D-minimal cell decomposition

joint with Madie Farris

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Let $\mathcal{R} = (R, <, \dots)$.

Definition. We say $\mathcal{R}$ is **o-minimal** if every definable subset of R is a finite union of intervals and points.

One of the foundational results in the study of o-minimality is that the finiteness condition in the definition of o-minimality holds uniformly in the parameters used to define the set.

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Definition. We say $\mathcal{R}$ is **strongly o-minimal** if for every $n \in \mathbb{N}$ and every definable subset $A \subseteq R^{n+1}$ there is $N \in \mathbb{N}$ such that for all $x \in R^n$ the fiber A_x is a union of N intervals and points.

Using cell decomposition, Knight, Pillay, and Steinhorn show that o-minimality implies strong o-minimality in arbitrary o-minimal structures.

Fix an expansion $\mathcal{R}$ of the real ordered field $\overline{\mathbb{R}} = (\mathbb{R}, <, +, \cdot)$.

Definition. (Miller) We say that $\mathcal{R}$ is **d-minimal** if every definable subset of $\mathbb{R}$ is a union of an open set and finitely many discrete sets.
We say that $\mathcal{R}$ is **strongly d-minimal** if for every $n \in \mathbb{N}$ and every definable set $A \subseteq \mathbb{R}^{n+1}$ there is $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^n$ either A_x has interior or is the union of N discrete sets.

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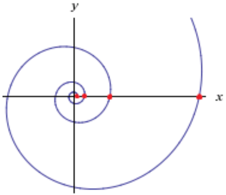
Examples.

1. $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ (van den Dries, '85),
2. $(\overline{\mathbb{R}}, S_\omega)$, where S_ω is a logarithmic spiral:

$$S_\omega = \{(e^t \cos(\omega t), e^t \sin(\omega t)) \ : \ t \in \mathbb{R}\}$$

(Miller-Speissegger,'00)



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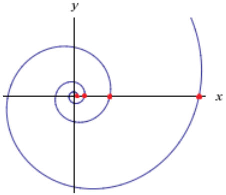

Question (Miller). Does d-minimality imply strong d-minimality?

D-minimal cell decomposition (Miller, 2005 and Thamrongthanyalak, 2019). Suppose $\mathcal{R}$ is strongly d-minimal. Then every definable subset of $\mathbb{R}^n$ is a union of countably many cells.

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Some previously known results on d-minimality include:

- D-Minimal Whitney Theorem (Miller and Thamrongthanyalak, 2018)
- D-Minimal Michael's Selection Theorem (Thamrongthanyalak, 2019)
- Countable Gromov-Yomdin Parametrization Theorem (Jäger, 2022)

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Locally o-minimal grid decomposition (H.-Walsberg 2019?). Let $\mathcal{S}$ be an expansion of $(\mathcal{R}, <, +)$ such that nowhere dense $\mathcal{S}$-definable subset of $\mathcal{R}$ is closed and discrete. Then $\mathcal{S}^\circ$ is locally o-minimal.

Fujita (2025): true for DC expansions of densely ordered groups.

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3. if $w_n = 1$, then there are $f, g \in \mathcal{C}_\infty(\pi(A))$ such that $f < g$ and $A = \big(f, g\big)$.

We view the one-point space $\mathbb{R}^0$ as a ()-cell, where () is the sequence of length 0.

Definition. Let $n \in \mathbb{N}_{\geq 1}$ and $w = (w_1, \ldots, w_n) \in \{0,1\}^n$. We define a **$w$-stack** recursively. For $n = 1$, a **0-stack** is a finite set of points in $\mathbb{R}$, and a **1-stack** is a union of finitely many disjoint open intervals in $\mathbb{R}$.

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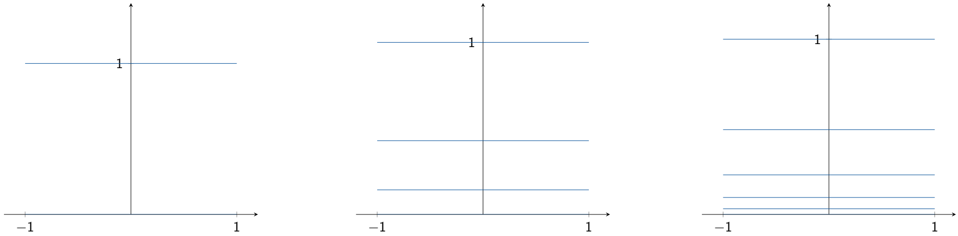
1. $\pi(A)$ is a $(w_1, \dots, w_{n-1})$-stack and
2. if $w_n = 0$, then for each connected component C of $\pi(A)$ there are $f_{A,1}, \dots, f_{A,m_C} \in \mathcal{C}(C)$ such that
   (a) $f_{A,1} < \cdots < f_{A,m_C}$ and
   (b) $A \cap [C \times \mathbb{R}] = \bigcup_{i=1}^{m_C} \Gamma(f_{A,i})$.
3. if $w_n = 1$, then for each connected component C of $\pi(A)$ there are $f_{A,1}, \dots, f_{A,m_C}, g_{A,1}, \dots, g_{A,m_C} \in \mathcal{C}_\infty(C)$ such that
   (a) $f_{A,i} < g_{A,i}$ for all $i = 1, \dots, m_C$ and either $g_{A,i} = f_{A,i+1}$ or $g_{A,i} < f_{A,i+1}$ for all $i = 1, \dots, m_C - 1$, and
   (b) $A \cap [C \times \mathbb{R}] = \bigcup_{i=1}^{m_C} \left(f_{A,i}, g_{A,_i}\right)$.

Again we consider $\mathbb{R}^0$ as the unique ()-stack.



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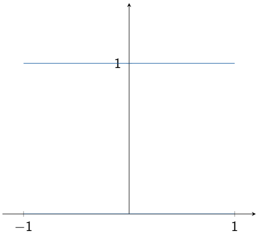
Definition. Let $A, B \subseteq \mathbb{R}^n$ be w-stacks. We say A is a **substack** of B if every connected component of A is a connected component of B.



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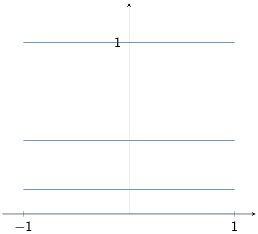

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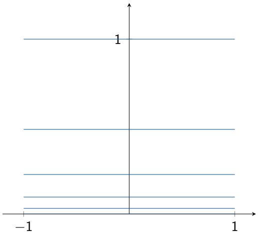

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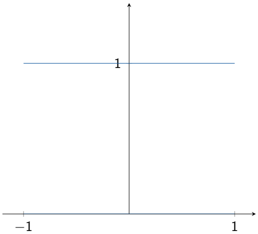

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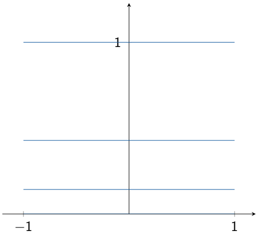

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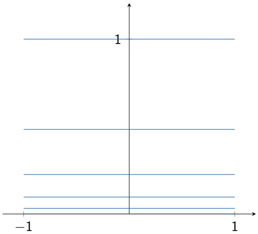

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Definition. We say $A \subseteq \mathbb{R}^n$ is **étale** at $x \in A$ if there is an open box $U \subseteq \mathbb{R}^n$ around x such that $\pi|_{A \cap U}$ is a homeomorphism onto its image in $\mathbb{R}^{n-1}$. We say that A is **étale** if A is étale at every $x \in A$.

Definition. Let $n \in \mathbb{N}_{\geq 1}$, $w = (w_1, \ldots, w_n) \in \{0,1\}^n$, and $D \subseteq (0,1]$ be a sequence set. Let $\mathcal{S} := (S_d)_{d \in D}$ be a definable family of w-stacks. We say $\mathcal{S}$ is a **(w, D)-grid** if

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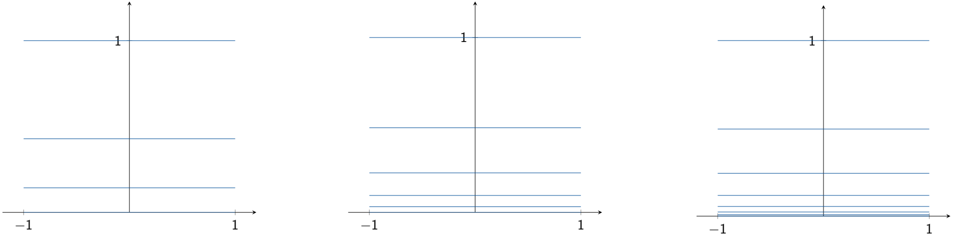
2. S_d is a substack of S_e for all $d, e \in D$ with $d > e$, and
3. $\bigcup_{d \in D} S_d$ is étale if $w_n = 0$.

We say $\bigcup_{d \in D} S_d$ is the **surface** of $\mathcal{S}$, and write $\operatorname{surf}(\mathcal{S})$.



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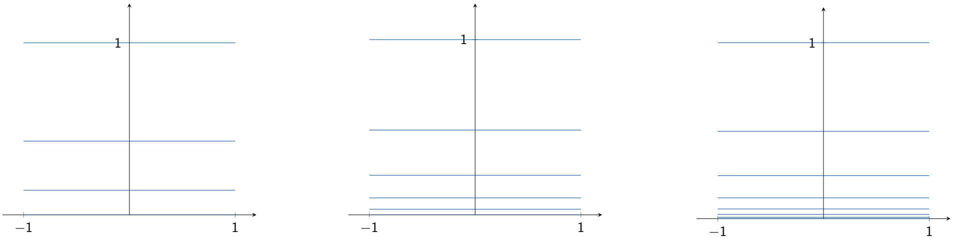
Stacks approximate grids. Let $\mathcal{S} = (S_d)_{d \in D}$ be a w-grid, and $C \subseteq \mathbb{R}^n$. Then C is a connected component of $\mathrm{surf}(\mathcal{S})$ if and only if there is $d \in D$ such that C is a connected component of S_d.



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Definition. We say $\mathcal{R}$ is **noiseless** if every definable subset of $\mathbb{R}$ either has interior or is nowhere dense.

Miller, 2005. Suppose that $\mathcal{R}$ is noiseless. Let $A \subseteq \mathbb{R}^m \times \mathbb{R}$ be definable. Then

1. every definable subset of $\mathbb{R}^n$ either has interior or is nowhere dense,
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Noiselessness of grids. Let $\mathcal{S} = (S_d)_{d \in D}$ be a grid and $Z \subseteq \mathrm{surf}(\mathcal{S})$ be definable. Then Z either has interior in $\mathrm{surf}(\mathcal{S})$ or is nowhere dense in $\mathrm{surf}(\mathcal{S})$.

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Fiber lemma for grid. Let $A \subseteq \mathbb{R}^n \times \mathbb{R}$ be such that $\pi(A)$ is a surface of a grid. Then

$$\{x \in \pi(A) : \mathrm{cl}(A_x) \neq \mathrm{cl}(A)_x\}$$

is nowhere dense in $\pi(A)$.

Definition. Let $A \subseteq \mathbb{R}$ be closed, and let λ be an ordinal. We define the set $A^{[\lambda]}$ as follows:

$$\begin{aligned} A^{[0]} &= A \\ A^{[\lambda+1]} &= A^{[\lambda]} \setminus \mathrm{isol}(A^{[\lambda]}) \\ A^{[\lambda]} &= A \setminus \bigcup_{\mu<\lambda} \mathrm{isol}(A^{[\mu]}) \quad \text{if } \lambda \text{ is a limit.} \end{aligned}$$

The **Cantor-Bendixson rank** of A, written $\mathrm{rk}(A)$, is the least ordinal λ such that $A^{[\lambda+1]} = A^{[\lambda]}$.

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Friedman and Miller, 2005. Let $A \subseteq \mathbb{R}$ be closed and $k \in \mathbb{N}$. Then the following are equivalent:

1. A is countable and $\mathsf{rk}(A) = k$,
2. k is the least number of discrete sets whose union is A.

Grid Decomposition Theorem Suppose $\mathcal{R}$ is d-minimal. Let $n \in \mathbb{N}$.

$(\mathrm{GD})_n$ Let $\mathcal{A}$ be a finite collection of definable subsets of $\mathbb{R}^n$. Then there is a grid decomposition of $\mathbb{R}^n$ compatible with $\mathcal{A}$.

$(\mathrm{UCB})_{n+1}$ Let $A \subseteq \mathbb{R}^{n+1}$ be definable such that A_x has no interior for all $x \in \pi(A)$. Then there is $k \in \mathbb{N}$ such that for all $x \in \pi(A)$

$$\mathrm{rk}(\mathrm{cl}(A_x)) \leq k.$$

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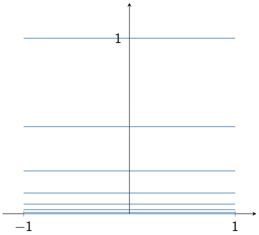
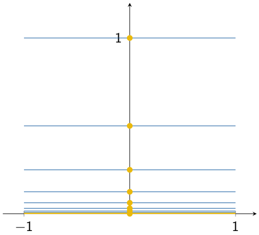
We will just discuss the proof of $(\mathrm{UCB})_2$.

CB-Rank of a grid over a cell Let $\mathcal{S} = (S_d)_{d \in D}$ be a $(w, 0)$-grid in $\mathbb{R}^n$ such that $\mathrm{surf}(\mathcal{S})$ is vertically bounded, and let $C \subseteq \mathbb{R}^{n-1}$ be a w-cell such that $\pi(S_d) = C$ for all $d \in D$. Then there is $k \in \mathbb{N}$ such that for all $x \in C$

$$\mathrm{rk}(\mathrm{cl}(\mathrm{surf}(\mathcal{S})_x)) = k.$$



$\mathrm{surf}(\bigcup_{d \in 2^{-\mathbb{N}}} S_d)$



$\{(x, y) \in \mathbb{R}^2 \ : \ y \in \mathrm{cl}(\mathrm{surf}(\bigcup_{d \in 2^{-\mathbb{N}}} S_d))\}$

$(\mathrm{UCB})_2$

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$(\mathrm{UCB})_2$

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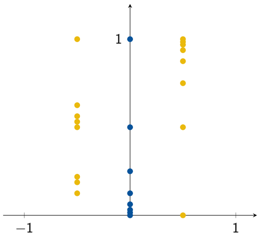
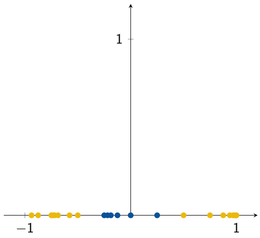
Case 1: $w = 0$

- Since $\pi(A)_x$ is discrete, we can map each of the fibers A_x into $\mathbb{R}$ in such a way that they don't overlap with $\pi(A)$ or each other. Call this set B.



A



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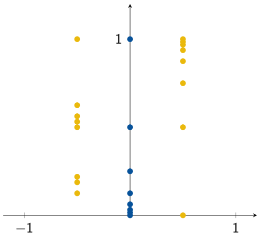

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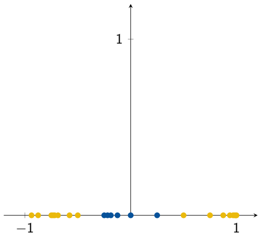

B

- Use $(\mathrm{UCB})_1$ to find a bound on $\mathrm{rk}(\mathrm{cl}(B))$ and transfer this bound to $\mathrm{rk}(\mathrm{cl}(A_x))$.

$(\mathrm{UCB})_2$

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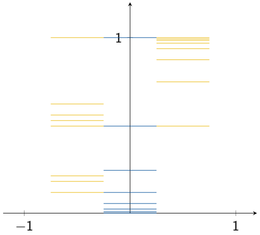
Let $A \subseteq [0,1]^2$ be definable such that A is the surface of $(w,0)$-grid.

Case 2: $w = 1$

- Define a set $A' = [\mathrm{Mid}(\pi(A)) \times \mathbb{R}] \cap A$.



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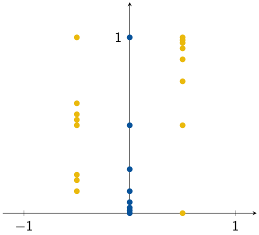

A'

- Then A' is the surface of a $(0,0)$-grid. Apply Case 1.

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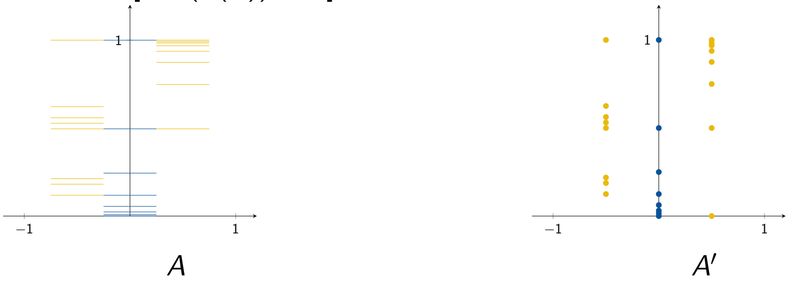

- Then A' is the surface of a $(0,0)$-grid. Apply Case 1.
- Rank of a grid is constant over a cell. Thus the rank of $\mathrm{cl}(A_x)$ is the same as the rank over the midpoint. This is uniformly bounded.

Suppose $\mathcal{R}$ is d-minimal.

Piecewise Continuity. Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$ be definable. Then there is a grid decomposition $\mathcal{G}$ of $\mathbb{R}^n$ compatible with A such that for every $\mathcal{S} \in \mathcal{G}$ with $\mathrm{surf}(\mathcal{S}) \subseteq A$ the restriction $f|_{\mathrm{surf}(\mathcal{S})}$ is continuous.

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Dimension. Let $A \subseteq \mathbb{R}^n$ be definable. Then

1. A is constructible and hence D_Σ, and
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'O-minimal' approximation. Let $A \subseteq \mathbb{R}^n$ be definable. Then there is a decreasing definable family $(A_d)_{d \in D}$ such that $A = \bigcup_{d \in D} A_d$ and for every $d \in D$

1. A_d has finitely many connected components, and
2. $\dim(A) = \dim(A_d)$.

Let $\mathbb{K}$ be an expansion of an ordered field $(K, <, +, \cdot)$.

Definition.(Miller) We say $\mathbb{K}$ is **definably complete** if every nonempty bounded subset of K definable in $\mathbb{K}$ has a supremum in K.

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Work in progress. If $\mathbb{K}$ is d-minimal, then $\mathbb{K}$ is strongly d-minimal.

Future Work

Countable Parametrization Theorem (Jäger, 2022) Let $\mathcal{R}$ be a d-minimal expansion of $\overline{\mathbb{R}}$. Every strongly bounded definable set $X \subseteq \mathbb{R}^m$ has a countable cellular k-parametrization for all $k \geq 1$.

Goal. D-minimal Pila-Wilkie. (Special cases: Boxall-Jones and Comte-Miller)

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D-minimal open core. Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$ such that every closed definable subset of $\mathbb{R}$ either has interior or is a finite union of open sets. Is the open core of $\mathcal{R}$ d-minimal?

Thank you!