

Strict C^p -triangulations - a new approach to desingularisation

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To start

Assume given an arbitrary o-minimal expansion of some real closed field R . For example, you can think of semialgebraic sets. We deal with subsets of spaces R^n ($n \in \mathbb{N}$) and mappings $f : A \rightarrow R^m$, where $A \subset R^n$, which are *definable with parameters* in this structure.

If \mathcal{K} is any family of subsets of a set X , then by a *refinement* of \mathcal{K} we understand any family \mathcal{L} of subsets of X such that each $L \in \mathcal{L}$ is contained in some $K \in \mathcal{K}$ and each $K \in \mathcal{K}$ is the union $\bigcup \mathcal{L}'$ of some subfamily $\mathcal{L}' \subset \mathcal{L}$.

Simplexes

We adopt a standard definition of a *simplex of dimension k* in R^n as the convex hull of $k + 1$ points a_0, \dots, a_k affinely independent in R^n ; i.e.

$$\Delta = [a_0, \dots, a_k] := \left\{ \sum_{i=0}^k \alpha_i a_i : \alpha_i \geq 0 \ (i \in \{0, \dots, k\}), \sum_{i=0}^k \alpha_i = 1 \right\}.$$

If $0 \leq i_0 < i_1 < \dots < i_l \leq k$, then the simplex $[a_{i_0}, \dots, a_{i_l}]$ is called a *face* of Δ of dimension l . The points a_0, \dots, a_k are called *vertices* of Δ . The *boundary* $\partial\Delta$ of a simplex Δ of dimension k is the union of all faces of Δ of dimension $< k$. Its *relative interior* is by definition

$$\begin{aligned} \overset{\circ}{\Delta} &:= \Delta \setminus \partial\Delta = (a_0, \dots, a_k) := \\ &\left\{ \sum_{i=0}^k \alpha_i a_i : \alpha_i > 0 \ (i \in \{0, \dots, k\}), \sum_{i=0}^k \alpha_i = 1 \right\}. \end{aligned}$$

By a (*finite*) *simplicial complex* in R^n we understand a finite family \mathcal{K} of simplexes in R^n such that for each $K \in \mathcal{K}$ all faces of K belong to \mathcal{K} and for each pair $K_1, K_2 \in \mathcal{K}$, $K_1 \cap K_2$ is empty or is a common face of both K_1 and K_2 . Then the set $|\mathcal{K}| := \bigcup \mathcal{K}$, called the *polyhedron of \mathcal{K}* , is of course a definable subset of R^n . If \mathcal{K} and \mathcal{L} are two simplicial complexes in R^n , we say that \mathcal{L} is a *subcomplex* of \mathcal{K} if simply $\mathcal{L} \subset \mathcal{K}$.

Let p be a positive integer and let $f : A \rightarrow R^m$ be a definable mapping defined on any definable subset $A \subset R^n$. We say that f is of class C^p or C^p -mapping if it admits an extension $\tilde{f} : U \rightarrow R^m$ of class C^p defined on an open definable neighborhood U of A in R^n . If A is a closed and bounded subset of R^n , then this definition is equivalent to existence of an extension $\tilde{f} : R^n \rightarrow R^m$ of class C^p (by definable C^p -partition of unity).

\mathcal{C}^p -triangulation and strict \mathcal{C}^p -triangulation

Let A be a definable closed bounded subset of \mathbb{R}^n . By a \mathcal{C}^p -triangulation of A we understand a pair (\mathcal{T}, h) , where \mathcal{T} is a finite simplicial complex in \mathbb{R}^n and h is a definable homeomorphism of $|\mathcal{T}|$ onto A such that for each simplex $\Delta \in \mathcal{T}$ the restriction $h|_{\overset{\circ}{\Delta}}$ is a \mathcal{C}^p -embedding of $\overset{\circ}{\Delta}$ into \mathbb{R}^n . If \mathcal{E} is a finite family of definable subsets of A we say that a triangulation (\mathcal{T}, h) is compatible with \mathcal{E} if for each $E \in \mathcal{E}$ the inverse image $h^{-1}(E)$ is a union of a family of some $\overset{\circ}{\Delta}$, where $\Delta \in \mathcal{T}$. It is a classical theorem that such a \mathcal{C}^p -triangulation always exists (Whitehead, Łojasiewicz, Giesecke, Hardt, Hironaka, van den Dries).

A \mathcal{C}^p -triangulation (\mathcal{T}, h) of A will be called a *strict \mathcal{C}^p -triangulation* of A if the mapping $h : |\mathcal{T}| \rightarrow \mathbb{R}^n$ is of class \mathcal{C}^p .

Theorem 1.

Let R be any real closed field expanded by some o-minimal structure. Let $f : A \rightarrow R^m$ be a definable continuous mapping defined on a definable, closed, bounded subset A of R^n . Let \mathcal{E} be a finite family of definable subsets of A .

Then, for any positive integer p , there exists a strict C^p -triangulation (\mathcal{T}, h) of A compatible with the family \mathcal{E} and such that $f \circ h$ is of class C^p .

Theorem 2.

Under the assumptions of Theorem 1, if additionally \mathcal{K} is any finite simplicial complex in R^n and $g : |\mathcal{K}| \rightarrow A$ is any definable homeomorphism, then there exists a strict C^p -triangulation (\mathcal{T}, h) of $|\mathcal{K}|$ such that \mathcal{T} is a refinement of \mathcal{K} , $h(\Lambda) = \Lambda$, for each $\Lambda \in \mathcal{K}$, $(\mathcal{T}, g \circ h)$ is a strict C^p -triangulation of A compatible with the family \mathcal{E} and such that $f \circ g \circ h$ is of class C^p .

By induction on the dimension n of the source space.

Case $n = 1$.

In this case we have given a definable continuous mapping $f : [a, b] \rightarrow R^d$. Assume $d = 1$ (for $d > 1$ reasoning is similar).

The first step is to find a definable strictly increasing homeomorphism $h : [c, d] \rightarrow [a, b]$ such that $|h'|$ and $|(f \circ h)'|$ (at points where they exist) are bounded by 1. This is achieved by a simple but beautiful **trick of Coste-Reguiat**. By dividing into smaller intervals the problem reduces to the case when f is C^1 on (a, b) and we have the following three subcases $|f'| \leq 1$ or $f' > 1$ or $f' < -1$ on (a, b) . In the first subcase we take $h = \text{id}_{[a,b]}$, in the second one $h = f^{-1} : [f(a), f(b)] \rightarrow [a, b]$ and in the third one $h(t) = f^{-1}(-t) : [-f(a), -f(b)] \rightarrow [a, b]$

The second step relies on another **trick by Yomdin-Gromov** which can be stated as the following lemma

Lemma.

If $\varphi : (0, 1] \rightarrow R$ is a definable function of class C^{p+1} such that $|\varphi'|$ is bounded and each of the derivatives $\varphi^{(\nu)}$ ($\nu \in \{2, \dots, p+1\}$) is either ≥ 0 or ≤ 0 , then the function $(0, 1] \ni t \mapsto \varphi(t^{p+1})$ extends to a C^p -function on $[0, 1]$ flat at 0.

Proof of lemma is based on the following elementary inequality

$$|\varphi^{(\nu)}(t)| \leq 2^{\binom{p+1}{2}-2} \sup |\varphi'| \frac{1}{t^{\nu-1}}, \quad \text{for } \nu \in \{2, \dots, p\} \text{ and } t \in (0, 1/2].$$

For $n > 1$ we use the same smoothing procedure but with parameters. To do this we need two auxiliary notions: *capsules* which are cells without vertical segments in the boundary and *detectors* which are special C^p -functions of choice. First we reduce to the case $f : K \rightarrow R$ where K is a capsule in R^n which can be considered as a family parameterized by an open subset D of R^{n-1} of vertical line segments parallel to x_n -axis shrinking to points over the boundary of D . To these line segments we apply the above smoothing procedure. As a result the partial derivative $\partial^p f / \partial x_n^p$ extends continuously by 0 to the boundary of the capsule and passing to some subdivision of K we can assume that f is C^p on $K \setminus (\partial D) \times R$. It is important that we obtain this substituting in f a homeomorphism which is of the form $\Phi(x', x_n) = (h(x'), \varphi(x', x_n))$, where $x' = (x_1, \dots, x_{n-1})$. Now we have to control other partial derivatives of order p . First consider $\partial^p f / \partial x_{n-1} \partial x_n^{p-1}$. We find a function $\omega : D \rightarrow R$ of class C^p which detects in every vertical fiber a point at which the maximum of $|\partial^p f / \partial x_{n-1} \partial x_n^{p-1}|$ over the fiber is attained (up to a factor 1/2).

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Now we apply the previous step to the function

$$\frac{\partial^{p-1}f}{\partial x_n^{p-1}}(x', \omega(x')) = \frac{\partial^{p-1}f}{\partial x_n^{p-1}}(x'', x_{n-1}, \omega(x'', x_{n-1})),$$

where $x'' = (x_1, \dots, x_{n-2})$ and where now x_{n-1} plays the role of a "vertical variable". Hence, there exists a homeomorphism of the form

$$\Psi(x'', x_{n-1}) = (g(x''), \psi(x'', x_{n-1}))$$

such that

$$\begin{aligned} & \frac{\partial}{\partial x_{n-1}} \left[\frac{\partial^{p-1}f}{\partial x_n^{p-1}}(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))) \right] = \\ & \frac{\partial^p f}{\partial x_{n-1} \partial x_n^{p-1}}(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))) \frac{\partial \psi}{\partial x_{n-1}} + \\ & \frac{\partial^p f}{\partial x_n^p}(g(x''), \psi(x'', x_{n-1}), \omega(g(x''), \psi(x'', x_{n-1}))) \frac{\partial}{\partial x_{n-1}} \omega(g(x''), \psi(x'', x_{n-1})). \end{aligned}$$

extends continuously by zero to the boundary of D .

(1) (T. Ohmoto and M. Shiota) Integration on definable subsets (here \mathcal{C}^1 -case sufficient).

Let A be a compact definable subset of \mathbb{R}^n of pure dimension d , whose \mathcal{C}^1 -regular part M is an oriented \mathcal{C}^1 -manifold of dimension d and let ω be a continuous differential d -form defined in a neighborhood of A , then we put

$$\int_A \omega := \sum_{\Delta \in \mathcal{T}} \int_{\Delta} f^* \omega,$$

where $f : |\mathcal{T}| \rightarrow A$ is a strict \mathcal{C}^1 -triangulation compatible with M .

In this way integration on definable sets is made very similar to integration on smooth manifolds, which allows quite easily to prove the Stokes formula and de Rham theorem.

This approach is used by A. Huber-Klawitter to study so-called *period numbers*.

(2) (J. Fernando, R. Ghiloni) Application to approximation theory

Theorem.

Let A and B be any definable closed bounded subsets of R^n and R^m respectively. Let $f : A \rightarrow B$ be a definable continuous mapping. Then, for every $\varepsilon > 0$ and every $p \in \mathbb{N}^$, there exists a C^p -mapping $g : A \rightarrow B$ such that*

$$|f(x) - g(x)| \leq \varepsilon, \quad \text{for each } x \in A.$$

Proof. By strict C^p -triangulation $h : |\mathcal{T}| \rightarrow B$ the proof reduces to the case where B is a polyhedron in R^m (in this case the theorem was proved by Fernando and Ghiloni).

Remark A. Carbone strengthened the last theorem proving that if $B = \text{Im}(f)$, then we can have $B = \text{Im}(g)$.

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