VC-Density in Pairs of Ordered Vector Spaces

Ebru Nayir
(joint work with Ayhan Günaydın)

University of Leeds

March 18, 2026

Overview

1. Dense Pairs of Ordered Vector Spaces
    - Dense Pairs
    - QE in T^d_{ovs}
2. VC-Density
    - Basics on VC-Density
    - Main Result
    - Sharpness

Overview

1. Dense Pairs of Ordered Vector Spaces
    - Dense Pairs
    - QE in T^d_{ovs}
2. VC-Density

Dense Pairs

- Let T be a complete o-minimal theory in a language L extending the language $\{+, -, 0, 1, <\}$ of ordered abelian groups with a distinguished positive element 1.
- A **dense pair** of models of T is a structure $(\mathcal{M}, \ldots, N)$ in the language $L^d := L \cup \{U\}$ extending L by a new unary predicate U, which is interpreted as N, where $\mathcal{M}, \mathcal{N} \models T$, $\mathcal{N} \prec \mathcal{M}$ with $N \neq M$, and N is dense in M. We denote a dense pair simply by $(\mathcal{M}, \mathcal{N})$.
- Note that being a dense pair is expressible in L^d; T^d is the L^d-theory of dense pairs.

Dense Pairs

Theorem 1 (van den Dries, [1])

The theory T^d is complete.

Theorem 2 (van den Dries, [1])

Let $(\mathcal{M}, \mathcal{N})$ be a model of T^d. Every L^d-definable subset of M^n can be defined by a boolean combination of formulas of the form

$$\exists y_1 \cdots \exists y_m (U(y_1) \wedge \cdots \wedge U(y_m) \wedge \phi(x_1, \ldots, x_n, y_1, \ldots, y_m)),$$

where ϕ is an L-formula.

T^d_{ovs}

Given an ordered field K, pairs of ordered K-vector spaces fall under the setting of dense pairs of o-minimal structures in the language L^d_{ovs}, where $L_{\text{ovs}} := \{+, -, 0, 1, <\} \cup \{\ell_\lambda : \lambda \in K\}$ is the language of ordered K-vector spaces.

Let T_{ovs} denote the L_{ovs}-theory of ordered K-vector spaces, so T^d_{ovs} is the theory of dense pairs of ordered K-vector spaces in the language L^d_{ovs}. We denote models of T^d_{ovs} by (V, W).

QE in T^d_{ovs}

Recall that the definable closure, **dcl**, in an o-minimal structure gives rise to a pregeometry. We call the independence given by dcl, the **dcl-independence**.

Let $(\mathcal{M},\mathcal{N})$ be a substructure of a model $(\mathcal{M}',\mathcal{N}') \models T^d$. We say $\mathcal{M}$ and $\mathcal{N}'$ are **free over $\mathcal{N}$ in $\mathcal{M}'$** if every subset $X \subseteq M$ dcl-independent over $\mathcal{N}$ is dcl-independent over $\mathcal{N}'$.

Proposition 1 (Günaydın-Nayir [2])

Let $(\mathcal{V},\mathcal{W})$ be a substructure of a model $(\mathcal{V}',\mathcal{W}') \models T^d_{\text{ovs}}$. Then $\mathcal{V}$ and $\mathcal{W}'$ are free over $\mathcal{W}$ in $\mathcal{V}'$.

Theorem 3 (van den Dries, [1])

Let $(\mathcal{M},\mathcal{N})$ be a common substructure of $(\mathcal{M}',\mathcal{N}') \models T^d$ and $(\mathcal{M}'',\mathcal{N}'') \models T^d$ such that $\mathcal{M}$ and $\mathcal{N}'$ are free over $\mathcal{N}$, and $\mathcal{M}$ and $\mathcal{N}''$ are free over $\mathcal{N}$ in $\mathcal{M}''$. See the elementary inclusion diagram:

$$\begin{tikzcd}
\mathcal{M}' \arrow[dr, dash, "\succsim"] \arrow[dd, dash, "\curlyvee"'] & & \mathcal{M}'' \arrow[dl, dash, "\precsim"'] \arrow[dd, dash, "\curlyvee"] \\
& \mathcal{M} \arrow[dd, dash, "\curlyvee"] & \\
\mathcal{N}' \arrow[dr, dash, "\succsim"] & & \mathcal{N}'' \arrow[dl, dash, "\precsim"'] \\
& \mathcal{N} &
\end{tikzcd}$$

Then $(\mathcal{M}',\mathcal{N}')$ and $(\mathcal{M}'',\mathcal{N}'')$ are elementarily equivalent over M, that is, they satisfy the same $L^d(M)$-sentences.

QE in T^d_{ovs}

Theorem 4 (Günaydın-Nayir [2])

The theory T^d_{ovs} admits quantifier elimination.

Overview

VC-Dimension

Let L be a first order language, T a complete L-theory with no finite models and let $\mathcal{A} \models T$. Let $\varphi(x; y)$ be a partitioned L-formula with $|x| = m$ and $|y| = n$.

The **shatter function** $\pi_\varphi(k)$ of φ in T is the map $\pi_\varphi : \mathbb{N} \to \mathbb{N}$ given by

$$\pi_\varphi(k) := \max \left\{ |\mathcal{S}_\varphi^\mathcal{A} \cap C| : C \subseteq A^m, |C| = k \right\},$$

where $\mathcal{S}_\varphi^\mathcal{A} = \{\varphi(\mathcal{A}; b) : b \in A^n\}$ and $\mathcal{S}_\varphi^\mathcal{A} \cap C = \{S \cap C : S \in \mathcal{S}_\varphi^\mathcal{A}\}$.

VC-Dimension

The **shatter function $\boldsymbol{\pi_\varphi(k)}$ of** φ in $\mathcal{A} \models T$ is the map $\pi_\varphi : \mathbb{N} \to \mathbb{N}$ given by

$$\pi_\varphi(k) := \max \left\{ |\mathcal{S}_\varphi^{\mathcal{A}} \cap C| : C \subseteq A^m, |C| = k \right\},$$

where $\mathcal{S}_\varphi^{\mathcal{A}} = \{\varphi(\mathcal{A}; b) : b \in A^n\}$ and $\mathcal{S}_\varphi^{\mathcal{A}} \cap C = \{S \cap C : S \in \mathcal{S}_\varphi^{\mathcal{A}}\}$.

The **VC-dimension of** φ in $\mathcal{A} \models T$ is the integer d if $\pi_\varphi(d) = 2^d$ and for any $k > d$, $\pi_\varphi(k) \neq 2^k$. We denote it by $\mathrm{VC}(\varphi) = d$. If there is no such d, we put $\mathrm{VC}(\varphi) = \infty$.

VC-Dimension

The **shatter function $\boldsymbol{\pi_\varphi(k)}$ of** φ in $\mathcal{A} \models T$ is the map $\pi_\varphi : \mathbb{N} \to \mathbb{N}$ given by

$$\pi_\varphi(k) := \max \left\{ |\mathcal{S}_\varphi^{\mathcal{A}} \cap C| : C \subseteq A^m, |C| = k \right\},$$

where $\mathcal{S}_\varphi^{\mathcal{A}} = \{\varphi(\mathcal{A}; b) : b \in A^n\}$ and $\mathcal{S}_\varphi^{\mathcal{A}} \cap C = \{S \cap C : S \in \mathcal{S}_\varphi^{\mathcal{A}}\}$.

The **VC-dimension of** φ in $\mathcal{A} \models T$ is the integer d if $\pi_\varphi(d) = 2^d$ and for any $k > d$, $\pi_\varphi(k) \neq 2^k$. We denote it by $\mathrm{VC}(\varphi) = d$. If there is no such d, we put $\mathrm{VC}(\varphi) = \infty$.

Theorem 5 (Laskowski, [2])

A complete theory is NIP if and only if every partitioned formula has finite VC-dimension.

VC-Density

Lemma 1 (Sauer [4], Shelah [1])

If $\mathrm{VC}(\varphi) = d$, then $\pi_\varphi(k) \leq \binom{k}{0} + \ldots + \binom{k}{d}$ for any k.

VC-Density

Lemma 1 (Sauer [4], Shelah [1])

If $\mathrm{VC}(\varphi) = d$, then $\pi_\varphi(k) \leq \binom{k}{0} + \ldots + \binom{k}{d}$ for any k.

If $\mathrm{VC}(\varphi) = d$, then the **VC-density of φ** in $\mathcal{A} \models T$ is given by

$$\mathrm{vc}(\varphi) := \inf\{r \in \mathbb{R}_{>0} : \pi_\varphi(k) = O(k^r)\}.$$

Remark

By the Lemma above, if $\mathrm{VC}(\varphi) = d$, then $\mathrm{vc}(\varphi) \leq \mathrm{VC}(\varphi)$.

The **VC-density of T** is a map $\mathrm{vc}^T : \mathbb{N} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ given by

$$\mathrm{vc}^T(n) := \sup\{\mathrm{vc}(\varphi) : \varphi(x; y) \text{ is a partitioned } L\text{-formula}, |y| = n\}.$$

Example 1

Consider the following formula with a single object variable x and parameter variable y of length n:

$$\psi(x;y) : \bigvee_{i=1}^{n} x = y_i.$$

In $\mathcal{A} \models T$, we have $\mathrm{VC}(\psi) = vc(\psi) = n$.

Example 2- Spoiler Alert

Consider the following L^d_{ovs}-formula in a single object variable x and a single parameter variable y:

$$\phi(x;y) : y \leq x < y+1 \vee U(x-y).$$

Proposition 2 (Günaydın-Nayir [2])

In any model of T^d_{ovs}, the VC-density of $\phi(x;y)$ is at least 2.

Proof

Let $(V,W) \models T^d_{\text{ovs}}$ and $A = \{a_1, \ldots, a_k\} \subseteq V$, where $k \geq 2$,

- $a_i + 1 < a_{i+1}$ for each i, and
- $a_i - a_j \notin W$ for $i \neq j$.

Example 2- Spoiler Alert

Consider the following L^d_{ovs}-formula in a single object variable x and a single parameter variable y:

$$\phi(x; y) : y \leq x < y + 1 \vee U(x - y).$$

- Let $(V, W) \models T^d_{\text{ovs}}$ and $A = \{a_1, \ldots, a_k\} \subseteq V$, where $k \geq 2$,
  - $a_i + 1 < a_{i+1}$ for each i, and
  - $a_i - a_j \notin W$ for $i \neq j$.
- Clearly, $\phi(V, a_i) \cap A = \{a_i\}$ for each i.
- Using the density of W, for each $i < j$, we may find $b \in V$ such that

$$a_i \in [b, b + 1) \text{ and } a_j \in b + W.$$

Therefore we obtain $\phi(V, b) \cap A = \{a_i, a_j\}$ for each $i < j$.

Example 2- Spoiler Alert

Consider the following L^d_{ovs}-formula in a single object variable x and a single parameter variable:

$$\phi(x; y) : y \leq x < y + 1 \vee U(x - y).$$

Proposition 2 (Günaydın-Nayir [2])

In any model of T^d_{ovs}, the VC-density of $\phi(x; y)$ is at least 2.

- It follows then

$$\pi_\phi(k) \geq \binom{k}{0} + \binom{k}{1} + \binom{k}{2} = \frac{k^2 + k + 2}{2},$$

and therefore $vc(\phi) \geq 2$.

UDTFS (Guingona, [3])

Let $\Delta(x;y)$ be a finite set of partitioned L-formulas. We say Δ has **uniform definability of types over finite sets (UDTFS) with d parameters** in $\mathcal{A}$ if there are finitely many families

$$\mathcal{F}_i = (\varphi^i(y;y_1,\dots,y_d))_{\varphi\in\Delta}$$

of L-formulas with $i \in I$, $|y_j| = |y|$ for each $j = 1,\dots,d$ such that for every finite $B \subseteq A^{|y|}$ and $q \in S^{\Delta}(B)$ there are $b_1,\dots,b_d \in B$ and some $i \in I$ such that for every $b \in B$ and for every $\varphi \in \Delta$,

$$\varphi(x;b) \in q \iff \mathcal{A} \models \varphi^i(b,b_1,\dots,b_d).$$

VC d Property

We say that $\mathcal{A}$ has the **VC *d* property** if every finite $\Delta(x;y)$ with $|x|=1$ has UDTFS with d parameters. A theory T has the VC d property if every model of T has the VC d property.

Theorem 6 (Aschenbrenner et al.,[3])

If $T = Th(\mathcal{A})$ has the VC d property, then $n \leq \mathrm{vc}^T(n) \leq dn$ for any n.

Theorem 7 (Aschenbrenner et al.,[3])

Let T be weakly o-minimal, then T has the VC 1 property.

VC d Property

Theorem 8 (Aschenbrenner et al.,[3])

Let Φ be a set of partitioned formulas in a single object variable x such that

- every partitioned formula with object variable x is equivalent in T to a boolean combination of formulas from Φ, and
- every finite subset of Φ has UDTFS with d parameters.

Then T has the VC d property.

Breadth

A collection $\mathcal{S}$ of sets is said to have **breadth *d*** if for any nonempty finite intersection $\bigcap_{i \in I} B_i$ of sets in $\mathcal{S}$ with $|I| \geq d$, there is $J \subseteq I$ with $|J| = d$ such that

$$\bigcap_{i \in I} B_i = \bigcap_{j \in J} B_j.$$

If there is no such d, then we say $\mathcal{S}$ has breadth ∞.

Lemma 3 (Aschenbrenner et al.,[3])

Let $\Delta(x; y)$ be a finite set of partitioned L-formulas and assume that $\mathcal{S}_\Delta := \{\varphi(A^{|x|}, b) : b \in A^{|y|}, \varphi \in \Delta\}$ has breadth d. Then Δ has *UDTFS* with d parameters.

Main Result

Theorem 9 (Günaydın-Nayir [2])

The theory T^d_{ovs} has the VC 2 property.

Sketch of Proof

- Let $(V, W) \models T^d_{\text{ovs}}$. Fix a single variable x, and let y vary among tuples of variables of various lengths. For any $\lambda_0 \in K$ and $\lambda \in K^{|y|}$, let Φ be the collection of the following partitioned L^d_{ovs}-formulas in variables x and y:
  1. $x = \lambda_0 + \lambda y$,
  2. $x < \lambda_0 + \lambda y$,
  3. $U(x + \lambda_0 + \lambda y)$,

  where λ_0 denotes $l_{\lambda_0}(1)$, and λy denotes $l_{\lambda_1}(y_1) + \cdots + l_{\lambda_n}(y_n)$.

Main Result

Theorem 9 (Günaydın-Nayir [2])

The theory T^d_{ovs} has the VC 2 property.

Sketch of Proof

- Let $(V, W) \models T^d_{\mathrm{ovs}}$. Fix a single variable x, and let y vary among tuples of variables of various lengths. For any $\lambda_0 \in K$ and $\lambda \in K^{|y|}$, let Φ be the collection of the following partitioned L^d_{ovs}-formulas in variables x and y:
  1. $x = \lambda_0 + \lambda y$,
  2. $x < \lambda_0 + \lambda y$,
  3. $U(x + \lambda_0 + \lambda y)$,

  where λ_0 denotes $l_{\lambda_0}(1)$, and λy denotes $l_{\lambda_1}(y_1) + \cdots + l_{\lambda_n}(y_n)$.
- Let $\Delta(x; y)$ be a finite subset of Φ and consider

$$\mathcal{S}_\Delta := \{\varphi(V; b) : b \in V^n, \varphi \in \Delta\}.$$

Sketch of Proof

- Let Φ be the collection of the following partitioned L^d_{ovs}-formulas in variables x and y:
    1. $x = \lambda_0 + \lambda y$,
    2. $x < \lambda_0 + \lambda y$,
    3. $U(x + \lambda_0 + \lambda y)$.
- Let $\Delta(x; y)$ be a finite subset of Φ and consider $\mathcal{S}_\Delta$.
- Let $A_1, \ldots, A_m \in \mathcal{S}_\Delta$ be such that $\bigcap_{i=1}^m A_i \neq \emptyset$ with $m \geq 2$.
- If there are A_i defined by formulas of the form (j), their intersection is given by only one of them, say A_{i_j}, where $j = 1, 2, 3$ and $i_j \in \{1, \ldots, m\}$.
- Hence we have either $\bigcap_{i=1}^m A_i = A_{i_j}$ for some $j = 1, 2, 3$, or

$$\bigcap_{i=1}^{m} A_i = A_{i_2} \cap A_{i_3}.$$

Sharpness

Corollary 1 (Günaydın, Nayir [2])

In any model of T^d_{ovs}, every partitioned L^d_{ovs}-formula $\varphi(x; y)$ satisfies

$$\text{vc}(\varphi) \leq 2|y|.$$

Therefore the L^d_{ovs}-formula

$$\phi(x; y) : y \leq x < y + 1 \vee U(x - y)$$

has the VC-density $2 = 2|y|$, and so the VC-density bound in Corollary 1 is optimal.

References

Lou van den Dries (1998)
Dense pairs of o-minimal Structures
Fundamenta Mathematicae.

Ayhan Günaydın and Ebru Nayir (2024)
VC-Density in Pairs of Ordered Vector Spaces.
arXiv:2405.10069

Matthias Aschenbrenner, Alf Dolich, Deidre Haskell, Dugald Macherson, and Sergei Starchenko (2016)
Vapnik-Chervonenkis Density in Some Theories Without the Independence Property, I
Transactions of the American Mathematical Society.

N. Sauer (1972)
On the density of families of sets
Journal of Combinatorial Theory. Series A.

References

S. Shelah (1972)
A combinatorial problem; stability and order for models and theories in infinitary languages
Pacific Journal of Mathematics.

Michael C. Laskowski (1992)
Vapnik-Chervonenkis Classes of Definable Sets
Journal of the London Mathematical Society.

Vincent Guingona (2012)
On uniform definability of types over finite sets
Journal of Symbolic Logic.