicative character is* TP_2.

- **But** Hrushovski ('21) managed to generalise Ax's work to $(\mathbb{F}_q, \Psi_q)$ using a mild version of continuous logic.

Generalisations of Ax's theorem

Question: In which language?

Fact (Hrushovski)

$\mathrm{Th}(\mathbb{F}_p, I_p)$ *decides* $(\mathbb{N}, +, \cdot)$ *for an interval* I_p *with* $|I_p|, |I_p^c|$ *unbounded*

Fact (Tran)

The theory of $(\bar{\mathbb{F}}_p, \lhd)$ *where* $\lhd$ *is a circular order stemming from a multiplicative character is* TP_2.

- **But** Hrushovski ('21) managed to generalise Ax's work to $(\mathbb{F}_q, \Psi_q)$ using a mild version of continuous logic.

Definition

$\mathcal{L}_{+,\times} := \mathcal{L}_{\mathrm{ring}} \cup \{\Psi\} \cup \{\chi\}$ where Ψ, χ are continuous logic predicates with allowed range $S^1 \cup \{0\} \subset \mathbb{C}$.

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

- **Terms** $t(\bar{x})$ are given by $\mathcal{L}_{\mathrm{ring}}$-terms.

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

- **Terms** $t(\bar{x})$ are given by $\mathcal{L}_{\mathrm{ring}}$-terms.
- **Atomic formulas** are of the form $\Psi(t(\bar{x})), \chi(t(\bar{x}))$ or $t_1(\bar{x}) = t_2(\bar{x})$ where the latter is $\{0, 1\}$-valued.

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

- **Terms** $t(\bar{x})$ are given by $\mathcal{L}_{\mathrm{ring}}$-terms.
- **Atomic formulas** are of the form $\Psi(t(\bar{x})), \chi(t(\bar{x}))$ or $t_1(\bar{x}) = t_2(\bar{x})$ where the latter is $\{0, 1\}$-valued.
- **Quantifier-free formulas** are continuous combinations of atomic formulas.

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

- **Terms** $t(\bar{x})$ are given by $\mathcal{L}_{\mathrm{ring}}$-terms.
- **Atomic formulas** are of the form $\Psi(t(\bar{x})), \chi(t(\bar{x}))$ or $t_1(\bar{x}) = t_2(\bar{x})$ where the latter is $\{0,1\}$-valued.
- **Quantifier-free formulas** are continuous combinations of atomic formulas.
- **Quantifers** are given by $\sup, \inf$ of the real (imaginary) part.

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

- **Terms** $t(\bar{x})$ are given by $\mathcal{L}_{\mathrm{ring}}$-terms.
- **Atomic formulas** are of the form $\Psi(t(\bar{x})), \chi(t(\bar{x}))$ or $t_1(\bar{x}) = t_2(\bar{x})$ where the latter is $\{0, 1\}$-valued.
- **Quantifier-free formulas** are continuous combinations of atomic formulas.
- **Quantifers** are given by $\sup, \inf$ of the real (imaginary) part.

An $\mathcal{L}_{+,\times}$-theory assigns to $\mathcal{L}_{+,\times}$-sentences (no free variables) values in $\mathbb{C}$.

Continuous logic

Formulas in $\mathcal{L}_{+,\times}$ are built inductively using

- **Terms** $t(\bar{x})$ are given by $\mathcal{L}_{\mathrm{ring}}$-terms.
- **Atomic formulas** are of the form $\Psi(t(\bar{x})), \chi(t(\bar{x}))$ or $t_1(\bar{x}) = t_2(\bar{x})$ where the latter is $\{0,1\}$-valued.
- **Quantifier-free formulas** are continuous combinations of atomic formulas.
- **Quantifers** are given by $\sup, \inf$ of the real (imaginary) part.

An $\mathcal{L}_{+,\times}$-theory assigns to $\mathcal{L}_{+,\times}$-sentences (no free variables) values in $\mathbb{C}$.

Definition

We define $(F, \Psi, \chi) \models PF^{+,\times}$, if:

(1) F is a field of characteristic 0 with Galois group $Gal(F) = \hat{\mathbb{Z}}$.

(2) $\chi : (F^{\times}, \cdot) \to S^1$ is a group homomorphism and $\chi(0) = 0$.

(3) $\Psi : (F, +) \to S^1$ is a group homomorphism.

(4) Geometric axiom (next slide)

E.g. (3) given by $\sup_{x,y} |\Psi(x+y) - \Psi(x)\Psi(y)| = 0$

Geometric axiom of $\mathrm{PF}^{+,\times}$

Definition

$\mathrm{PF}^{+,\times}(4)$ states: Given any absolutely irreducible curve $C \subset \mathbb{A}^n$ defined over F, $\alpha : \mathbb{A}^n \to \mathbb{A}^k$ an integral linear map and $\beta : \mathbb{G}_m^n \to \mathbb{G}_m^l$ an integral multiplicative map such that (with $C' = \mathbb{G}_m^n \cap C$)

- $C_1 := \alpha(C)$ is not contained in any rational hyperplane over F,
- $C_2 := \beta(C')$ is not contained in any multiplicative coset over F.

Then,

$$(\Psi^{(k)} \times \chi^{(l)})(\Delta(F)) \subseteq_{\mathrm{dense}} \mathbb{T}^{k+l}$$

where $\Delta(F) := \{(\alpha(\bar{z}), \beta(\bar{z})) \mid \bar{z} \in C'(F)\}$.

Geometric axiom of $\mathrm{PF}^{+,\times}$

Definition

$\mathrm{PF}^{+,\times}(4)$ states: Given any absolutely irreducible curve $C \subset \mathbb{A}^n$ defined over F, $\alpha : \mathbb{A}^n \to \mathbb{A}^k$ an integral linear map and $\beta : \mathbb{G}_m^n \to \mathbb{G}_m^l$ an integral multiplicative map such that (with $C' = \mathbb{G}_m^n \cap C$)

- $C_1 := \alpha(C)$ is not contained in any rational hyperplane over F,
- $C_2 := \beta(C')$ is not contained in any multiplicative coset over F.

Then,

$$(\Psi^{(k)} \times \chi^{(l)})(\Delta(F)) \subseteq_{\mathrm{dense}} \mathbb{T}^{k+l}$$

where $\Delta(F) := \{(\alpha(\bar{z}), \beta(\bar{z})) \mid \bar{z} \in C'(F)\}$.

Lemma

The above axiom is expressible in $\mathcal{L}_{+,\times}$ using expressions of the following form:

Geometric axiom of $\mathrm{PF}^{+,\times}$

Definition

$\mathrm{PF}^{+,\times}(4)$ states: Given any absolutely irreducible curve $C \subset \mathbb{A}^n$ defined over F, $\alpha : \mathbb{A}^n \to \mathbb{A}^k$ an integral linear map and $\beta : \mathbb{G}_m^n \to \mathbb{G}_m^l$ an integral multiplicative map such that (with $C' = \mathbb{G}_m^n \cap C$)

- $C_1 := \alpha(C)$ is not contained in any rational hyperplane over F,
- $C_2 := \beta(C')$ is not contained in any multiplicative coset over F.

Then,

$$(\Psi^{(k)} \times \chi^{(l)})(\Delta(F)) \subseteq_{\mathrm{dense}} \mathbb{T}^{k+l}$$

where $\Delta(F) := \{(\alpha(\bar{z}), \beta(\bar{z})) \mid \bar{z} \in C'(F)\}$.

Lemma

The above axiom is expressible in $\mathcal{L}_{+,\times}$ using expressions of the following form:

For certain (C, h) with $h \in \mathbb{Q}[\bar{Y}, \bar{Y}^{-1}, \bar{Z}, \bar{Z}^{-1}]$ real-valued on $\mathbb{T}^{2n}$:

$$\sup\{h(\Psi^{(n)}(\bar{x}), \chi^{(n)}(\bar{x})) \mid \bar{x} \in C'(F)\} \geq 0.$$

Quantifier reduction for $\mathrm{PF}^{+,\times}$

Proposition

Let $(F_1, \Psi_1, \chi_1), (F_2, \Psi_2, \chi_2)$ *be two models of* $\mathrm{PF}^{+,\times}$ *and* E *a common relatively algebraically closed substructure, then*

$$(F_1, \Psi_1, \chi_1) \equiv_E (F_2, \Psi_2, \chi_2).$$

Quantifier reduction for $\mathrm{PF}^{+,\times}$

Proposition

Let $(F_1, \Psi_1, \chi_1), (F_2, \Psi_2, \chi_2)$ *be two models of* $\mathrm{PF}^{+,\times}$ *and* E *a common relatively algebraically closed substructure, then*

$$(F_1, \Psi_1, \chi_1) \equiv_E (F_2, \Psi_2, \chi_2).$$

In particular, the completions of $\mathrm{PF}^{+,\times}$ *are determined by the isomorphism types of* $F \cap \bar{\mathbb{Q}}$ *together with the corresponding restrictions of* Ψ *and* χ *for* $(F, \Psi, \chi) \models \mathrm{PF}^{+,\times}$.

Quantifier reduction for $\mathrm{PF}^{+,\times}$

Proposition

Let $(F_1, \Psi_1, \chi_1), (F_2, \Psi_2, \chi_2)$ *be two models of* $\mathrm{PF}^{+,\times}$ *and* E *a common relatively algebraically closed substructure, then*

$$(F_1, \Psi_1, \chi_1) \equiv_E (F_2, \Psi_2, \chi_2).$$

In particular, the completions of $\mathrm{PF}^{+,\times}$ *are determined by the isomorphism types of* $F \cap \bar{\mathbb{Q}}$ *together with the corresponding restrictions of* Ψ *and* χ *for* $(F, \Psi, \chi) \models \mathrm{PF}^{+,\times}$.

Corollary

$\mathrm{PF}^{+,\times}$ *is model-complete (after naming some constants).*

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\subseteq$: **To show**: For any ultraproduct $(F, \Psi, \chi) = \prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q)$ with $\mathrm{ord}(\Psi) = \infty = \mathrm{ord}(\chi)$ we have $(F, \Psi, \chi) \models \mathrm{PF}^{+,\times}$.

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\mathrm{lim}}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\subseteq$: **To show**: For any ultraproduct $(F, \Psi, \chi) = \prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q)$ with $\mathrm{ord}(\Psi) = \infty = \mathrm{ord}(\chi)$ we have $(F, \Psi, \chi) \models \mathrm{PF}^{+,\times}$.

- The difficult case is Axiom (4): To show for $(C, \alpha, \beta, \Delta)$ as before,

$$(\Psi^{(k)} \times \chi^{(l)})(\Delta(F)) \subseteq_{\mathrm{dense}} \mathbb{T}^{k+l}$$

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\subseteq$: **To show**: For any ultraproduct $(F, \Psi, \chi) = \prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q)$ with $\mathrm{ord}(\Psi) = \infty = \mathrm{ord}(\chi)$ we have $(F, \Psi, \chi) \models \mathrm{PF}^{+,\times}$.

- The difficult case is Axiom (4): To show for $(C, \alpha, \beta, \Delta)$ as before,

$$(\Psi^{(k)} \times \chi^{(l)})(\Delta(F)) \subseteq_{\mathrm{dense}} \mathbb{T}^{k+l}$$

Main ingredient:

Weil bounds: $\forall r \in \mathbb{N}\ \exists K(r) > 0$, s.th for C abs. irr. curve defined over $\mathbb{F}_q$ and $g, h \in \mathbb{F}_q(C)$, such that there are no $r, s \in \overline{\mathbb{F}}_q(C)$, $a \in \mathbb{F}_q$ with both $g = a(s^q - s)$ and $h = r^d$, then $\mathrm{complexity}(C, g, h) \leq r$ implies

$$\Big| \sum_{\bar{x} \in C(\mathbb{F}_q)} \Psi_q(g(\bar{x}))\chi_q(h(\bar{x})) \Big| \leq K \cdot q^{\frac{1}{2}}$$

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\supseteq$: **To show**: For any completion T of $\mathrm{PF}^{+,\times}$ there is $\prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q) \models T$.

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\supseteq$: **To show**: For any completion T of $\mathrm{PF}^{+,\times}$ there is $\prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q) \models T$.

- Completions of $\mathrm{PF}^{+,\times}$ are determined by $(\bar{\mathbb{Q}} \cap F, \Psi, \chi)$ up to isomorphism

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\supseteq$: **To show**: For any completion T of $\mathrm{PF}^{+,\times}$ there is $\prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q) \models T$.

- Completions of $\mathrm{PF}^{+,\times}$ are determined by $(\bar{\mathbb{Q}} \cap F, \Psi, \chi)$ up to isomorphism
- By Ax (Čebotarev density theorem) we find

$$\prod_{\mathcal{U}} \mathbb{F}_q \cap \bar{\mathbb{Q}} \cong_{\mathrm{ring}} F \cap \bar{\mathbb{Q}}$$

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\supseteq$: **To show**: For any completion T of $\mathrm{PF}^{+,\times}$ there is $\prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q) \models T$.

- Completions of $\mathrm{PF}^{+,\times}$ are determined by $(\bar{\mathbb{Q}} \cap F, \Psi, \chi)$ up to isomorphism
- By Ax (Čebotarev density theorem) we find

$$\prod_{\mathcal{U}} \mathbb{F}_q \cap \bar{\mathbb{Q}} \cong_{\mathrm{ring}} F \cap \bar{\mathbb{Q}}$$

- Compactness $\rightarrow$ work in $L/\mathbb{Q}$ finite Galois.

Ax's Theorem for $(\mathbb{F}_q, \Psi_q, \chi_q)$

Theorem (L., 25)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

$\supseteq$: **To show**: For any completion T of $\mathrm{PF}^{+,\times}$ there is $\prod_{\mathcal{U}}(\mathbb{F}_q, \Psi_q, \chi_q) \models T$.

- Completions of $\mathrm{PF}^{+,\times}$ are determined by $(\bar{\mathbb{Q}} \cap F, \Psi, \chi)$ up to isomorphism
- By Ax (Čebotarev density theorem) we find

$$\prod_{\mathcal{U}} \mathbb{F}_q \cap \bar{\mathbb{Q}} \cong_{\mathrm{ring}} F \cap \bar{\mathbb{Q}}$$

- Compactness $\to$ work in $L/\mathbb{Q}$ finite Galois.
- Choose the Ψ_q, χ_q using quantative Weyl equidistribution (Erdős-Turán-Koksma inequality).

Model-theoretic tameness of $\mathrm{PF}^{+,\times}$

Theorem (L.)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

Proposition

The theory $\mathrm{PF}^{+,\times}$ *is simple.*

Model-theoretic tameness of $\mathrm{PF}^{+,\times}$

Theorem (L.)

We have $\mathrm{Th}_{\lim}((\mathbb{F}_q, \Psi_q, \chi_q)_q) = \mathrm{PF}^{+,\times}$

Proposition

The theory $\mathrm{PF}^{+,\times}$ is simple.

Proposition

$\mathrm{PF}^{+,\times}$ is decidable in the sense of continuous logic,
i.e., for every rational $\epsilon > 0$ and an $\mathcal{L}_{+,\times}$-sentence ϕ, we can uniformly compute $c \in \mathbb{Q}$ such that $|\phi - c| \leq \epsilon$.

Definable counting in PF

Theorem (Chatzidakis-van den Dries-Macintyre)

For all $\mathcal{L}_{ring}$-formulas $\phi(\bar{x}, \bar{y})$, there are $C > 0$, $\mathcal{L}_{ring}$-formulas $\psi_1(\bar{y}), \ldots, \psi_k(\bar{y})$ and a finite set $D_\phi \subseteq \{0, \ldots, |\bar{x}|\} \times \mathbb{Q}_{>0} \cup \{0, 0\}$ such that for all $\mathbb{F}_q$, $\bar{a} \in \mathbb{F}_q^{|\bar{y}|}$:

$$||\phi(\mathbb{F}_q, \bar{a})| - \mu_i q^{d_i}| \leq C q^{d_i - \frac{1}{2}} \iff \bar{a} \in \psi_i(\mathbb{F}_q) \qquad (\text{and } \forall \bar{a}\, \exists i : \quad \bar{a} \in \psi_i(\mathbb{F}_q)).$$

Definable counting in PF

Theorem (Chatzidakis-van den Dries-Macintyre)

For all $\mathcal{L}_{ring}$-formulas $\phi(\bar{x}, \bar{y})$, there are $C > 0$, $\mathcal{L}_{ring}$-formulas $\psi_1(\bar{y}), \ldots, \psi_k(\bar{y})$ and a finite set $D_\phi \subseteq \{0, \ldots, |\bar{x}|\} \times \mathbb{Q}_{>0} \cup \{0, 0\}$ such that for all $\mathbb{F}_q$, $\bar{a} \in \mathbb{F}_q^{|\bar{y}|}$:

$$||\phi(\mathbb{F}_q, \bar{a})| - \mu_i q^{d_i}| \leq C q^{d_i - \frac{1}{2}} \iff \bar{a} \in \psi_i(\mathbb{F}_q) \qquad (\text{and } \forall \bar{a}\, \exists i : \ \bar{a} \in \psi_i(\mathbb{F}_q)).$$

- Fix $\Theta(\bar{x}, \bar{y})$ an $\mathcal{L}_{ring}$-formula. Let $\phi(\bar{x}, \bar{y})$ be an $\mathcal{L}_{ring}$-formula.

Definable counting in PF

Theorem (Chatzidakis-van den Dries-Macintyre)

For all $\mathcal{L}_{ring}$-formulas $\phi(\bar{x}, \bar{y})$, there are $C > 0$, $\mathcal{L}_{ring}$-formulas $\psi_1(\bar{y}), \dots, \psi_k(\bar{y})$ and a finite set $D_\phi \subseteq \{0, \dots, |\bar{x}|\} \times \mathbb{Q}_{>0} \cup \{0, 0\}$ such that for all $\mathbb{F}_q$, $\bar{a} \in \mathbb{F}_q^{|\bar{y}|}$:

$$||\phi(\mathbb{F}_q, \bar{a})| - \mu_i q^{d_i}| \leq C q^{d_i - \frac{1}{2}} \iff \bar{a} \in \psi_i(\mathbb{F}_q) \qquad (\text{and } \forall \bar{a}\, \exists i : \ \bar{a} \in \psi_i(\mathbb{F}_q)).$$

- Fix $\Theta(\bar{x}, \bar{y})$ an $\mathcal{L}_{ring}$-formula. Let $\phi(\bar{x}, \bar{y})$ be an $\mathcal{L}_{ring}$-formula.
- Consider the map $f_q^\phi : \mathbb{F}_q^{\bar{y}} \to \mathbb{R}$ given by

$$f_q^\phi(\bar{a}) := \frac{|(\phi \wedge \Theta)(\mathbb{F}_q^{\bar{x}}, \bar{a})|}{|\Theta(\mathbb{F}_q^{\bar{x}}, \bar{a})|}$$

Definable counting in PF

Theorem (Chatzidakis-van den Dries-Macintyre)

For all $\mathcal{L}_{ring}$-formulas $\phi(\bar{x},\bar{y})$, there are $C>0$, $\mathcal{L}_{ring}$-formulas $\psi_1(\bar{y}),\dots,\psi_k(\bar{y})$ and a finite set $D_\phi \subseteq \{0,\dots,|\bar{x}|\}\times\mathbb{Q}_{>0}\cup\{0,0\}$ such that for all $\mathbb{F}_q$, $\bar{a}\in\mathbb{F}_q^{|\bar{y}|}$:

$$||\phi(\mathbb{F}_q,\bar{a})|-\mu_i q^{d_i}|\leq Cq^{d_i-\frac{1}{2}} \iff \bar{a}\in\psi_i(\mathbb{F}_q) \qquad (\text{and } \forall\bar{a}\,\exists i:\ \bar{a}\in\psi_i(\mathbb{F}_q)).$$

- Fix $\Theta(\bar{x},\bar{y})$ an $\mathcal{L}_{ring}$-formula. Let $\phi(\bar{x},\bar{y})$ be an $\mathcal{L}_{ring}$-formula.
- Consider the map $f_q^\phi:\mathbb{F}_q^{\bar{y}}\to\mathbb{R}$ given by

$$f_q^\phi(\bar{a}) := \frac{|(\phi\wedge\Theta)(\mathbb{F}_q^{\bar{x}},\bar{a})|}{|\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})|}$$

Corollary

For all $\phi(\bar{x},\bar{y}),\epsilon>0$ there is an $\mathcal{L}_{ring}$-definable function g such that for all finite fields $\mathbb{F}_q$ and all $\bar{a}\in\mathbb{F}_q$:

$$|g(\bar{a})-f_q^\phi(\bar{a})|<\epsilon$$

Definable integration in $\mathrm{PF}^{+,\times}$

- Fix $\Theta(\bar{x},\bar{y})$ an $\mathcal{L}_{ring}$-formula. Let $\phi(\bar{x},\bar{y})$ be an $\mathcal{L}_{+,\times}$-formula.
- Consider the map $f_q^{\phi} : \mathbb{F}_q^{\bar{y}} \to \mathbb{R}$ given by

$$f_q^{\phi}(\bar{a}) := \frac{|\sum_{\bar{x}\in\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})} \phi(\bar{x},\bar{a})|}{|\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})|}$$

Definable integration in $\mathrm{PF}^{+,\times}$

- Fix $\Theta(\bar{x}, \bar{y})$ an $\mathcal{L}_{ring}$-formula. Let $\phi(\bar{x}, \bar{y})$ be an $\mathcal{L}_{+,\times}$-formula.
- Consider the map $f_q^{\phi} : \mathbb{F}_q^{\bar{y}} \to \mathbb{R}$ given by

$$f_q^{\phi}(\bar{a}) := \frac{\left|\sum_{\bar{x}\in\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})} \phi(\bar{x}, \bar{a})\right|}{|\Theta(\mathbb{F}_q^{\bar{x}}, \bar{a})|}$$

Theorem (L., '25)

For all $\phi(\bar{x}, \bar{y}), \epsilon > 0$ there is an $\mathcal{L}_{+,\times}$-formula g, such that for all $(\mathbb{F}_q, \Psi_q, \chi_q)$ and $\bar{a} \in \mathbb{F}_q$:

$$|g(\bar{a}) - f_q^{\phi}(\bar{a})| < \epsilon$$

Definable integration in $\mathrm{PF}^{+,\times}$

- Fix $\Theta(\bar{x},\bar{y})$ an $\mathcal{L}_{ring}$-formula. Let $\phi(\bar{x},\bar{y})$ be an $\mathcal{L}_{+,\times}$-formula.
- Consider the map $f_q^{\phi} : \mathbb{F}_q^{\bar{y}} \to \mathbb{R}$ given by

$$f_q^{\phi}(\bar{a}) := \frac{\left|\sum_{\bar{x}\in\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})} \phi(\bar{x},\bar{a})\right|}{|\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})|}$$

Theorem (L., '25)

For all $\phi(\bar{x},\bar{y}), \epsilon > 0$ there is an $\mathcal{L}_{+,\times}$-formula g, such that for all $(\mathbb{F}_q, \Psi_q, \chi_q)$ and $\bar{a} \in \mathbb{F}_q$:

$$|g(\bar{a}) - f_q^{\phi}(\bar{a})| < \epsilon$$

This entails **character sums** over definable sets (h_1, h_2 rational functions)

$$\sum_{\bar{x}\in\Theta(\mathbb{F}_q^{\bar{x}},\bar{a})} \Psi(h_1(\bar{x}))\chi(h_2(\bar{x}))$$

Towards applications: Tao's algebraic regularity lemma

- In the following let V, W be $\mathcal{L}_{\mathrm{ring}}(A)$-definable sets in $\mathbb{F}_q$ and $f : V \times W \to \{0,1\}$ an $\mathcal{L}_{\mathrm{ring}}(A)$-definable set.

Towards applications: Tao's algebraic regularity lemma

- In the following let V, W be $\mathcal{L}_{\mathrm{ring}}(A)$-definable sets in $\mathbb{F}_q$ and $f : V \times W \to \{0,1\}$ an $\mathcal{L}_{\mathrm{ring}}(A)$-definable set.

Theorem (Tao, 12)

For any $K \in \mathbb{N}$ there exists $C = C(K) \in \mathbb{N}$ such that whenever $\mathrm{complexity}(V, W, f) \leq K$, then there are $V = V_1 \cup \cdots \cup V_a$ and $W = W_1 \cup \cdots \cup W_b$ such that for all $1 \leq i \leq a$ and $1 \leq j \leq b$:

- ***(large sets)*** $|V_i| \geq \frac{|V|}{C}$ *and* $|W_j| \geq \frac{|W|}{C}$.
- ***(definability)*** *All V_i and W_j are $\mathcal{L}_{\mathrm{ring}}(\mathrm{acl}(A))$-definable of complexity bounded by C.*
- ***($q^{-\frac{1}{4}}$-regularity)*** *For all subsets $A \subseteq V_i$, $B \subseteq W_j$ we have*

$$\left| \sum_{A \times B} f(\bar{x}, \bar{y}) - r_{ij}|A||B| \right| \leq C q^{-\frac{1}{4}} |V_i||W_j|$$

where $r_{ij} = |V_i|^{-1}|W_j|^{-1} \sum_{V_i \times W_j} f(\bar{x}, \bar{y})$.

Algebraic regularity for $\mathrm{PF}^{+,\times}$-definable functions

- In the following let V, W be $\mathcal{L}_{\mathrm{ring}}(A)$-definable sets in $\mathbb{F}_q$ and $f : V \times W \to \mathbb{C}$ an $\mathcal{L}_{+,\times}$-definable set.

Algebraic regularity for $\mathrm{PF}^{+,\times}$-definable functions

- In the following let V, W be $\mathcal{L}_{\mathrm{ring}}(A)$-definable sets in $\mathbb{F}_q$ and $f : V \times W \to \mathbb{C}$ an $\mathcal{L}_{+,\times}$-definable set.

Theorem (?, (very much) ongoing work)

For any $K \in \mathbb{N}$ and $\delta > 0$ there exists $C = C(\delta, K) \in \mathbb{N}$ such that whenever $\mathrm{complexity}(V, W, f) \leq K$, then there are $V = V_1 \cup \cdots \cup V_a$ and $W = W_1 \cup \cdots \cup W_b$ such that for all $1 \leq i \leq a$ and $1 \leq j \leq b$:

- ***(large sets)*** $|V_i| \geq \frac{|V|}{C}$ *and* $|W_j| \geq \frac{|W|}{C}$.
- ***(weak-definability)*** *All V_i and W_j are complements of $\mathcal{L}_{+,\times}(\mathrm{acl}(A))$-zerosets of complexity bounded by C.*
- ***($\delta + q^{-\frac{1}{4}}$-regularity)*** *For all subsets $A \subseteq V_i$, $B \subseteq W_j$ we have*

$$\left| \sum_{A \times B} f(\bar{x}, \bar{y}) - r_{ij}|A||B| \right| \leq C(\delta + q^{-\frac{1}{4}})|V_i||W_j|$$

where $r_{ij} = |V_i|^{-1}|W_j|^{-1} \sum_{V_i \times W_j} f(\bar{x}, \bar{y})$.

(We can replace $C\delta|V_i||W_j|$ by $C\delta \cdot \min\{|A||B|^{\frac{1}{2}}|W|^{\frac{1}{2}}, |A|^{\frac{1}{2}}|V|^{\frac{1}{2}}\}|B|$.)