Conic *o*-minimal sheaves

Duarte Costa

on-going work with Mário Edmundo and Luca Prelli

University of Lisbon

March 18, 2026

FCT - UID/04561/2025

1. What was already done?
2. What else can be done?
3. What are Conic sheaves?
4. How to adapt this to the o-minimal context?
5. What did we do so far?
6. What's next?

Recalling sheaves

Definition (Presheaf)

X a topological space, $\mathrm{Op}(X)$ the collection of open subsets of X.

Recalling sheaves

Definition (Presheaf)

X a topological space, $\mathrm{Op}(X)$ the collection of open subsets of X. An assignment

$$\mathrm{Op}(X) \ni U \stackrel{\mathcal{F}}{\longmapsto} \mathcal{F}(U)$$

is called a **presheaf of abelian groups** if:

Recalling sheaves

Definition (Presheaf)

X a topological space, $\mathrm{Op}(X)$ the collection of open subsets of X. An assignment

$$\mathrm{Op}(X) \ni U \xmapsto{\mathcal{F}} \mathcal{F}(U)$$

is called a **presheaf of abelian groups** if:

- $\mathcal{F}(U)$ is an abelian group.

Recalling sheaves

Definition (Presheaf)

X a topological space, $\mathrm{Op}(X)$ the collection of open subsets of X. An assignment

$$\mathrm{Op}(X) \ni U \overset{\mathcal{F}}{\longmapsto} \mathcal{F}(U)$$

is called a **presheaf of abelian groups** if:

- $\mathcal{F}(U)$ is an abelian group.
- If $U \subset V$, there is an associated homomorphism of abelian groups

  $$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U), \quad s \mapsto s|_U,$$

  called the **restriction morphism**, such that if $W \subset U$ is open, then $(s|_U)|_W = s|_W$.

Recalling sheaves

Definition (Presheaf)

X a topological space, $\mathrm{Op}(X)$ the collection of open subsets of X. An assignment

$$\mathrm{Op}(X) \ni U \stackrel{\mathcal{F}}{\longmapsto} \mathcal{F}(U)$$

is called a **presheaf of abelian groups** if:

- $\mathcal{F}(U)$ is an abelian group.
- If $U \subset V$, there is an associated homomorphism of abelian groups

$$\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U), \quad s \mapsto s|_U,$$

  called the **restriction morphism**, such that if $W \subset U$ is open, then $(s|_U)|_W = s|_W$.

Terminology: $s \in \mathcal{F}(U)$ is called **a section of $\mathcal{F}$ over** U.

Recalling sheaves

Definition (Sheaf)

A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if, for every open cover $\{U_i\}_{i\in I}$ of an open set $U \subset X$, the following conditions hold:

Recalling sheaves

Definition (Sheaf)

A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if, for every open cover $\{U_i\}_{i \in I}$ of an open set $U \subset X$, the following conditions hold:

- (*Locality*) If $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.

Recalling sheaves

Definition (Sheaf)

A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if, for every open cover $\{U_i\}_{i \in I}$ of an open set $U \subset X$, the following conditions hold:

- (*Locality*) If $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (*Gluing*) If for each $i \in I$ we have $s_i \in \mathcal{F}(U_i)$ satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ whenever $U_i \cap U_j \neq \emptyset$, then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Recalling sheaves

Definition (Sheaf)

A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if, for every open cover $\{U_i\}_{i \in I}$ of an open set $U \subset X$, the following conditions hold:

- (*Locality*) If $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (*Gluing*) If for each $i \in I$ we have $s_i \in \mathcal{F}(U_i)$ satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ whenever $U_i \cap U_j \neq \emptyset$, then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Examples:

(i) $U \mapsto C^0(U, \mathbb{R}) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$

Recalling sheaves

Definition (Sheaf)

A presheaf $\mathcal{F}$ on a topological space X is a **sheaf** if, for every open cover $\{U_i\}_{i \in I}$ of an open set $U \subset X$, the following conditions hold:

- (*Locality*) If $s, t \in \mathcal{F}(U)$ satisfy $s|_{U_i} = t|_{U_i}$ for all $i \in I$, then $s = t$.
- (*Gluing*) If for each $i \in I$ we have $s_i \in \mathcal{F}(U_i)$ satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ whenever $U_i \cap U_j \neq \emptyset$, then there exists a unique section $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$ for all $i \in I$.

Examples:

(i) $U \mapsto C^0(U, \mathbb{R}) = \{f : U \to \mathbb{R} \mid f \text{ continuous}\}$

(ii) $U \mapsto C^\infty(U, \mathbb{R}) = \{f : U \to \mathbb{R} \mid f \text{ smooth}\}$

What is "a sheaf theory"?

Given a continuous map $f : X \to Y$ between topological spaces, we have six (Grothendieck) operations:

What is "a sheaf theory"?

Given a continuous map $f : X \to Y$ between topological spaces, we have six (Grothendieck) operations:

- $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ with (f^{-1}, f_*) forming an adjunction pair.

What is "a sheaf theory"?

Given a continuous map $f : X \to Y$ between topological spaces, we have six (Grothendieck) operations:

- $f_* : \mathrm{Sh}(X) \to \mathrm{Sh}(Y)$ and $f^{-1} : \mathrm{Sh}(Y) \to \mathrm{Sh}(X)$ with (f^{-1}, f_*) forming an adjunction pair.
- $\mathrm{Hom}(\bullet, \bullet) : \mathrm{Sh}(X)^{\mathrm{op}} \times \mathrm{Sh}(X) \to \mathrm{Sh}(X)$ and $\bullet \bigotimes \bullet : \mathrm{Sh}(X)^2 \to \mathrm{Sh}(X)$ such that $(\bullet \bigotimes \mathcal{F}, \mathrm{Hom}(\mathcal{F}, \bullet))$ form an adjunction pair.
- $Rf_! : D^b(X) \to D^b(Y)$ and $f^! : D^b(Y) \to D^b(X)$ with $(Rf_!, f^!)$ forming an adjunction pair.
- and some other "compatibility conditions" between them...

Our general setting

We fix an *o*-minimal expansion of a real closed field, $\mathcal{R} = (R, \mathcal{L}^{\mathcal{R}})$, where $\mathcal{L} = (+\,, \cdot\,, 0\,, 1\,, <\,, \ldots)$ and let:

Our general setting

We fix an o-minimal expansion of a real closed field, $\mathcal{R} = (R, \mathcal{L}^{\mathcal{R}})$, where $\mathcal{L} = (+\,,\cdot\,,0\,,1\,,<\,,\ldots)$ and let:

- $X \subset R^n$ be a definable set (with parameters in R), defined by an $\mathcal{L}$-formula with parameters $\overline{b} = (b_1, \ldots, b_r) \in R^r$, $\phi_X(\overline{x}, \overline{b})$.

Our general setting

We fix an *o*-minimal expansion of a real closed field, $\mathcal{R} = (R, \mathcal{L}^{\mathcal{R}})$, where $\mathcal{L} = (+\,,\cdot\,,0\,,1\,,<\,,\dots)$ and let:

- $X \subset R^n$ be a definable set (with parameters in R), defined by an $\mathcal{L}$-formula with parameters $\overline{b} = (b_1, \dots, b_r) \in R^r$, $\phi_X(\overline{x}, \overline{b})$.
- $\widetilde{X}$ be the *o*-minimal spectrum of X, $\widetilde{X} = \{p \in S_n^{\mathcal{R}}(R) \mid \phi_X(\overline{x}, \overline{b}) \in p\}$; Basis for the topology: $[\psi_U] := \{p \in \widetilde{X} \mid \psi_U \in p\}$ whenever ψ_U defines an open subset of X.

Our general setting

We fix an *o*-minimal expansion of a real closed field, $\mathcal{R} = (R, \mathcal{L}^{\mathcal{R}})$, where $\mathcal{L} = (+\,,\cdot\,,0\,,1\,,<\,,\dots)$ and let:

- $X \subset R^n$ be a definable set (with parameters in R), defined by an $\mathcal{L}$-formula with parameters $\overline{b} = (b_1, \dots, b_r) \in R^r$, $\phi_X(\overline{x}, \overline{b})$.
- $\widetilde{X}$ be the *o*-minimal spectrum of X, $\widetilde{X} = \{p \in S_n^{\mathcal{R}}(R) \mid \phi_X(\overline{x}, \overline{b}) \in p\}$; Basis for the topology: $[\psi_U] := \{p \in \widetilde{X} \mid \psi_U \in p\}$ whenever ψ_U defines an open subset of X.
- We're interested in studying $\mathrm{Sh}(\widetilde{X})$, the category of *o*-minimal sheaves.

Recalling the o-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$

Recalling the o-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$
- $\widetilde{X}$ is quasi-compact and is a spectral space, i.e:

Recalling the o-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$
- $\widetilde{X}$ is quasi-compact and is a spectral space, i.e:
  1. It has a basis of quasi-compact open subsets ($\widetilde{U}$, for $U \subset X$ open and definable) that is closed under finite intersections (because $\widetilde{U \cap V} = \widetilde{U} \cap \widetilde{V}$)

Recalling the o-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$
- $\widetilde{X}$ is quasi-compact and is a spectral space, i.e:
  1. It has a basis of quasi-compact open subsets ($\widetilde{U}$, for $U \subset X$ open and definable) that is closed under finite intersections (because $\widetilde{U \cap V} = \widetilde{U} \cap \widetilde{V}$)
  2. The irreducible closed subsets are closures of unique points

Recalling the o-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$
- $\widetilde{X}$ is quasi-compact and is a spectral space, i.e:
  1. It has a basis of quasi-compact open subsets ($\widetilde{U}$, for $U \subset X$ open and definable) that is closed under finite intersections (because $\widetilde{U \cap V} = \widetilde{U} \cap \widetilde{V}$)
  2. The irreducible closed subsets are closures of unique points
- Topology of $\widetilde{X}$ is controllable:

Recalling the *o*-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$
- $\widetilde{X}$ is quasi-compact and is a spectral space, i.e:
  1. It has a basis of quasi-compact open subsets ($\widetilde{U}$, for $U \subset X$ open and definable) that is closed under finite intersections (because $\widetilde{U \cap V} = \widetilde{U} \cap \widetilde{V}$)
  2. The irreducible closed subsets are closures of unique points
- Topology of $\widetilde{X}$ is controllable:
  1. X is definably connected iff $\widetilde{X}$ is connected

Recalling the o-minimal spectrum

- If $Y \subset X$ is a definable subset, since $\widetilde{Y} \simeq [\phi_Y] \subset \widetilde{X}$, we identify $\widetilde{Y} \subset \widetilde{X}$
- $\widetilde{X}$ is quasi-compact and is a spectral space, i.e:
  1. It has a basis of quasi-compact open subsets ($\widetilde{U}$, for $U \subset X$ open and definable) that is closed under finite intersections (because $\widetilde{U \cap V} = \widetilde{U} \cap \widetilde{V}$)
  2. The irreducible closed subsets are closures of unique points
- Topology of $\widetilde{X}$ is controllable:
  1. X is definably connected iff $\widetilde{X}$ is connected
  2. X is definably normal iff $\widetilde{X}$ is normal

o-minimal sheaf theory

Given a definable continuous map $f : X \to Y$, we have six (Grothendieck) operations (mainly by Edmundo-Prelli, 2020):

- $\widetilde{f}_* : \mathrm{Sh}(\widetilde{X}) \to \mathrm{Sh}(\widetilde{Y})$ and $\widetilde{f}^{-1} : \mathrm{Sh}(\widetilde{Y}) \to \mathrm{Sh}(\widetilde{X})$ with $(\widetilde{f}^{-1}, \widetilde{f}_*)$ forming an adjunction pair.
- $\mathrm{Hom}(\bullet, \bullet) : \mathrm{Sh}(\widetilde{X})^{\mathrm{op}} \times \mathrm{Sh}(\widetilde{X}) \to \mathrm{Sh}(\widetilde{X})$ and $\bullet \bigotimes \bullet : \mathrm{Sh}(\widetilde{X})^2 \to \mathrm{Sh}(\widetilde{X})$ such that $(\bullet \bigotimes \mathcal{F}, \mathrm{Hom}(\mathcal{F}, \bullet))$ form an adjunction pair.
- $R\widetilde{f}_{\wr} : D^b(\widetilde{X}) \to D^b(\widetilde{Y})$ and $\widetilde{f}^{\,\wr} : D^b(\widetilde{Y}) \to D^b(\widetilde{X})$ with $(R\widetilde{f}_{\wr}, \widetilde{f}^{\,\wr})$ forming an adjunction pair.
- and some other "compatibility conditions" between them...

Motivation for conic sheaves

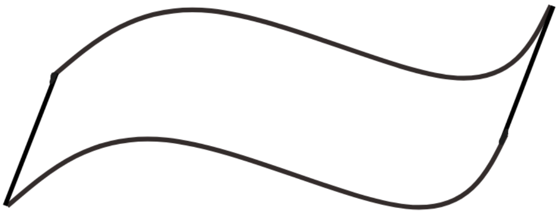


Figure 1: "Propagation" on a manifold.

Motivation for conic sheaves

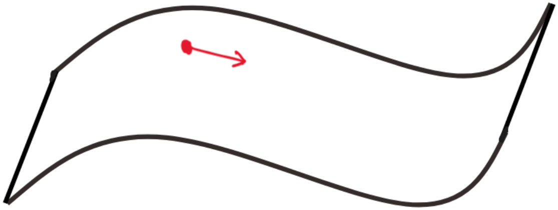


Figure 2: "Propagation" on a manifold.

Point $\to x_0$

Motivation for conic sheaves

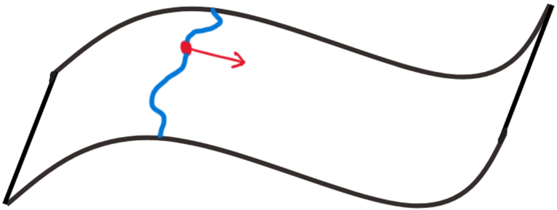


Figure 3: "Propagation" on a manifold.

Point $\rightarrow x_0$

Curve $\rightarrow \{f = 0\}$

Motivation for conic sheaves

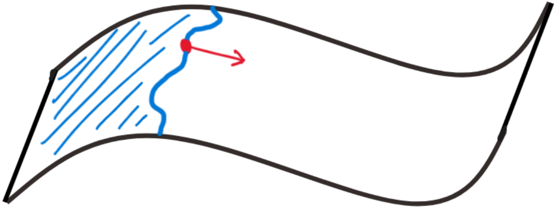


Figure 4: "Propagation" on a manifold.

Point $\to x_0$

Curve $\to \{f = 0\}$

Open $\to \{f < 0\} =: U$

Motivation for conic sheaves

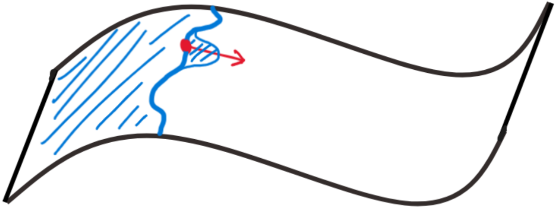


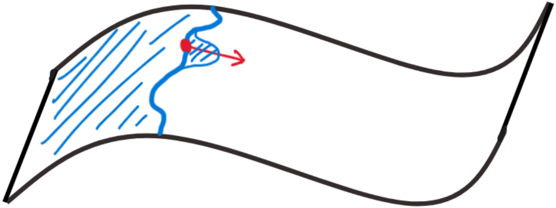
Figure 5: "Propagation" on a manifold.

Point $\to x_0$

Curve $\to \{f = 0\}$

Open $\to \{f < 0\} =: U$

Motivation for conic sheaves



$$(R\Gamma_{\{f\geq 0\}}(\mathcal{F}))_{x_0} = 0$$

$$\Updownarrow$$

$$\varinjlim_{x_0\in V} H^j(U\cup V,\mathcal{F}) \simeq H^j(U,\mathcal{F})$$

Figure 5: "Propagation" on a manifold.

Point $\rightarrow x_0$

Curve $\rightarrow \{f=0\}$

Open $\rightarrow \{f<0\} =: U$

Motivation for conic sheaves

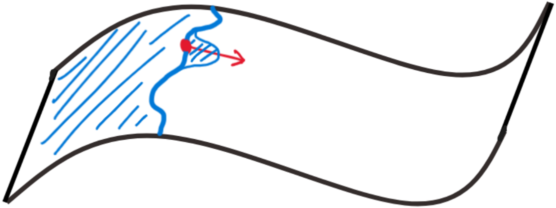


Figure 5: "Propagation" on a manifold.

Point $\to x_0$

Curve $\to \{f = 0\}$

Open $\to \{f < 0\} =: U$

$$(R\Gamma_{\{f \geq 0\}}(\mathcal{F}))_{x_0} = 0$$

$$\Updownarrow$$

$$\varinjlim_{x_0 \in V} H^j(U \cup V, \mathcal{F}) \simeq H^j(U, \mathcal{F})$$

$$(R\Gamma_{\{\mathrm{Re}(f) \geq 0\}}(\mathcal{F}))|_{f^{-1}(0)}$$

$$\downarrow \wr$$

$$s'^{-1} \underbrace{\mu_Y(\mathcal{F})}_{\text{conic sheaf}}$$

Classical conic sheaves

- X topological space

Classical conic sheaves

- X topological space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action

Classical conic sheaves

- X topological space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $S \subset X$ is conic if $S = \mathbb{R}^+ S := \mu(\mathbb{R}^+ \times S)$

Classical conic sheaves

- X topological space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $S \subset X$ is conic if $S = \mathbb{R}^+ S := \mu(\mathbb{R}^+ \times S)$
- $X_{\mathbb{R}^+}$ setwise X, but $U \subset X_{\mathbb{R}^+}$ is open if ($U \subset X$ is open and $\mathbb{R}^+ U = U$)

Classical conic sheaves

- X topological space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $S \subset X$ is conic if $S = \mathbb{R}^+ S := \mu(\mathbb{R}^+ \times S)$
- $X_{\mathbb{R}^+}$ setwise X, but $U \subset X_{\mathbb{R}^+}$ is open if ($U \subset X$ is open and $\mathbb{R}^+ U = U$)
- $\eta : X \to X_{\mathbb{R}^+}$

Classical conic sheaves

- X topological space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $S \subset X$ is conic if $S = \mathbb{R}^+ S := \mu(\mathbb{R}^+ \times S)$
- $X_{\mathbb{R}+}$ setwise X, but $U \subset X_{\mathbb{R}+}$ is open if ($U \subset X$ is open and $\mathbb{R}^+ U = U$)
- $\eta : X \to X_{\mathbb{R}+}$

Definition

$S \subset X$ is $\mathbb{R}^+$-connected if $S \cap \mathbb{R}^+ x$ is always connected.

Classical conic sheaves

- X topological space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $S \subset X$ is conic if $S = \mathbb{R}^+ S := \mu(\mathbb{R}^+ \times S)$
- $X_{\mathbb{R}^+}$ setwise X, but $U \subset X_{\mathbb{R}^+}$ is open if ($U \subset X$ is open and $\mathbb{R}^+ U = U$)
- $\eta : X \to X_{\mathbb{R}^+}$

Definition

$S \subset X$ is $\mathbb{R}^+$-connected if $S \cap \mathbb{R}^+ x$ is always connected.

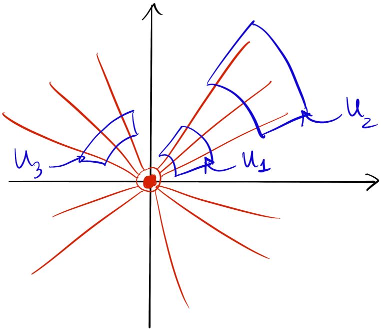


Figure 6: $\mu : \mathbb{R}^+ \times \mathbb{R}^2 \to \mathbb{R}^2$, $(t, (x, y)) \mapsto t(x, y)$

Classical conic sheaves

Conic sheaves (Kashiwara-Schapira, 1990)

$\mathcal{F} \in \mathrm{Sh}(X)$ is conic if any of the following equivalent conditions holds:

1. $\mu^{-1}\mathcal{F} \simeq p^{-1}\mathcal{F}$, with $p : \mathbb{R}^+ \times X \to X$ projection
2. $\mathcal{F}|_{\mathbb{R}^+x} =$ constant
3. $\mathcal{F}(\mathbb{R}^+U) \xrightarrow{\sim} \mathcal{F}(U)$ if U is $\mathbb{R}^+$-connected
4. $\mathcal{F} \simeq \eta^{-1}\eta_* \mathcal{F}$

Classical conic sheaves

Conic sheaves (Kashiwara-Schapira, 1990)

$\mathcal{F} \in \mathrm{Sh}(X)$ is conic if any of the following equivalent conditions holds:

1. $\mu^{-1}\mathcal{F} \simeq p^{-1}\mathcal{F}$, with $p : \mathbb{R}^+ \times X \to X$ projection
2. $\mathcal{F}|_{\mathbb{R}^+x} =$ constant
3. $\mathcal{F}(\mathbb{R}^+U) \xrightarrow{\sim} \mathcal{F}(U)$ if U is $\mathbb{R}^+$-connected
4. $\mathcal{F} \simeq \eta^{-1}\eta_*\mathcal{F}$

Equivalence of categories (Kashiwara-Schapira, 1990)

If $\mathrm{Sh}_{\mathbb{R}^+}(X)$ is the category of conic sheaves,

$$\mathrm{Sh}_{\mathbb{R}^+}(X) \simeq \mathrm{Sh}(X_{\mathbb{R}^+})$$

Classical conic sheaves

Conic sheaves (Kashiwara-Schapira, 1990)

$\mathcal{F} \in \mathrm{Sh}(X)$ is conic if any of the following equivalent conditions holds:

1. $\mu^{-1}\mathcal{F} \simeq p^{-1}\mathcal{F}$, with $p : \mathbb{R}^+ \times X \to X$ projection
2. $\mathcal{F}|_{\mathbb{R}^+x} =$ constant
3. $\mathcal{F}(\mathbb{R}^+U) \xrightarrow{\sim} \mathcal{F}(U)$ if U is $\mathbb{R}^+$-connected
4. $\mathcal{F} \simeq \eta^{-1}\eta_*\mathcal{F}$

Equivalence of categories (Kashiwara-Schapira, 1990)

If $\mathrm{Sh}_{\mathbb{R}^+}(X)$ is the category of conic sheaves,

$$\mathrm{Sh}_{\mathbb{R}^+}(X) \simeq \mathrm{Sh}(X_{\mathbb{R}^+})$$

Example:

$U \mapsto C^0_{\mathbb{R}^+}(U, \mathbb{R}) := \{f : U \to \mathbb{R} \mid f \text{ continuous and } \textbf{locally } f(\mu(t,x)) = tf(x)\}$

is a conic sheaf.

1. What was already done?
2. What else can be done?
3. What are Conic sheaves?
4. **How to adapt this to the o-minimal context?**
5. What did we do so far?
6. What's next?

Subanalytic site

General topological scenario:

- X topological space
- $\mathrm{Op}(X) \to$ open subsets of X
- $\mathrm{Cov}(U) \to$ coverings of U by objects in $\mathrm{Op}(X)$

Subanalytic scenario:

- X real analytic manifold
- $\mathrm{Op}(X_{sa}) \to$ open subanalytic subsets of X
- $\mathrm{Cov}_{sa}(U) \to$ **locally finite** coverings of U by objects in $\mathrm{Op}(X_{sa})$

Subanalytic site

General topological scenario:

- X topological space
- $\mathrm{Op}(X) \to$ open subsets of X
- $\mathrm{Cov}(U) \to$ coverings of U

  by objects in $\mathrm{Op}(X)$

Subanalytic scenario:

- X real analytic manifold
- $\mathrm{Op}(X_{\mathsf{sa}}) \to$ open subanalytic subsets of X
- $\mathrm{Cov}_{\mathsf{sa}}(U) \to$ **locally finite** coverings of U

  by objects in $\mathrm{Op}(X_{\mathsf{sa}})$

Theorem (Kashiwara-Schapira, 2001)

There is an equivalence of categories:

$$\mathsf{Sh}(X_{\mathsf{sa}}) \simeq \mathrm{Ind}_{\mathbb{R}-c}(X)$$

which means we can always write

$$\mathcal{F} \simeq \varinjlim_{i} \rho_* \mathcal{F}_i \quad , \rho : X \to X_{\mathsf{sa}}$$

Subanalytic sheaf theory

Theorem (Kashiwara-Schapira, 2001)

There is an equivalence of categories:

$$\mathrm{Sh}(X_{\mathrm{sa}}) \simeq \mathrm{Ind}_{\mathbb{R}-c}(X)$$

which means we can always write

$$\mathcal{F} \simeq \varinjlim_{i} \rho_* \mathcal{F}_i \quad , \rho : X \to X_{\mathrm{sa}}$$

Subanalytic sheaf theory

Theorem (Kashiwara-Schapira, 2001)

There is an equivalence of categories:

$$\mathsf{Sh}(X_{\mathsf{sa}}) \simeq \mathrm{Ind}_{\mathbb{R}-c}(X)$$

which means we can always write

$$\mathcal{F} \simeq \varinjlim_i \rho_* \mathcal{F}_i \quad , \rho : X \to X_{\mathsf{sa}}$$

Given $f : X \to Y$ a real analytic map:

$$f_{!!}\mathcal{F} := \varinjlim_i \rho_* f_! \mathcal{F}_i$$

Conic subanalytic sheaves

- $X_{\mathbb{R}^+}$
- Opens are conic opens of X
- $\eta : X \to X_{\mathbb{R}^+}$

- $X_{sa,\mathbb{R}^+}$
- Opens are conic opens of X_{sa}
- $\eta_{sa} : X_{sa} \to X_{sa,\mathbb{R}^+}$

Conic subanalytic sheaves

- $X_{\mathbb{R}^+}$
- Opens are conic opens of X
- $\eta : X \to X_{\mathbb{R}^+}$

- $X_{sa,\mathbb{R}^+}$
- Opens are conic opens of X_{sa}
- $\eta_{sa} : X_{sa} \to X_{sa,\mathbb{R}^+}$

Conic subanalytic sheaves

$\mathcal{F} \in \mathrm{Sh}(X_{sa})$ is conic if any of the following equivalent conditions holds:

1. ~~$\mu^{-1}\mathcal{F} \simeq p^{-1}\mathcal{F}$, with $p : \mathbb{R}^+ \times X \to X$ projection~~
2. ~~$\mathcal{F}|_{\mathbb{R}^+x}$ = constant~~
3. $\mathcal{F}(\mathbb{R}^+U) \xrightarrow{\sim} \mathcal{F}(U)$ if U is $\mathbb{R}^+$-connected
4. $\mathcal{F} \simeq \eta_{sa}^{-1}\, \eta_{sa\,*}\, \mathcal{F}$

Conic subanalytic sheaves

- $X_{\mathbb{R}^+}$
- Opens are conic opens of X
- $\eta : X \to X_{\mathbb{R}^+}$

- $X_{sa,\mathbb{R}^+}$
- Opens are conic opens of X_{sa}
- $\eta_{sa} : X_{sa} \to X_{sa,\mathbb{R}^+}$

Conic subanalytic sheaves

$\mathcal{F} \in \mathrm{Sh}(X_{sa})$ is conic if any of the following equivalent conditions holds:

1. ~~$\mu^{-1}\mathcal{F} \simeq p^{-1}\mathcal{F}$, with $p : \mathbb{R}^+ \times X \to X$ projection~~
2. ~~$\mathcal{F}|_{\mathbb{R}^+x}$ = constant~~
3. $\mathcal{F}(\mathbb{R}^+U) \xrightarrow{\sim} \mathcal{F}(U)$ if U is $\mathbb{R}^+$-connected
4. $\mathcal{F} \simeq \eta_{sa}^{-1}\, \eta_{sa\,*}\, \mathcal{F}$

Equivalence of categories (L. Prelli, 2013)

If $\mathrm{Sh}_{\mathbb{R}^+}(X_{sa})$ is the category of conic **subanalytic** sheaves,

$$\mathrm{Sh}_{\mathbb{R}^+}(X_{sa}) \simeq \mathrm{Sh}(X_{sa,\mathbb{R}^+})$$

o-minimal site

General topological scenario:

- X topological space
- $\mathrm{Op}(X) \to$ open subsets of X
- $\mathrm{Cov}(U) \to$ coverings of U

  by objects in $\mathrm{Op}(X)$

Definable scenario:

- X definable set
- $\mathrm{Op}(X_{\mathrm{def}}) \to$ open definable subsets of X
- $\mathrm{Cov}_{\mathrm{def}}(U) \to$ **finite** coverings of U

  by objects in $\mathrm{Op}(X_{\mathrm{def}})$

o-minimal site

General topological scenario:

- X topological space
- $\mathrm{Op}(X) \to$ open subsets of X
- $\mathrm{Cov}(U) \to$ coverings of U by objects in $\mathrm{Op}(X)$

Definable scenario:

- X definable set
- $\mathrm{Op}(X_{\mathrm{def}}) \to$ open definable subsets of X
- $\mathrm{Cov}_{\mathrm{def}}(U) \to$ **finite** coverings of U by objects in $\mathrm{Op}(X_{\mathrm{def}})$

Theorem (Edmundo-Prelli, 2016)

There is an equivalence of categories:

$$\mathsf{Sh}(X_{\mathrm{def}}) \simeq \mathsf{Sh}(\widetilde{X})$$

o-minimal adaptation

Topological case

- X top. space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $X_{\mathbb{R}^+}$
- $\eta : X \to X_{\mathbb{R}^+}$

Subanalytic case

- X analytic manifold
- $\mu : \mathbb{R}^+ \times X \to X$ analytic action
- X_{sa}
- $X_{\mathsf{sa},\mathbb{R}^+}$
- $\eta_{\mathsf{sa}} : X_{\mathsf{sa}} \to X_{\mathsf{sa},\mathbb{R}^+}$

o-minimal case

- X definable set
- $\mu : R^+ \times X \to X$ def., continuous action.
- X_{def}
- X_{def,R^+}
- $\eta_{\mathrm{def}} : X_{\mathrm{def}} \to X_{\mathrm{def},R^+}$

o-minimal adaptation

Topological case

- X top. space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $X_{\mathbb{R}^+}$
- $\eta : X \to X_{\mathbb{R}^+}$

Subanalytic case

- X analytic manifold
- $\mu : \mathbb{R}^+ \times X \to X$ analytic action
- X_{sa}
- $X_{\mathsf{sa},\mathbb{R}^+}$
- $\eta_{\mathsf{sa}} : X_{\mathsf{sa}} \to X_{\mathsf{sa},\mathbb{R}^+}$

o-minimal case

- X definable set
- $\mu : R^+ \times X \to X$ def., continuous action.
- X_{def}
- X_{def,R^+}
- $\eta_{\mathrm{def}} : X_{\mathrm{def}} \to X_{\mathrm{def},R^+}$
- $\widetilde{\eta_{\mathrm{def}}} : \widetilde{X} \to \widetilde{X_{\mathrm{def},R^+}}$

o-minimal adaptation

Topological case

- X top. space
- $\mu : \mathbb{R}^+ \times X \to X$ continuous action
- $X_{\mathbb{R}^+}$
- $\eta : X \to X_{\mathbb{R}^+}$

Subanalytic case

- X analytic manifold
- $\mu : \mathbb{R}^+ \times X \to X$ analytic action
- X_{sa}
- $X_{\mathsf{sa},\mathbb{R}^+}$
- $\eta_{\mathsf{sa}} : X_{\mathsf{sa}} \to X_{\mathsf{sa},\mathbb{R}^+}$

o-minimal case

- X definable set
- $\mu : R^+ \times X \to X$ def., continuous action.
- X_{def}
- X_{def,R^+}
- $\eta_{\mathrm{def}} : X_{\mathrm{def}} \to X_{\mathrm{def},R^+}$
- $\widetilde{\eta_{\mathrm{def}}} : \widetilde{X} \to \widetilde{X_{\mathrm{def},R^+}}$

Theorem

There is an equivalence of categories:

$$\mathsf{Sh}(X_{\mathrm{def},R^+}) \simeq \mathsf{Sh}(\widetilde{X_{\mathrm{def},R^+}})$$

Conic *o*-minimal sheaves

Definition

$\mathcal{F} \in \mathrm{Sh}(X_{\mathrm{def}}) \simeq \mathrm{Sh}(\widetilde{X})$ is **conic** if:

- $\mathcal{F}(\widetilde{R^+U}) \xrightarrow{\sim} \mathcal{F}(\widetilde{U})$ if U is R^+-definably-connected

Theorem (Equivalence of categories)

If $\mathrm{Sh}_{R^+}(X_{\mathrm{def}})$ is the category of conic *o*-minimal sheaves,

$$\mathrm{Sh}_{R^+}(X_{\mathrm{def}}) \underset{\eta_{\mathrm{def}}^{-1}}{\overset{\eta_{\mathrm{def}\,*}}{\rightleftarrows}} \mathrm{Sh}(X_{\mathrm{def},R^+})$$

$$\mathrm{Sh}_{R^+}(X_{\mathrm{def}}) \simeq \mathrm{Sh}(X_{\mathrm{def},R^+})$$

Characterization of conic *o*-minimal sheaves

Theorem (Characterizations of conic *o*-minimal sheaves)

$\mathcal{F} \in \mathrm{Sh}(X_{\mathrm{def}}) \simeq \mathrm{Sh}(\widetilde{X})$ is conic if any of the following equivalent conditions holds:

1. $\widetilde{\mu}^{-1}\mathcal{F} \simeq \widetilde{p}^{-1}\mathcal{F}$, with $p : R^+ \times X \to X$ projection
2. $\mathcal{F}^{\mathcal{S}}|_{\widetilde{S^+a}} =$ constant, for any elementary extension $\mathcal{R} \preceq \mathcal{S}$ and $a \in X(\mathcal{S})$
3. $\mathcal{F}|_{\widetilde{\eta_{\mathrm{def}}}^{-1}(\beta)} =$ constant, for any $\beta \in \widetilde{X_{\mathrm{def},R^+}}$
4. $\mathcal{F}(\widetilde{R^+U}) \xrightarrow{\sim} \mathcal{F}(\widetilde{U})$ if U is R^+-definably-connected (definition)
5. $\mathcal{F} \simeq \eta_{\mathrm{def}}^{-1}\, \eta_{\mathrm{def}\,*}\, \mathcal{F}$

1. What was already done?
2. What else can be done?
3. What are Conic sheaves?
4. How to adapt this to the o-minimal context?
5. What did we do so far?
6. What's next?

What's next?

What's next?

- Finish checking everything

What's next?

- Finish checking everything
- Stability (of conic sheaves) under the 6 Grothendieck operations

What's next?

- Finish checking everything
- Stability (of conic sheaves) under the 6 Grothendieck operations
  - Topological context: **always stable**

What's next?

- Finish checking everything
- Stability (of conic sheaves) under the 6 Grothendieck operations
  - Topological context: **always stable**
  - Subanalytic context: **Not stable** only under $f_{!!}$

What's next?

- Finish checking everything
- Stability (of conic sheaves) under the 6 Grothendieck operations
  - Topological context: **always stable**
  - Subanalytic context: **Not stable** only under $f_{!!}$
- Fourier-Sato Transform: $D^{+}_{\mathbb{R}^+}(E) \simeq D^{+}_{\mathbb{R}^+}(E^*)$

What's next?

- Finish checking everything
- Stability (of conic sheaves) under the 6 Grothendieck operations
  - Topological context: **always stable**
  - Subanalytic context: **Not stable** only under $f_{!!}$
- Fourier-Sato Transform: $D^{+}_{\mathbb{R}^{+}}(E) \simeq D^{+}_{\mathbb{R}^{+}}(E^{*})$
- etc...

References I

[EP16] Mário J. Edmundo and Luca Prelli.
Sheaves on $\mathcal{T}$-topologies.
J. Math. Soc. Japan, 68(1):347–381, 2016.

[EP20] Mário J. Edmundo and Luca Prelli.
The six Grothendieck operations on o-minimal sheaves.
Math. Z., 294(1-2):109–160, 2020.

[KS90] Masaki Kashiwara and Pierre Schapira.
Sheaves on manifolds, volume 292 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*.
Springer-Verlag, Berlin, 1990.
With a chapter in French by Christian Houzel.

[KS01] Masaki Kashiwara and Pierre Schapira.
Ind-sheaves, volume 271.
Société mathématique de France, 2001.

[Pre08] Luca Prelli.
Sheaves on subanalytic sites.
Rend. Semin. Mat. Univ. Padova, 120:167–216, 2008.

References II

[Pre13] Luca Prelli.
Microlocalization of subanalytic sheaves.
Mém. Soc. Math. Fr. (N.S.), (135):vi+91, 2013.

[vdD98] Lou van den Dries.
Tame topology and o-minimal structures, volume 248 of *London Mathematical Society Lecture Note Series*.
Cambridge University Press, Cambridge, 1998.

Thanks!