

1-basedness in t -minimal structures

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Zilber, CHL

- Totally categorical theories are not finitely axiomatizable.
- First key property: local modularity of the (unique, up to non-orthogonality) strongly minimal set.
- Second key property: locally modular non-trivial strongly minimal sets are “Module like”.
- Cherlin-Harrington-Lachlan: \aleph_0 -categorical \aleph_0 -stable theories are coordinatized by locally modular strongly minimal types.

CHL moral: \aleph_0 -categorical \aleph_0 -stable theories share important structural properties with totally categorical theories.

Hrushovski

Locally modular regular types

- Non-trivial locally modular regular types are, essentially, linear spaces over a division ring.
- A group chunk theorem and a suitable group configuration theorems are needed for the construction.

With Cherlin (“Finite structures with few types”): the CHL moral extends beyond the stable setting to structures coordinatized by linear SU-minimal types.

Hrushovski-Pillay

1-based groups

- 1 Abelian by finite.
- 2 Rigid (no definable families of infinite subgroups).
- 3 Linear: definable subsets of G^n are finite boolean combinations of cosets of definable subgroups.
- 4 Wagner proves analogous results for 1-based simple groups.

Recall also (Bouscaren-Hrushovski): (roughly) in a 1-based theory any non-trivial type can be associated with a type definable group.

Peterzil

Linear o-minimal structures

- 1 O-minimal structures are geometric (so “local modularity makes sense”).
- 2 There are non-locally modular o-minimal structures that are locally linear.
- 3 A new notion of linearity captures precisely those o-minimal structures where a group is type-definable, but no field is.
- 4 Maalouf: An analogous notion of linearity* produces a (type)-definable group in geometric C-minimal structures.

* The statement assumes local modularity, but the proof seems to go through for linear structures.

Peterzil-Loveys

The structure of linear o-minimal groups¹

¹I am unaware of analogous results in the C-minimal setting.

O-minimal linear groups are:

- 1 Abelian (well, all o-minimal groups are).
- 2 Have no definable families of definable subgroups.
- 3 Every definable subset of G^n is, locally – at a generic point – the generic of a coset of a definable subgroup.
- 4 Families of germs of (unary) functions are 1-dimensional.

In fact: a linear o-minimal local group is a reduct of an ordered vector space over an ordered division ring. This extends to the above type-def. groups.

Many notions of linearity out there

Welcome to the Zoo

- Local modularity, 1-basedness, (almost) canonical bases of definable plane curves are 1-dimensional, no large families of (almost) normal families of plane curves, no (complete) quasi-designs, no type-definable pseudo-planes, weak 1-basedness, weak local modularity... aimed to capture variants of linearity in various tame contexts.
- Much work went into understanding the relations between these notions in various settings.
- I am unaware of significant structural results associated with those notions of linearity (besides those already mentioned).

The goal

The big organizing idea: Zilber's Trichotomy is, essentially, a topological phenomenon.

- Find a uniform *topological* proof of the group construction theorems for the linear o-minimal and C-minimal setting.
- Prove structural results in the spirit of Hrushovski-Pillay and Peterzil-Loveys.
- Expand the proof to *non-geometric* topological settings (e.g., dense weakly o-minimal structures).

Larger aim: find an axiomatic topological framework where this can be carried out.

No need to look far

t-minimality

- L. Mathews (~ 1995): A structure \mathcal{M} is t-minimal if it admits a uniformly definable basis for a Hausdorff topology such that a definable $S \subseteq M$ is infinite if and only if $\text{Int}(S) \neq \emptyset$.
- Examples include: dense (weakly) o-minimal structures, dense C-minimal structures, dp-minimal valued fields. 1-h-minimal fields...
- Problem: with no further assumptions, results are fairly weak.
- A t-minimal structure is *visceral* if the topology is *uniform*.
- Dolich-Goodrick: considerably stronger results for visceral theories (still with additional assumptions).

Not quite there yet

Visceral is too strong

- Dense o-minimal and C-minimal theories need not be visceral.
- The standard source of uniform structures:
 - 1 G a (definable) group.
 - 2 A (definable) neighbourhood base at e .
 - 3 Declare translations homeomorphisms.
- Primary goal: construct a group, not assume one exists to begin with!

Johnson: visceral theories without assumptions. Dimension theory works for general t -minimal.

Dimension is not enough

Example (The all purpose counter-example: the Sorgenfrey line)

Take $(\mathbb{R}, +, \cdot, <)$ with the topology generated by intervals of the form $[a, b)$. It is:

- 1 t-minimal.
- 2 geometric.
- 3 The function $x \mapsto -x$ is definable and nowhere continuous.

Definition

A t-minimal \mathcal{M} has the *independent neighborhood property* if for any tuple $a \in M^n$, any parameter set A , and any neighborhood U of a , there is a neighborhood V of a such that $V \subset U$ and V is definable over a parameter t with $\dim(a/At) = \dim(a/A)$.

Many natural examples

Example

The following are t -minimal with the independent neighbourhood property:

- 1 Visceral theories (Johnson).
- 2 Dense C -minimal and weakly o -minimal theories.

The INP implies, e.g., generic continuity, the local homeomorphism property...

A useful observation

The INP provides some replacement for exchange, supplying “enough” additive pairs satisfying

$$\dim(ab/A) = \dim(a/A) + \dim(b/Aa).$$

How are groups constructed?

In model theory, often, groups are constructed in two steps:

- 1 Construct a *group chunk*: multiplication is defined only generically.
- 2 Recover a group from the group chunk.
 - Step (2) is standard: Weil, Hrushovski, van den Dries, Pillay, Eleftheriou...
 - For step (1) use composition of definable bijections within definable families.
 - For better results, work with *germs* of definable bijections.

Caution: In different settings germs may have different meanings.

The usual definition

Definition

Let $a \in M^n$.

- 1 The *germ* $g := \text{germ}_a(X)$ of X at a is the equivalence class realized by X among all subsets of M^n modulo agreement in a neighborhood of a .
- 2 A *definable germ* at a is a germ at a realized by some definable subset of M^n .
- 3 If g is a definable germ at a , then $\dim(g)$, is the smallest value of $\dim(X)$ where X is a definable set realizing g .

Caution: This is **not** the dimension of g as an element of \mathcal{M}^{eq} .

Germ of types exist

Fact

If \mathcal{M} is t -minimal then for all $a \in M^n$ and parameter set A $\text{germ}_a(X)$ is eventually constant as X ranges over $\text{tp}(a/A)$.

This is $\text{germ}(a/A)$.

A crucial implication of the generic neighbourhood property is that germs characterize independence:

Fact

Let $a \in M^n$. Then $\dim(a/A) = \dim(a/B)$ iff $\text{germ}(a/A) = \text{germ}(a/B)$.

Some calculus of germs

Infinitesimal neighbourhoods

For $a \in M^n$ and a parameter set A we let $\mu(a/A) := \text{tp}(a/A) \cap \mu(a)$, where $\mu(a)$ is the infinitesimal neighbourhood of a .

Fact

- $\text{germ}_a(\mu(a/A)) = \text{germ}(a/A)$.
- $\mu(ab/A)_b = \mu(a/Ab)$ and $\text{germ}(ab/A)_b = \text{germ}(a/Ab)$ follows.
- *A trace of exchange: If $B \supset A$ and $\dim(ab/B) = \dim(ab/A)$ then $\dim(a/Bb) = \dim(a/Ab)$.*

What does 1-basedness mean?

In the stable setting:

- Recall: a stable theory is 1-based if for any $a, A = \text{acl}(A)$, $\text{Cb}(a/A) \subseteq \text{acl}(a)$.
- Any A -definable set where a is generic is determined uniquely “almost everywhere”, by the fact that a belongs to it.
- Compare with: a point incident to a line of the form $ax + by - c = 0$ in \mathbb{R}^2 .

This does not make sense already in o-minimal theories.

Solution

- Think of “ X, Y agree generically” in the stable setting as “ $X \cap U = Y \cap U$ for some Zariski open U ” in AG.
- In Hausdorff topologies this reads: $\text{germ}_a(X) = \text{germ}_a(Y)$.

Topologically 1-based types

Definition

Let $a \in M^m$, $b \in M^n$, and A a parameter set. We say that $\text{tp}(a/Ab)$ is *topologically 1-based over A* if

$$\dim(b/Aa) = \dim(b/A \text{germ}(a/Ab)).$$

This reads as: b is independent from $\text{germ}(a/Ab)$ over Aa .

Theorem (The main structure theorem)

TFAE for $a, b \subseteq M$ and any A :

- 1 $\text{tp}(a/Ab)$ is topologically 1-based over A .
- 2 The map $y \mapsto \text{germ}(a/Ay)$ is constant for $y \in \mu(b/Aa)$.
- 3 Any two fibers in the projection $\mu(ab/A) \rightarrow \mu(b/A)$ are either equal or disjoint.

topologically 1-based theories

Definition

A t -minimal \mathcal{M} with the INP is topologically 1-based if $\text{tp}(a/Ab)$ is for all $a, b \subseteq M$ and parameter set A .

Fact (\mathcal{M} topologically 1-based)

- 1 *Stable under naming constants and admissible reducts (preserving the topology and INP).*
- 2 *No definable infinite fields in \mathcal{M} .*
- 3 *If \mathcal{M} is geometric it is weakly 1-based (and vice versa).*
- 4 *So linear o -minimal and geometric C -minimal structures are topologically 1-based with INP.*

Summary of the results

Theorem

Let \mathcal{M} be a sufficiently saturated t -minimal structure with the independent neighborhood property. Assume that \mathcal{M} is non-trivial and topologically 1-based. Then there are a countable parameter set A , and an A -type-definable abelian group G , such that:

- 1 G is open in \mathcal{M} .
- 2 G is a topological group with the topology inherited from \mathcal{M} .
- 3 G is locally linear.

Remark

The group existence theorem is local. For the structural part more than one non-trivial topologically 1-based type is needed.

Strategy of the proof

A combinatorial group(oid) configuration theorem

- A *groupoid spine* is a certain category (possibly missing some id_X arrows) whose objects are sets, whose morphisms are bijections, and with a linear order on its collection of objects.
- We assume that for all $i < j$ the set $\text{Mor}(X_i, X_j)$ is given (satisfying obvious assumptions), and ask when a groupoid spine can be extended to a groupoid.

Our combinatorial group configuration theorem is:

Theorem

A symmetric groupoid spine on more than two objects extends to a groupoid.

Rigidity

When can a groupoid spine be extended to a symmetric groupoid spine?

Definition

Let $(I, <, \mathcal{X}, R, \text{Mor})$ be a groupoid spine. We say that $(I, <, \mathcal{X}, R, \text{Mor})$ is *regular* if for all $(i, j) \in R$, and all $x \in X_i$ and $y \in X_j$, there is exactly one morphism $f \in \text{Mor}(i, j)$ with $f(x) = y$.

Theorem

Let $(I, <, \mathcal{X}, R, \text{Mor})$ be a regular groupoid spine. Then there is a symmetric set $S \supset R$ such that $(I, \mathcal{X}, R, \text{Mor})$ extends to S . If the data is type-definable, the group action $(\text{Mor}(X_i, X_i), X_i)$ is type-definable for any object $X_i \in \mathcal{X}$.

The construction of the group

Aim: construct a regular type-definable group-spine (assuming topological 1-basedness).

Definition

- 1 \mathcal{M} is *non-trivial* if for some A , there are $a, b, c \in M$ each A -algebraic over the others and $\dim(abc/A) = 2$.
- 2 A 1-dimensional topological pre-group configuration (a, b, c, t, u) is a topological group configuration if $\text{tp}(ac/tu)$ is topologically 1-based.

Theorem

Let (a, b, c, t, u) be a 1-dimensional topological group configuration in \mathcal{M} . Then there is an \mathcal{M} -type-definable group structure on the set $\mu(a/\emptyset)$ with identity element a .

- 1 $\mu(abt/\emptyset)$ is the graph of a regular family of homeomorphisms $\mu(a/\emptyset) \rightarrow \mu(b/\emptyset)$.
- 2 Similarly for $\mu(b, c, u/\emptyset)$, $\mu(a, u, tc/\emptyset)$.
- 3 $\mu(a/\emptyset)$, $\mu(b/\emptyset)$ and $\mu(c/\emptyset)$ with the above families of homeomorphisms form a regular groupoid spine.

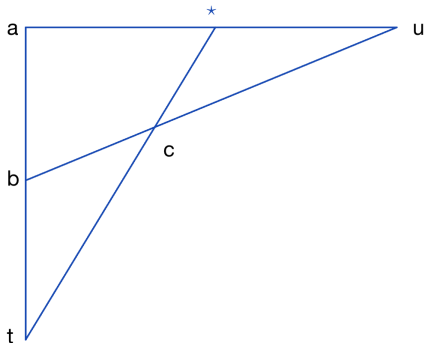


Figure: A pre-group configuration

The structural results

- 1 Topologize the group using Marikova's method.
- 2 We can replace the group to get one that is topological with the affine topology.
- 3 Local linearity and local commutativity are “topologized” adaptations of the analogous results from stability theory.

Remark: with a little more effort we can make the group visceral (in an appropriate sense).

Thank you!