New developments on algebraically bounded theories with generic derivations

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Beyond differentially closed fields

The study of differentially closed fields (DCF_0) has been very fruitful.

- Model-theoretically tame (ω-stable, eliminates quantifiers and imaginaries).
- Good understanding of geometries, definable groups, and so on.
- Many applications to the algebra of differential equations.

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What can be said for fields (or expansions of fields) that are not (just) algebraically closed, expanded by one or more derivations?

Key examples: ordered fields, valued fields, PAC fields, large fields...

In this talk, fields are always of characteristic zero.

A brief history

- Singer (1978): Closed ordered differential fields (CODF).
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- Tressl (2005): The *uniform companion* for large, model complete fields with several commuting derivations.
- León Sánchez–Tressl (2024): Differentially large fields.
- Fornasiero–K. (2021): An o-minimal generalization of CODF.
- Other frameworks *assuming* existence of a model companion:
  - Derivation-like theories (León Sánchez–Mohamed 2025).
  - Certain geometric theories of fields (Point 2025^{+}).

Algebraically bounded theories

The notion of an *algebraic boundedness*, introduced by van den Dries (1989), generalizes many of the above classes of fields.

Definition (Johnson–Ye 2023)

An $\mathcal{L}$-theory T of fields is ***algebraically bounded*** *if*

$$a \in \mathrm{acl}_{\mathcal{L}}(B) \iff \mathrm{trdeg}(a|\mathbb{P}(B)) = 0 \qquad (\mathbb{P} := \mathrm{dcl}_{\mathcal{L}}(\emptyset)).$$

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- Examples: real closed fields, henselian valued fields, bounded PAC and PRC fields, ACF with a generic predicate.
- More generally, any large field that is model complete by constants is algebraically bounded (recall we assume characteristic zero).
- Algebraically bounded fields have a *definable dimension theory*. In particular, they are perfect.

Our notation

- T is a complete algebraically bounded $\mathcal{L}$-theory of characteristic zero fields, where $\mathcal{L}$ extends the language of rings.
- $\mathcal{L}^\delta := \mathcal{L} \cup \{\delta\}$, where δ is a new unary function symbol.
- $\mathcal{L}^\Delta := \mathcal{L} \cup \Delta$, where $\Delta := (\delta_1, \ldots, \delta_m)$
- T^δ is the $\mathcal{L}^\delta$-theory of structures $K \models T$, equipped with a derivation δ. Recall, this means

$$\delta(a+b) = \delta a + \delta b \qquad \delta(ab) = a\delta b + b\delta a$$

- $T^{\Delta,\mathrm{nc}}$ is the $\mathcal{L}^\Delta$-theory of models of T expanded by m derivations.
- T^Δ extends $T^{\Delta,\mathrm{nc}}$ by axioms stating that the derivations commute.
- If $\mathbb{P}$ is not algebraic over $\mathbb{Q}$, we "hardcode" the restriction of the derivations to $\mathbb{P}$.

Generic derivations

Definition (Fornasiero–Terzo 2025)

Let $(K, \delta) \models T^\delta$. *Then* δ *is* ***generic*** *if for all* $\mathcal{L}(K)$*-definable* $X \subseteq K^{n+1}$ *with* $\dim_{\mathcal{L}} \pi_n(X) = n$*, there is* $a \in K$ *with* $(a, \delta a, \ldots, \delta^n a) \in X$*.*

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We write

$$T_g^\delta := T^\delta + \text{“}\delta \text{ is generic”}.$$

Likewise for $T_g^{\Delta,\mathrm{nc}}$ and T_g^Δ.

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Theorem (Fornasiero–Terzo 2025)

If T is model complete, then T_g^δ is the model completion of T^δ. Likewise for $T_g^{\Delta,\mathrm{nc}}$ and T_g^Δ.

Differential largeness and genericity

Definition (León Sánchez–Tressl 2024)

A differential field (K, Δ) *(with* Δ *commuting) is* ***differentially large*** *if*

$$(K, \Delta) \preccurlyeq_{\exists} \big(K((t_1, \dots, t_m)), \Delta + (\tfrac{d}{dt_1}, \dots, \tfrac{d}{dt_m})\big).$$

Equivalently, (K, Δ) *is differentially large if* K *is large and*

$$(K, \Delta) \subseteq (L, \Delta) \text{ and } K \preccurlyeq_{\exists} L \implies (K, \Delta) \preccurlyeq_{\exists} (L, \Delta).$$

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- There are large fields that are not algebraically bounded (Fehm) and algebraically bounded fields that are not large (Johnson–Ye).
- Let K be a large, algebraically bounded pure field. Is (K,Δ) differentially large if and only if Δ is generic?
- For (K,Δ), (L,Δ) differentially large, do we have

$$(K,\Delta) \subseteq (L,\Delta) \text{ and } K \preccurlyeq L \implies (K,\Delta) \preccurlyeq (L,\Delta)?$$

Model complete fields and éz-fields

- León Sánchez and Tressl show that Tressl's *uniform companion* axiom scheme axiomatizes the differentially large fields.
- Thus, for large pure fields K that are *model complete* (possibly with constants), genericity coincides with differential largeness.

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- Walsberg and Ye (2023) isolated the class of *éz-fields*. These are large, algebraically bounded, and topologically tame with respect to the étale-open topology.

Theorem (K.–Kesting 2026)

Suppose K is a large, algebraically bounded pure field. If δ is generic, then (K, δ) is differentially large. For éz-fields, the converse holds.

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To what extent is the model theory of $T_g^{\Delta,\mathrm{nc}}$ and T_g^{Δ} determined by the model theory of T?

In the remainder of the talk, I'll discuss some recent results in the following areas:

1. Classical model theory
2. Neostability
3. Definable groups and fields
4. Definable dimensions

§1. Classical transfer results

Some basic transfer properties

Theorem (Fornasiero–Terzo 2025)

- T_g^{Δ} *is consistent and complete.*
- *If* T *eliminates quantifiers, then so does* T_g^{Δ}*.*
- *For every* $\mathcal{L}_{\Delta}$*-formula* φ*, there is an* $\mathcal{L}$*-formula* ψ *such that*

$$T_g^{\Delta} \models \forall x \big(\varphi(x) \leftrightarrow \psi(\mathrm{Jet}_{\Delta}(x))\big),$$

where $\mathrm{Jet}_{\Delta}(x) = \{\delta_{i_1}\delta_{i_2}\cdots\delta_{i_n}x : n \in \mathbb{N},\ i_k \in \{1,\ldots,m\}\}$.

Likewise for $T_g^{\Delta,\mathrm{nc}}$*.*

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Likewise for $T_g^{\Delta,\mathrm{nc}}$.

Big question: Do $T_g^{\Delta,\mathrm{nc}}$ and T_g^Δ eliminate imaginaries modulo T^{eq}?

Relative elimination of imaginaries

Theorem (Fornasiero–Terzo 2025⁺)

If T has a suitable definable topology (e.g. a t-henselian topology), then $T_g^{\Delta,\mathrm{nc}}$ and T_g^{Δ} eliminate imaginaries modulo T^{eq}.

The proof, suggested by Tressl, follows from an open core result (every $\mathcal{L}^{\Delta}$-definable open set is $\mathcal{L}$-definable). This was previously used to great effect by Cubides Kovacsics and Point.

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Theorem (Fornasiero–Terzo, Fornasiero–K.–Matthews 2026⁺)

Assume T is simple. Then $T_g^{\Delta,\mathrm{nc}}$ and T_g^{Δ} have geometric elimination of imaginaries (and full EI modulo T^{eq}).

The proof uses results of Yoneda (2009).

§2. Neostability

Properties defined by sequences

Combinatorial properties defined using sequences can be easily transferred using the characterization $\varphi(x) \leftrightarrow \psi(\mathrm{Jet}_\Delta(x))$.

Theorem (Fornasiero–Terzo 2025, 2025⁺)

If T is stable, NIP, or distal, then so are T_g^Δ and $T_g^{\Delta,\mathrm{nc}}$.

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Proof for NIP: Suppose the $\mathcal{L}^\Delta$-formula $\varphi(x;y)$ has IP, so there is an $\mathcal{L}^\Delta(\emptyset)$-indiscernible $(a_i)_{i<\omega}$ and b such that

$$\models \varphi(a_i, b) \iff i \text{ is even.}$$

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Take an $\mathcal{L}$-formula ψ with $\varphi(x; y) \leftrightarrow \psi(\mathrm{Jet}_\Delta(x); \mathrm{Jet}_\Delta(y))$. Then $(\mathrm{Jet}_\Delta(a_i))_{i<\omega}$ is $\mathcal{L}(\emptyset)$-indiscernible and

$$\models \psi(\mathrm{Jet}_\Delta(a_i), \mathrm{Jet}_\Delta(b)) \iff i \text{ is even.} \qquad \square$$

Properties defined by independence relations

Theorem (Fornasiero–Terzo 2025+, León Sánchez–Mohamed 2025, Fornasiero–K.–Matthews 2026+)

- *If T is simple, then so are T_g^Δ and $T_g^{\Delta,\mathrm{nc}}$.*
- *If T is NSOP$_1$, then so are T_g^Δ and $T_g^{\Delta,\mathrm{nc}}$.*
- *If T_g^Δ and $T_g^{\Delta,\mathrm{nc}}$ have GEI, then they are rosy.*

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Conjecture: If T is rosy, then so are T_g^{Δ} and $T_g^{\Delta,\mathrm{nc}}$.

This likely requires some sort of relative elimination of imaginaries.

Ranks and supersimplicity

If $|\Delta| > 1$, then $T_g^{\Delta,\mathrm{nc}}$ is never supersimple, superrosy, or strong.

Theorem (Fornasiero–K.–Matthews 2026+)

The following are equivalent:

- T *is simple,*
- T *is supersimple of SU-rank* 1*,*
- T_g^{Δ} *is supersimple of SU-rank* ω^m*, where* $m = |\Delta|$*,*
- T_g^{Δ} *is simple,*
- T_g^{Δ} *is strong.*

Moreover, if T_g^{Δ} *has GEI, then it is superrosy of* $U^{þ}$*-rank* ω^m

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We bound the $U^{þ}$ and SU-ranks of types by the *Kolchin polynomial*.

Tree properties

A **tree property** for a formula φ is given by a tree or array (b_η) of parameters, where certain configurations $\{\varphi(x; b_\eta) : \eta \in I\}$ are consistent while others are inconsistent.

- TP_2 is given by an infinite array with inconsistent rows and consistent downward paths.
- ATP is given by a binary tree with inconsistent chains and consistent antichains.

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Theorem (Kaplan–Kesting 2026)

If T has NTP_2, then so does T_g^δ. If T has NATP, then so does T_g^δ.

- Earlier work of Point (2018) established preservation of NTP_2 for certain topological fields.
- This should hold for several derivations.

§3. Definable groups and fields

Definable groups

Theorem (Pillay–Point–Rideau-Kikuchi 2025⁺)

Any $\mathcal{L}^{\Delta}$-definable group in T_g^{Δ} admits an $\mathcal{L}^{\Delta}$-definable embedding into an $\mathcal{L}$-interpretable group in T.

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Peterzil, Pillay, and Point (2025) also showed in certain instances that any *finite-dimensional* $\mathcal{L}^{\delta}$-definable group in T_g^{δ} is $\mathcal{L}^{\delta}$-definably isomorphic to a group of the form

$$(G, s)^{\sharp} := \{g \in G : s(g) = (g, \delta g)\}$$

for G an $\mathcal{L}$-definable group and $s \colon G \to \tau(G)$ an $\mathcal{L}$-definable group section.

The latter result also holds in the o-minimal setting.

Definable fields

Conjecture

Any $\mathcal{L}^\Delta$-definable field in T_g^Δ admits an $\mathcal{L}^\Delta$-definable embedding into an $\mathcal{L}$-definable field in T.

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- Wang (2024⁺) proves the conjecture for the case of a single derivation, assuming T is NIP.
- With Eleftheriou and Point (2026⁺), we establish several instances of the conjecture for finite-dimensional $\mathcal{L}^{\delta}$-definable fields in T_g^{δ} (also in the o-minimal setting).

§4. Definable dimensions

Kolchin polynomials

Let $(K, \Delta) \models T_g^{\Delta}$. For a tuple y in an extension of K and $t \in \mathbb{N}$, set

$$\mathrm{Jet}_{\Delta}^{\leqslant t}(y) := \{\delta_1^{r_1} \cdots \delta_m^{r_m}(y) : r_1 + \cdots + r_m \leqslant t\}.$$

There is a polynomial $\omega_{y|K}(T)$, the **Kolchin polynomial**, such that

$$\omega_{y|K}(t) = \mathrm{trdeg}(\mathrm{Jet}_{\Delta}^{\leqslant t}(y)|K) \quad \text{for } t \gg 0.$$

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For an $\mathcal{L}^\Delta(K)$-definable set X, set $\omega_X \coloneqq \max\{\omega_{y|K} : y \in X^*\}$.

Theorem (Fornasiero–K.–Terzo 2026⁺)

If $(X_a)_{a \in A}$ is an $\mathcal{L}^\Delta(K)$-definable family, then the map $a \mapsto \omega_{X_a}$ has finite image and definable fibers.

This was shown in DCF$_{0,m}$ by Freitag, León Sánchez, and Li (2020).

Multivariate Kolchin polynomials

For $\boldsymbol{s} = (s_1, \dots, s_m) \in \mathbb{N}^m$, set $\mathrm{Jet}_\Delta^{\leqslant \boldsymbol{s}}(y) := \{\delta_1^{r_1} \cdots \delta_m^{r_m}(y) : \forall i (r_i \leqslant s_i)\}$.

By Levin (2001), Fornasiero–K. (2024), there is $\omega_{y|K}^{\mathrm{mv}}(T_1, \dots, T_m)$ with

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There is no natural way to extend this to definable sets. However the **dominant term** map $y \mapsto \mathfrak{d}(\omega_{y|K}^{\mathrm{mv}})$ extends to a well-defined map $X \mapsto \mathfrak{d}_X$ from $\mathcal{L}^\Delta(K)$-definable sets into the idempotent semiring of *dominant term polynomials*.

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Theorem (Fornasiero–K.–Terzo 2026⁺)

The map $X \mapsto \mathfrak{d}_X$ is a generalized definable dimension function: it is definable and additive in families, and thus invariant under definable bijections.

An example

Let $\Delta = \{\delta_1, \delta_2\}$ and let $C_i := \ker(\delta_i)$ for $i = 1, 2$. Our generalized definable dimension gives

$$\mathfrak{d}_K = T_1 T_2, \qquad \mathfrak{d}_{C_1} = T_2, \qquad \mathfrak{d}_{C_2} = T_1.$$

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T_1 and T_2 are incomparable in this semiring, so no $\mathcal{L}^\Delta$-definable map $f\colon C_1 \to C_2$ can be injective or surjective.

In conclusion

- Underpinning the last results is a "Δ-cell decomposition theorem," which should prove useful in upgrading a number of the single-derivation results.
- Better (geometric/relative) axioms would also help with this; see the axioms for $\mathrm{DCF}_{0,m}$ by León Sánchez (2012).
- There is still much to be done, especially around imaginaries.
- Most (not all!) of this works in the o-minimal differential setting. What other settings can we consider?

In conclusion

- Underpinning the last results is a “Δ-cell decomposition theorem,” which should prove useful in upgrading a number of the single-derivation results.
- Better (geometric/relative) axioms would also help with this; see the axioms for $\mathrm{DCF}_{0,m}$ by León Sánchez (2012).
- There is still much to be done, especially around imaginaries.
- Most (not all!) of this works in the o-minimal differential setting. What other settings can we consider?

Thank you!