Linearity in geometric structures and in quasiminimal pregeometry structures

Alexander Berenstein (joint work with E. Vassiliev)

Universidad de los Andes

Leeds 2026

Outline of the talk

Strongly minimal theories and linearity.

Linearity in some geometric structures.

Quasimimimal pregeometry structures.

Linearity in quasimimimal pregeometry structures.

Strongly minimal theories

A complete theory T in a countable language $\mathcal{L}$ is said to be strongly minimal if for every $M \models T$ and definible $A \subseteq M$, A is either finite or cofinite.

Proposition

If T is strongly minimal, $M \models T$, then (M, acl) defines a pregeometry. So $\mathrm{acl} : \mathcal{P}(M) \to \mathcal{P}(M)$ satisfies:

Mono. $M \models T$, $A \subseteq B \subseteq M$, $A \subseteq \mathrm{acl}(A) \subseteq \mathrm{acl}(B)$.

Idem. $M \models T$, $A \subseteq M$, $\mathrm{acl}(\mathrm{acl}(A)) = \mathrm{acl}(A)$.

F.char. $M \models T$, $A \subseteq M$, $\mathrm{acl}(A) = \bigcup_{B \subset_{finite} A} \mathrm{acl}(B)$.

Exch. $M \models T$, $A \subseteq M$, $b, c \in M$ and $b \in \mathrm{acl}(A \cup \{c\}) \setminus \mathrm{acl}(A)$, then $c \in \mathrm{acl}(A \cup \{b\})$.

Strongly minimal theories and pregeometries

Definition

Let T be str. minimal, $M \models T$ and $A \subseteq B \subseteq M$, $\vec{c} \in M^n$. $\dim(\vec{c}/A) =$ length of a maximally acl-independent subset $\vec{c}$ after localizing acl in A.

$dim(\vec{c}/A) > dim(\vec{c}/B)$ iff $tp(\vec{c}/B)$ forks over A. So algebraic independence captures a natural notion of independence.

If T is strongly minimal, it eliminates the quantifier $\exists^\infty$, that is, for each formula $\varphi(x, \vec{y})$ there is $\theta(\vec{y})$ such that

$$M \models \theta(\vec{a}) \text{ if and only if } |\varphi(M, \vec{a})| = \infty$$

Equivalently, for any $\psi(\vec{x}, \vec{y})$ with $|\vec{x}| = n$ there are $\theta_i(\vec{y})$ with $i \leq n$ such that

$$M \models \theta_k(\vec{a}) \text{ if and only if } \dim(\varphi(M, \vec{a})) = k$$

Different types of pregeometries

Strongly minimal theories come in three flavors: desintegrated, locally modular and non locally modular.

Example.
$\mathcal{L} = \emptyset$, $T =$ there are infinitely many elements. If $M \models T$, $\mathrm{acl}(A) = A$. Disintegrated: $\mathrm{acl}(A) = \bigcup_{a \in A} \mathrm{acl}(a)$.

Example.
$\mathcal{L} = \{+, 0, \{\lambda_f\}_{f \in \mathbb{F}}\}$, $T =$ theory of vector spaces over $\mathbb{F}$.
It is modular: if $X, Y \subseteq M$ are finite dimensional and *closed*, so $\mathrm{acl}(X) = X$, $\mathrm{acl}(Y) = Y$, then

$$\dim(X \cup Y) + \dim(X \cap Y) = \dim(X) + \dim(Y)$$

Equivalently,

$$X \underset{X \cap Y}{\perp} Y$$

Different types of pregeometries

Example.
Affine spaces. It is locally modular: after localizing in a finite set it is modular.

Examples.
$\mathcal{L} = \{+, \cdot, 0, 1\}$, $T = ACF_0$ it is not locally modular.

Hrushovski constructions.

The linear case

Definition
A stable theory T is 1-based if for any $M \models T$ and $A \subseteq M$, $\vec{b} \in M$, $Cb(stp(\vec{b}/A)) \in \mathrm{acl}^{eq}(\vec{b})$.

If T is strongly minimal and $M \models T$, a *plane curve* C is a strongly minimal subset of M^2. If the canonical parameter of every such C has dim ≤ 1, we say T is linear.

Theorem Let T be strongly minimal. The following are equivalent:

1. T is locally modular
2. T is 1-based.
3. T is linear.

Expansions

Let P be a new unary predicate. If we have $P(M) \preceq M$, then we call (M, P) an *elementary pair*.

We say (M, P) is a dense-codense pair if

i) $M \models T$, $A \subseteq M$ is finite dimensional and $p(x) \in S(A)$ is non-algebraic, there is $b \models p(x)$, $b \in M \setminus \mathrm{acl}(A \cup P(M))$.

ii) $M \models T$, $A \subseteq M$ is finite dimensional and $p(x) \in S(A)$ is non-algebraic, there is $b \models p(x)$ with $b \in P(M)$.

If it is a dense-codense elementary pair, we call (M, P) a *beautiful pair*.

In general the class of beautiful pairs is not a first order class, it is the class of $\aleph_0$-saturated models of the common theory of beautiful pairs.

Expansions

Example.
$\mathcal{L} = \{+, 0, 1, \times\}$, $T = ACF_0$. The pair $(\mathbb{C}, \overline{\mathbb{Q}(e_i : i \in \omega)})$ is a beautiful pair.

Theorem (Buechler)
If T is trivial, then $RM(Th(M, P)) = 1$.
If T is loc. modular non-trivial, then $RM(Th(M, P)) = 2$.
If T is not loc. modular then $RM(Th(M, P)) = \omega$.

Theorem (Vassiliev)
Assume T has Q.E. Then T is locally modular iff $Th(M, P)$ is model complete.

Simple theories of SU-rank one

Assume now T is a simple theory of SU-rank one. Then acl has the exchange property, algebraic dimension is definable and acl-independence captures forking. We can use the same definition as before and obtain a dense-codense elementary pair now called a *lovely pair*.

Theorem (Vassiliev) Let T be of SU-rk one with Q.E.
- If T is trivial, then $SU(Th(M, P)) = 1$.
- If T is 1-based non-trivial, then $SU(Th(M, P)) = 2$.

Simple theories of SU-rank one

Theorem (Vassiliev)
T of SU-rk one with Q.E. (M, P) a lovely pair. TFAE:
-$SU(Th(M,P)) \leq 2$
-$Th(M,P)$ is model complete
-$\mathrm{acl}_{\mathcal{L}} = \mathrm{acl}_{\mathcal{L}_P}$
-$scl(-) = \mathrm{acl}(- \cup P(M))$ is modular.
-T is 1-based

But it is **NOT** equivalent to local modularity, consider a generic subset of a vector space.

Lesson: there is a strong connection between the complexity of the pregeometry and the complexity of the associated lovely pair. In which other setting does this apply?

o-minimal theories

Assume now M is *dense* o-minimal, then acl has the exchange property and algebraic dimension is definable.

Defn(Hasson, Onshuus, Peterzil) A definable family of plane curves is almost normal, if each curve has infinite intersection with only finitely many other curves.

An o-minimal structure M is *generically linear*, if any almost normal family of plane curves in M has dimension ≤ 1.

o-minimal theories

We can use the same definition as before and obtain a dense-codense elementary pair called now a *dense pair* of o-minimal theories.

Theorem (B-Vassiliev) Let T be dense o-minimal, $M \models T$. TFAE

- M does not interpret a field.
- M is generically linear.
- $Th(M, P)$ is model complete.
- $\mathrm{acl}_{\mathcal{L}} = \mathrm{acl}_{\mathcal{L}_P}$.
- $scl(-) = \mathrm{acl}(- \cup P(M))$ is modular.
- T is weakly 1-based: For all $\vec{a} \in M$ and $B \subseteq M$ and all $\vec{c} \models \mathrm{tp}(\vec{a}/B)$ with $\vec{c} \mathop{\smile\!\!\!\!\!\!\mid}_B \vec{a}$ we have $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\vec{c}} B$.
- M is weakly locally modular: for all finite $\vec{a}, \vec{b} \in M^n$ there is a finite $C \mathop{\smile\!\!\!\!\!\!\mid} \vec{a}\vec{b}$ such that $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}_{cl(C\vec{a}) \cap (C\vec{b})} \vec{b}$

It is **NOT** equivalent to local modularity, one can consider an example by Peterzil-Loveys: $(\mathbb{R}, +, 0, <, \pi \restriction_{[-1,1]})$.

Common framework: geometric structures

Assume now T is geometric: acl has the exchange property and algebraic dimension is definable. We can use the same definition as before and obtain a *lovely pair* of geometric structures.

Theorem (B-Vassiliev)
Let T be geometric, $M \models T$. TFAE
- $Th(M, P)$ is model complete.
- $\mathrm{acl}_{\mathcal{L}} = \mathrm{acl}_{\mathcal{L}_P}$.
- *scl* is modular.
- T is weakly 1-based: For all $\vec{a} \in M$ and $B \subseteq M$ and all $\vec{c} \models \mathrm{tp}(\vec{a}/B)$ with $\vec{c} \mathop{\smile\!\!\!\!\!\!\,|}_B \vec{a}$ we have $\vec{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{c}} B$.

We do not need the independence notion to extend to the imaginary sorts.

Quasiminimal pregeometry structures

Defn. Let M be an $\mathcal{L}$-structure for a countable language $\mathcal{L}$, equipped with a pregeometry cl. It is a *quasiminimal pregeometry structure* if it satisfies.

QM1 If $tp(a, \overline{b}) = tp(a', \overline{c})$, $a \in cl(\overline{b})$ iff $a' \in cl(\overline{c})$.

QM2 Each $M \in \mathcal{C}$ is infinite dimensional.

QM3 If $A \subset M$ is finite, $cl(A)$ is countable.

QM4 If $A, A' \subset M$ are countable and enumerated s.t. $tp(A) = tp(A')$ and $a \notin cl(A)$, $a' \notin cl(A')$, then $tp(A, a) = tp(A', a')$

QM5 If $A, A' \subset M$ are closed countable enumerated so that $tp(A) = tp(A')$ and $\vec{b}$, $\vec{c}$ finite tuples such that

$$tp(A, \vec{b}) = tp(A', \vec{c})$$

then if $a \in cl(A, \vec{b})$ there is $a' \in cl(A, \vec{c})$ s.t.

$$tp(A, \vec{b}, a) = tp(A', \vec{c}, a')$$

Quasiminimal-Examples

Example.
T strongly minimal, $M \models T$ at least $\aleph_0$-saturated, $cl = \text{acl}$.

Example.
$\mathcal{L} = \{E\}$, $\mathcal{C} =$ structures where E is an equiv relation with infinite classes, each one countable. If $M \models T$ and $A \subseteq M$, $cl(A) =$ classes that interest A. It is trivial.

Example.
$\mathcal{L} = \{+, \times, 0, 1, exp\}$, $\mathcal{C} =$ Zilber fields, i.e. $(\mathbb{B}, +, \times, 0, 1, exp)$ (modulo Schanuel conjecture is the complex field with exponential closure). It is not modular.

Closed embeddings

Closed embedding

Given (M_1, cl_{M_1}) and (M_2, cl_{M_2}) quasiminimal pregeometry $\mathcal{L}$-structures, an $\mathcal{L}$-embedding $\theta : M_1 \to M_2$ is *closed embedding* if for each $A \subseteq M_1$ we have $\theta(cl_{M_1}(A)) = cl_{M_2}(\theta(A))$.

Quasiminimal classes.

Given a quasiminimal pregeometry structure M, let $\mathcal{C}$ be the smallest class of $\mathcal{L}$-structures which contain M and it is closed under the following operations: closed substructures, isomorphisms, and unions of chains of closed embeddings.

Axiomatization

Theorem (Zilber; Bays, Hart, Hyttinen, Kesälä and Kirby). Let $\mathcal{C}$ be a quasiminimal class. Then $\mathcal{C}$ has a unique structure for each cl-dimension. The class is axiomatizable by a single $\mathcal{L}_{\omega_1\omega}(Q)$-sentence.

$\mathcal{L}_{\omega_1\omega}$: you can take countable conjunctions and disjunctions of formulas.

$Q(x)\varphi(x) =$ there are uncountably many realizations for $\varphi(x)$

Theorem Definability of dimension (Kirby):
For each $n \in \mathbb{N}$ there is an $\mathcal{L}_{\omega_1\omega}$- formula $\pi_n(x, y_1, \ldots, y_n)$ such that for each $M \in \mathcal{C}$, $a \in M$, $\vec{b} = (b_1, ..., b_n) \in M$

$$a \in cl(\vec{b}) \quad iff \quad M \models \pi_n(a, \vec{b})$$

More analogies

For T strongly minimal, $M \models T$, $A \subseteq B \subseteq M$, $\vec{c} \in M$, we have:

$dim(\vec{c}/A) = \dim(\vec{c}/B)$ iff $tp(\vec{c}/B)$ does not fork over A.

Instead of forking-independence: Lascar splitting.
Let M be a model, let $B \subseteq M$ and $\vec{c} \in M$. We say that $tp(\vec{c}/B)$ *Lascar splits* over $A \subseteq B$ if there are finite tuples $\vec{d}_1$ and $\vec{d}_2$ in B with $Lstp(\vec{d}_1/A) = Lstp(\vec{d}_2/A)$ but $tp(\vec{d}_1/A \cup \vec{c}) \neq tp(\vec{d}_2/A \cup \vec{c})$.

Theorem
$dim_{cl}(\vec{c}/A) = \dim_{cl}(\vec{c}/B)$ iff $tp(\vec{c}/B)$ does not Lascar split over A.

$U_{Lsp}(\vec{c})$ foundational rank for Lascar splitting.

Expansions

Let P be a new unary predicate. we say (M, P) is a dense-codense pair if

i) $M \in \mathcal{C}$, $A \subseteq M$ is finite dimensional, there is $b \in M \setminus cl(A \cup P(M))$.

ii) $M \in \mathcal{C}$, $A \subseteq M$ is finite dimensional, there is $b \in P(M) \setminus cl(A)$.

If we have $P(M) \in \mathcal{C}$ and it is closed in M, then (M, P) is a *beautiful pair*. The class of beautiful pairs is $\mathcal{L}_{\omega_1\omega}(Q)$-axiomatizable.

We can count types over countable models: the class $\mathcal{C}_P$ of beautiful pairs (resp. $\mathcal{C}_H$ of H-structures) is ω-stable.

Buechler's Theorem, so far

Let $\mathcal{C}$ is a quisimimimal class and (M, P) a beautiful pair.
1. If (M, cl) is trivial, $c \in M$, $A \subseteq M$, $U_{Lsp}(c/A) \leq 1$.
2. If (M, cl) is locally modular non-trivial, $c \in M$, $A \subseteq M$, $U_{Lsp}(c/A) \leq 2$.
Moreover $U_{Lsp}(c/A) = 2$ for generics.

Characterizations of linearity

Let $\mathcal{C}$ is a quisimimimal class and (M, P) a beautiful pair of structures in $\mathcal{C}$. The following are equivalent:
1. If (M, cl) is locally modular.
2. Model completeness of beautiful pairs: If $M, N \in \mathcal{C}$ are such that $N \preceq M$. If $(N, P) \subseteq (M, P)$ are beautiful pairs then $(N, P) \preceq (M, P)$.

Furthermore, if (M, cl) is locally modular then $bcl_P = cl$.

Thank you

Thank you!!!