Distality in Oriented Abelian Groups and their Pairs

Melissa Özsahakyan

Workshop on Tame Geometry and Combinatorics

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Outline

Oriented Abelian Groups

- An *orientation* on an abelian group A is a ternary relation $\mathcal{O}$ on A such that
  1. $\{(x, y) \in A^2, x \neq y : \mathcal{O}(0, x, y)\}$ is a strict linear order on the set $A \setminus \{0\}$.
  2. $\mathcal{O}(a, b, c) \Leftrightarrow \mathcal{O}(b, c, a) \Leftrightarrow \mathcal{O}(c, a, b)$.
  3. $\mathcal{O}(a, b, c) \Rightarrow \mathcal{O}(a + d, b + d, c + d)$
  4. $\mathcal{O}(a, b, c) \Leftrightarrow \mathcal{O}(-c, -b, -a)$

  for every $a, b, c, d \in A$.

  An abelian group equipped with an orientation is called *oriented abelian group*.
- Oriented abelian groups are topological groups and their basic open sets are of the form $(a, b)_{\mathcal{O}} := \{c \in A : \mathcal{O}(a, c, b)\}$ for $a, b \in A$.
- The unit circle $\mathbb{S}$ is an oriented abelian group.

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Theory of Regularly Dense Oriented Abelian Groups

Let $L_{\mathcal{O}} = \{+, -, 0, \mathcal{O}\}$.

Lemma/Definition

For an oriented abelian group A, if the chain of successive torsion subgroups

$$\mathrm{Tor}_{p^0}(A) \leq \mathrm{Tor}_{p^1}(A) \leq \ldots \leq \mathrm{Tor}_{p^e}(A) \leq \ldots \leq \mathrm{Tor}_{p^{e+k}}(A) \leq \ldots,$$

terminates, then there is some $e \in \mathbb{N}$ such that $\mathrm{Tor}_{p^{e+k}}(A) = \mathrm{Tor}_{p^e}(A)$ for all $k \in \mathbb{N}$. In this case, we put $e(A, p) = e$. Otherwise, we put $e(A, p) = \infty$.

Lemma (A.Günaydın)

Let A and B be regularly dense oriented abelian groups. Then $A \equiv B$ iff for every prime p, we have $|A/pA| = |B/pB|$ and $e(A, p) = e(B, p)$.

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Definition

For $i \in \mathbb{N}^{>0}$, let p_i denote the i-th prime number. Let $\vec{d} = (d_i)_i$ and $\vec{e} = (e_i)_i$, where $d_i, e_i \in \mathbb{N} \cup \{\infty\}$.
We define $T(\vec{d}, \vec{e})$ to be the $L_{\mathcal{O}}$-theory whose models are structures $(A, +, -, 0, \mathcal{O})$ such that:

- A is a regularly dense oriented abelian group;
- for every prime p_i, we have

$$|A/p_iA| = p_i^{d_i} \quad \text{and} \quad e(A, p_i) = e_i.$$

Corollary

$T(\vec{d}, \vec{e})$ is complete.

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Let $T'(\vec{1}, \vec{0})$ be the definitional extension of $T(\vec{1}, \vec{0})$ to an $L'_{\mathcal{O}} := L_{\mathcal{O}} \cup \{D_{m,n}\}_{m,n \in \mathbb{Z}}$ theory with defining axioms

$$D_{m,n}(x, y) \leftrightarrow \exists s \exists t (x = ms \wedge y = nt \wedge \mathcal{O}(0, s, t))$$

for each $m, n \in \mathbb{Z}$.

Lemma

The theory $T'(\vec{1}, \vec{0})$ has quantifier elimination in the language $L'_{\mathcal{O}}$.

Remark

By Tran and Walsberg, $T(\vec{1}, \vec{0})$ is dp-minimal, hence has NIP.

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VC Calculations and Results in Oriented Abelian Groups

Lemma

The theories $T(\vec{0}, \vec{\infty})$ and $T(\vec{0}, \vec{0})$ admit quantifier elimination.

Theorem (joint work with Ebru Nayir)

The theories $T(\vec{0}, \vec{\infty})$ and $T(\vec{0}, \vec{0})$ have the VC 1 property.

Corollary

In any model of these theories, any partitioned $L_{\mathcal{O}}$-formula has VC-density at most $|y|$.

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Pairs of Oriented Abelian Groups

- $L_{\mathcal{O}}(V) = L_{\mathcal{O}} \cup \{V\}$
- $(A, G) \models T((\vec{d_1}, \vec{e_1}), (\vec{d_2}, \vec{e_2}))$: $A \models T(\vec{d_1}, \vec{e_1})$, $G \models T(\vec{d_2}, \vec{e_2})$ where G is a dense subgroup of A and A/G is infinite.

Lemma

Theories $T((\vec{0}, \vec{\infty}), (\vec{0}, \vec{e}))$ where $\vec{e}$ is either $\vec{\infty}$ or $\vec{0}$ and $T((\vec{0}, \vec{0}), (\vec{0}, \vec{0}))$ admit quantifier elimination.

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Corollary

In any models of the above theories, any partitioned $L^{p}_{\mathcal{O}}$*-formula in parameter variables* y *has the VC-density at most* $2|y|$.

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The theories $T\big((\vec{0},\vec{\infty}),(\vec{0},\vec{\infty})\big)$, $T\big((\vec{0},\vec{\infty}),(\vec{0},\vec{0})\big)$ *and* $T\big((\vec{0},\vec{0}),(\vec{0},\vec{0})\big)$ *are not dp-minimal.*

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Distality

Definition

An L-theory T is distal if for every small parameter set $B \subseteq \mathbb{M}$, every indiscernible sequence $a = (a_i)_{i \in I}$ and every $i \in I$, the following holds: if

1. $I_{<i} := \{j \in I : j < i\}$ and $I_{>i} := \{j \in I : i < j\}$ are infinite,
2. $a_{I \setminus \{i\}}$ is B-indiscernible,

then a is B-indiscernible. We say that an L-structure is distal if its theory is distal.

Distality in Divisible Oriented Abelian Groups

Theorem (joint work with Charlotte Kestner)

The theories $T(\vec{0}, \vec{\infty})$ and $T(\vec{0}, \vec{0})$ are distal.

Fact (P.Simon)

If T is dp-minimal and every non-constant indiscernible sequence of singletons is not totally indiscernible, then T is distal.

Proof of Theorem

By QE, any formula $\varphi(x; y)$ is equivalent to Boolean combinations of atomic formulas $\mathcal{O}(0, x, ky)$ and $x = ky$.
Let $(a_i)_{i \in I}$ be a non-constant indiscernible sequence. For given $i_0 < j_0$ we may assume that $\mathcal{O}(0, a_{i_0}, a_{j_0})$. Then by indiscernibility, we should have $\mathcal{O}(0, a_i, a_j)$ for every $i < j$.

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Proof Continued

Consider the formula $\varphi(x; y) := \mathcal{O}(0, x, y)$. Then

$$\varphi(a_i, a_j) \iff \mathcal{O}(0, a_i, a_j) \iff i < j.$$

Hence φ orders the sequence I, which means $(a_i)_{i \in I}$ is not totally indiscernible.
Therefore the theories $T(\vec{0}, \vec{\infty})$ and $T(\vec{0}, \vec{0})$ are distal.

Pairs are not Distal

Theorem (joint work with Charlotte Kestner)

The theories $T\left((\vec{d_1}, \vec{e_1}), (\vec{d_2}, \vec{e_2})\right)$ are not distal.

Fact (Aschenbrenner et. al.)

The following are equivalent:

1. *T is not distal.*
2. *There is an indiscernible sequence $a = (a_i)_{i \in \mathbb{Q}}$ in $\mathbb{M}_x$ and some $b \in \mathbb{M}_y$ such that $a_{\mathbb{Q} \setminus \{0\}}$ is b-indiscernible, and some partitioned L-formula $\varphi(x; \vec{y})$ such that*

$$\models \varphi(a_i; \vec{b}) \iff i \neq 0.$$

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$$\models \varphi(a_i; \vec{b}) \iff i \neq 0.$$

Proof.

Since the proof is the same for all the theories stated above, wlog we may assume that $(A, G) \models T\big((\vec{0}, \vec{\infty}), (\vec{0}, \vec{0})\big)$. Let $(a_i)_{i \in \mathbb{Q}}$ be an indiscernible sequence in $A \setminus G$ such that $a_i - a_j \notin G$ for some $i \neq j$. By indiscernibility we obtain $a_i - a_j \notin G$ for all $i \neq j$.
Let $\varphi(x; y) := \neg V(x - y)$ and $b = a_0$. Then

$$\varphi(a_i; b) \iff a_i - a_0 \notin G \iff i \neq 0.$$

□

The Next Question

Definition (R.Walker)

We say that T is m-distal if the following holds:
Whenever for every subset $B \subseteq A$ with $|B| = m$, the sequence obtained by inserting the elements of B into their respective cuts in I is indiscernible, then the sequence obtained by inserting all elements of A simultaneously

$$I_0 + a_0 + I_1 + a_1 + \ldots + I_{n-1} + a_{n-1} + I_n$$

is also indiscernible. Distality rank of a theory T is the smallest integer m such that T is m-distal.

Question

What are the distality ranks of the theories mentioned above?

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What are the distality ranks of the theories mentioned above?

Pierre Simon.
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