Dense pairs of rings.

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Dense pairs

A. Tarski asked whether the pair $(\mathbb{R}, \mathbb{Q}^r)$, where $\mathbb{Q}^r$ denotes the real-closure of $\mathbb{Q}$ in $\mathbb{R}$, is decidable.

This was answered positively by A. Robinson (1959) by axiomatizing the theory of dense pairs of real closed fields and showing it is complete. He also described a language in which this theory of pairs is model-complete.

Later, C. Toffalori (1978) proved a model-completeness result for *sheaves of pairs of real-closed fields.*

In this talk, we want to describe complete, respectively model-complete theories of rings whose corresponding theory of pairs is complete, respectively model-complete, using Feferman-Vaught techniques. We also observe that the framework of differential rings helps, as in the case of certain dense pairs of fields. Then we also consider two other properties: the open core property and elimination of imaginaries.

A first step: Dense pairs of models of a geometric theory

Let T be a complete, geometric $\mathcal{L}$-theory of integral domains, $\mathcal{L}$ contains the ring language $\mathcal{L}_{rings} := \{+, -, \cdot, 0, 1\}$.

T geometric means, in this setting, that acl satisfies the exchange. $\rightsquigarrow$ a useful notion of dimension $\dim$ on definable sets.

To the language $\mathcal{L}$ one adds a unary predicate P; let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$.
[Fornasiero, 2011] Let $M \models T$ and let $X \subset M$, X definable, then X is dense if for every definable $U \subset M$, $\dim(U) = 1$, $X \cap U \neq \emptyset$.

Let T_P be the $\mathcal{L}_P$-theory $Th(K, P(K))$ where $K \models T$, $P(K)$ is acl-closed and dense (in K).

[Fornasiero, 8.3, 8.5, 2011] Let $(K, P(K)) \models T_P$. Then $P(K) \models T$, $P(K) \preceq K$ and the theory T_P is complete.
Moreover any $\mathcal{L}_P$-formula in T_P is equivalent to a boolean combination of formulas of the form $\exists \bar{y}\, (P(\bar{y}) \wedge \psi(\bar{x}, \bar{y}))$, where ψ is an $\mathcal{L}$-formula.

Von Neumann regular rings

[Lipchitz, Saracino, 1973] The model-companion of the theory of commutative rings without nilpotent elements is the model-completion of the theory of commutative von Neumann regular rings.

[Macintyre, Weispfenning, 1973/74] The model-companion of the theory of commutative lattice ordered f-rings without nilpotent elements is the model-completion of the theory of commutative von Neumann regular lattice ordered f-rings.

A von Neumann (commutative) regular ring R is a commutative ring $(R, +, -, \cdot, 0, 1)$ satisfying $\forall x \exists y \; (x^2 y = x \;\&\; y^2 x = y)$.

Denote by $B(R)$ the boolean subring consisting of the idempotents of R. Let X be the Stone space of $B(R)$, namely the space of ultrafilters on $B(R)$. It is a Hausdorff space with a basis of clopen (=open and closed) subsets. (The space X of ultrafilters of $B(R)$ is homeomorphic to the space of prime ideals of R.)

Von Neumann regular rings

Note that a commutative ring without nilpotent elements is a subdirect product (indexed by its prime ideals) of domains and a regular ring is isomorphic to a boolean product of fields.

So, Feferman-Vaught techniques apply in order to transfer results for existentially closed fields to the class of *existentially closed* regular rings with atomless boolean algebra of idempotents.

Subdirect products

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures.

We consider subdirect products $\mathcal{A} := \prod^s_{x \in X} \mathcal{A}_x$ of elements $\mathcal{A}_x \in \mathcal{C}$ over some index set X, namely $\mathcal{L}$-substructures of direct products of elements of $\mathcal{C}$ with the additional property that for any $x \in X$, and any $a_x \in A_x$, there is $a := (a(y))_{y \in X} \in \mathcal{A}$ such that $a(x) = a_x$.

Notation: Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures and let $\mathcal{A} := \prod^s_{x \in X} \mathcal{A}_x$ be a subdirect product of elements $\mathcal{A}_x \in \mathcal{C}$ over some index set X. Let $\varphi(x_1, \ldots, x_n)$ be an $\mathcal{L}$-formula and let $\bar{f} := (f_1, \ldots, f_n) \in \prod^s_{x \in X} \mathcal{A}_x$. Then $[\varphi(\bar{f})] := \{x \in X \colon \mathcal{A}_x \models \varphi(f_1(x), \ldots, f_n(x))\}$.

Second step: Boolean products [Burris-Werner, 1978]

Fix $\tilde{T}$ a theory of Boolean algebras. Let $B \models \tilde{T}$, then letting X be the space of ultrafilters of B, using Stone duality, we have that the set of continuous functions from $X \to \{0,1\}$ is a boolean ring X^* isomorphic to B.

Let $\mathcal{C}$ be a class of $\mathcal{L}$-structures.

• The class $\Gamma^a_{\tilde{T}}(\mathcal{C})$ of $\mathcal{L}$-structures consists of all subdirect products $\mathcal{A} = \prod^s_{x \in X} \mathcal{A}_x$ of elements of $\mathcal{C}$, with X a boolean space and $X^* \models \tilde{T}$, which in addition satisfy the following:

atomic extension property: if φ is an atomic $\mathcal{L}$-formula, for any tuple $\bar{f} \in A$, then $[\varphi(\bar{f})]$ is a clopen subset of X,

patchwork property: for any $f, g \in A$ and any clopen subset U of X, there is $h \in A$ such that $U \subseteq [f = h]$ and $X \setminus U \subseteq [g = h]$.

Second step: Boolean products [Burris-Werner, 1978]

• The subclass $\Gamma^e_{\tilde{T}}(\mathcal{C})$ consists of those elements of $\Gamma^a_{\tilde{T}}(\mathcal{C})$ which satisfy in addition:
elementary extension property: given φ an $\mathcal{L}$-formula and a tuple $\bar{f} \in A$, then $[\varphi(\bar{f})]$ is a clopen subset of X.

For $\mathcal{A} \in \Gamma^a_{\tilde{T}}(\mathcal{C})$, we will denote by $X(A)$ the underlying boolean space (or by X if this is no ambiguity) and sometimes we will omit the subscript $\tilde{T}$ (when it is clear).

Feferman-Vaught

Let $\varphi(\bar{u})$ be an $\mathcal{L}$-formula with a determining sequence

$$(\Phi^*(z_1, \ldots, z_\ell), \psi_1(\bar{u}), \ldots, \psi_\ell(\bar{u})),$$

where Φ^* is a boolean algebra formula and ψ_i, $1 \leq i \leq \ell$ are $\mathcal{L}$-formulas.

Then for $\mathcal{A} \in \Gamma^e(\mathcal{C})$ and $\bar{f} \in A$, we have

$$\mathcal{A} \models \varphi(\bar{f}) \leftrightarrow X(A)^* \models \Phi^*([\psi_1(\bar{f})], \ldots, [\psi_\ell(\bar{f})]).$$

The determining sequence is constructed by induction on the complexity of φ: one puts φ in prenex normal form and one describes the procedure first for atomic formulas, then how to proceed with negations and conjunctions and finally with formulas with one existential quantifier.

Some technicalities on the language

Denote by $\mathcal{C}^p$ the class of expansions of elements of $\mathcal{C}$ in the language $\mathcal{L}_p := \mathcal{L} \cup \{p(.,.)\}$, where $p(u, v)$ is defined by $p(a, b) = a$ if $b = 0$ and $p(a, b) = 0$ otherwise.

Note that in the class of von Neumann regular rings, the expansion $\mathcal{L}_p$ is an expansion by definitions of $\mathcal{L}$.
Indeed in any boolean product of integral domains, we can define $p(a, b)$ as follows:

$$p(a, b) = c \leftrightarrow \exists d \; (b\, d\, b = b \wedge b\, c = 0 \wedge (c - a)\,(1 - b\, d) = 0). \quad (1)$$

(One expresses that the supports of b and c are disjoint and on the complement of the support of b, c is equal to a.) Moreover the defining formula is a (positive primitive) existential $\mathcal{L}$-formula.

Some technicalities on the language

On each n-ary relation $r(\bar{x})$, one requires that there is a positive existential $\mathcal{L}$-formula $\varphi_r(\bar{x})$ such that

$$(\dagger) \quad T \models \forall \bar{x}(\neg r(\bar{x}) \leftrightarrow \varphi_r(\bar{x})).$$

Lemma

In $\mathcal{C}$ we have:
$\mathcal{C}^p \models (u = 0 \vee v = 0) \leftrightarrow p(u,v) = u,$
$\mathcal{C}^p \models (u = 0 \wedge v = 0) \leftrightarrow p(u,v) + v = 0,$
$\mathcal{C}^p \models (u = 0 \vee v \neq 0) \leftrightarrow p(u,v) = 0.$

In particular, in $\mathcal{C}^p$, any open $\mathcal{L}$-formula not containing relation symbols, is equivalent either to an atomic $\mathcal{L}_p$-formula or the negation of an atomic $\mathcal{L}_p$-formula. Further if $\mathcal{C}$ is a class of rings, $u \neq 0$ iff $p(1,u) = 0$.

Transfer results-EC

Given a class $\mathcal{C}$ of $\mathcal{L}$structures, we denote by $SP(\mathcal{C})$ the class obtained from $\mathcal{C}$ by closing under direct products and substructures.

[Burris-Werner, 1979]

• Let $\mathcal{C}$ be an elementary class with a complete theory. If $\tilde{T}$ is a complete theory of boolean algebras, then $\Gamma^e_{\tilde{T}}(\mathcal{C})$ has a complete theory.

• Let $\mathcal{C}$ be an elementary class with a model-complete, complete theory of integral domains. Then the existentially closed models of $Th(SP(\mathcal{C}))$ form the elementary class $\Gamma^e_{T_a}(\mathcal{C})$ (with an explicit axiomatisation, obtained from an axiomatisation of $\mathcal{C}$).

Transfert results-EI

Starting with (complete) theories of boolean algebras, R. Wencel obtained the following results (and in fact a characterization).

[Wencel, 2005] Let $\tilde{T}$ be respectively the theory of atomic boolean algebras and the theory of atomless boolean algebras (or the theory of boolean algebras with finitely many atoms).
Then $\tilde{T}$ admits weak elimination of imaginaries.

(For atomless boolean algebras, one can use the strong small index property of the countable atomless boolean algebra proven by J. Truss and $\aleph_0$-categoricity).

Theorem (Derakshan-Hrushovski, 2023)

Let $\tilde{T}$ be respectively the theory of atomic boolean algebras and the theory of atomless boolean algebras. Suppose T_0 admits elimination of imaginaries, then the theory $\Gamma^e_{\tilde{T}}(T_0)$ admits weak elimination of imaginaries.

Boolean products of dense pairs

Suppose $\mathcal{L}$ contains at least one constant symbol.
Let $\mathcal{C}_P$ be the class of pairs $(\mathcal{A}, \mathcal{D})$ of elements of $\mathcal{C}$ with $\mathcal{D}$ an $\mathcal{L}$-substructure of $\mathcal{A}$. We view the elements of $\mathcal{C}_P$ as the expansions of elements of $\mathcal{C}$ in $\mathcal{L}_P$ with the predicate P interpreted by a proper $\mathcal{L}$-substructure.

Let $\mathcal{A} \in \Gamma^a(\mathcal{C}_P)$ with $\mathcal{A} = \prod^s_{x \in X} \mathcal{A}_x$ and $\mathcal{A}_x \in \mathcal{C}_P$. Define $D := \{(a_x) \in \prod^s_{x \in X} \mathcal{A}_x \colon \mathcal{A}_x \models P(a_x)\}$.

Lemma

Then, $\mathcal{D} \in \Gamma^a(\mathcal{C})$ (with $X(\mathcal{D}) = X$) and whenever $\mathcal{A} \in \Gamma^e(\mathcal{C}_P)$, $\mathcal{D} \in \Gamma^e(\mathcal{C})$.

Boolean products of dense pairs

Now assume that $\mathcal{C}$ is the class of models of a complete geometric theory T (extending the theory of integral domains). Recall that T_P be the $\mathcal{L}_P$-theory $Th(K, P(K))$ where $K \in \mathcal{C}$, $P(K)$ is acl-closed and dense (in K). Denote by $\mathcal{C}_{P,d}$ the class of models of T_P.

Let $\mathcal{A} \in \Gamma^e(\mathcal{C}_{P,d})$ and consider the expansion $(\mathcal{A}, P(\mathcal{A}))$ with $P(A) := \{(a_x) : \mathcal{A}_x \models P(a_x)\}$.

Lemma

Then $(\mathcal{A}, P(\mathcal{A}))$ is an elementary pair of elements of $\Gamma^e(\mathcal{C})$. Let $\tilde{T}$ be a complete theory of Boolean algebras. Then $Th(\Gamma^e_{\tilde{T}}(\mathcal{C}_{P,d}))$ is a complete $\mathcal{L}_P$-theory.

For the first part, we use that each formula has a determining sequence.
For the second part we use the fact that $T_{P,d}$ is complete and we apply the transfer result of Burris-Werner for Γ^e.

Dense pairs of Boolean products

Now assume that the models of T are fields K (of characteristic 0) endowed with a definable topology (a basis of neighbourhoods of 0 is given by $\{\chi(K, b)$ with b a tuple varying in $K\}$). Further we assume that χ is equivalent to a positive primitive existential formula χ_p (in the expansion of $\mathcal{L}$ by the projector p).

Digression: [W. Johnson, 2016/2020] If $\mathcal{K} := (K, +, \cdot, 0, 1, \ldots)$ expands an infinite field and is of finite dp-rank but not strongly minimal, then one can put on K a definable non-discrete field topology which is a V-topology.

[d'Elbée, Halevi, Johnson, 2025] A dp-minimal integral domain is a local ring, a dp-minimal commutative ring is the direct product of a finite ring and a dp-minimal henselian local ring and a ring of finite dp-rank is a direct product of finitely many henselian local rings.

Dense pairs of Boolean products

Let $\mathcal{A} \models \Gamma^a(T)$, $\mathcal{D} \subseteq_{\mathcal{L}} \mathcal{A}$. For $x \in X(\mathcal{A})$, let $D_x := \{u \in A_x : \exists d \in D d(x) = u\}$. Then we say that $\mathcal{D}$ is acl-closed if for each $x \in X(\mathcal{A})$ D_x is acl-closed in $\mathcal{A}_x$.

Definition

Let $\mathcal{A} \models \Gamma^a(T)$, $\mathcal{D} \subseteq_{\mathcal{L}} \mathcal{A}$. Then the pair $(\mathcal{A}, \mathcal{D})$ is a dense pair if

- $X(\mathcal{D})^* \preceq X(\mathcal{A})^*$,
- $\mathcal{D}$ is acl-closed in $\mathcal{A}$,
- for every tuple b in A, $\chi_p(A, b) \cap D \neq \emptyset$.
- $\forall e \in X(\mathcal{A})^* \; \exists \tilde{e} \in X(\mathcal{D})^* \; (e(x) \neq 0 \to \tilde{e} \subset e \wedge \tilde{e}(x) \neq 0)$.

Note that the class of dense pairs $(\mathcal{A}, \mathcal{D})$ with $\mathcal{A} \in \Gamma^e(T)$ is elementary.

Dense pairs of Boolean products

Lemma

Let $\mathcal{A} \models \Gamma^e(T)$, $\mathcal{D} \subseteq_{\mathcal{L}} \mathcal{A}$. Suppose that the pair $(\mathcal{A}, \mathcal{D})$ is a dense pair, then $\mathcal{D} \preceq \mathcal{A}$.

Theorem

Let $\mathcal{A} \models \Gamma^a(T)$ and let $(\mathcal{A}, \mathcal{D})$ be a dense pair. Suppose the theory $T_{P,d}$ is positively model complete (in some expansion of $\mathcal{L}$). Then the theory of dense pairs is model-complete.

Differential expansions

We then consider the theories T_δ of differential expansions of models of T by a derivation δ, namely $T_\delta := T \cup \{\delta(x+y) = \delta(x) + \delta(y),\ \delta(xy) = x\delta(y) + \delta(x)y\}$, on which we impose no interactions with the topology.

Note that in models K of T_δ, we always get a pair of fields (K, C_K), where $C_K := \{x \in K \colon \delta(x) = 0\}$. Conversely, given a pair of fields (K, L) of characteristic 0, there is a derivation δ such that $L = C_K$.

So using this detour, it can be used to recover transfer results from T to T_P (like NIP, open core, elimination of $\exists^\infty$, and $\mathrm{acl} = \mathrm{acl}_\delta$ and if T is distal, one gets that T_P has a distal expansion).

Existentially closed differential expansions

Suppose that the class of existentially closed models of T_δ is elementary, and let T_δ^* be an axiomatisation.

For instance, if T is model-complete + under some additional conditions, T_δ^* is the theory T_δ together with the following scheme of axioms (DL):

for $\mathcal{K} \models T_\delta$, for every differential polynomial $P(x) \in K\{x\}$ with $\ell(x) = 1$ and $\mathrm{ord}_x(P) = m \geqslant 1$, for $y = (y_0, \ldots, y_m)$, we have that for any neighbourhood W of 0 in K^{m+1},

$$(\forall y)\big((P^*(y) = 0 \wedge s_P^*(y) \neq 0) \rightarrow \exists x \\ \big(P(x) = 0 \wedge s_P(x) \neq 0 \wedge (\bar{\delta}^m(x) - y) \in W, \big)\big)$$

where $s_P^* := \partial_{y_n} P^*$ and $s_P := \partial_{\delta^m(x)} P(x)$.

Existentially closed differential expansions of T_δ

[Guzy-P (2010); Cubides-P (2022)] When consistent the theory T_δ^* axiomatizes the theory of existentially closed models of T_δ.

We get consistency of T_δ^*, when working in the class of large fields, a notion due to F. Pop.

[Pop, 1996] A field K is large if every smooth curve over K has infinitely many K-points, provided it has at least one. Equivalently K is large if K is existentially closed in $K((t))$.

(Cubidès-P.) Let $\mathcal{K} \models T_\delta^*$, then the subfield C_K is dense and co-dense in K and $C_K \preceq_{\mathcal{L}} K$. So $(K, C_K) \models T_P$.
$\rightsquigarrow$ For any model $(K, F) \models T_P$, there is a model K^* of T_δ^* such that $(K, F) \preceq (K^* \restriction \mathcal{L}, C_{K^*})$.

Open core

Definition (Dolich-Miller-Steinhorn, 2010)

Let $\tilde{L}$ be an extension of $\mathcal{L}$, let $\tilde{K}$ be an $\tilde{L}$-expansion of a topological field K which is an $\mathcal{L}$-structure. Then $\tilde{K}$ has $\mathcal{L}$-open core if every $\tilde{L}$-definable subset of K^n is $\mathcal{L}$-definable.

Theorem (Cubides-P., 2022)

T_δ^* has $\mathcal{L}$-open core.

We use the fact that we can associate with an $\mathcal{L}_\delta$-definable set X, an $\mathcal{L}$-definable set Y where differential tuples from elements of X are dense and where any differential tuple is coming from X.
(A differential tuple is a tuple consisting of successive derivatives (up to some order) of an ordinary tuple.)
And the properties of the dimension on $\mathcal{L}$-definable subsets in models of T, in particular that $\dim(Fr(X)) < \dim(X)$.

Open core

Theorem

Assume that T admits quantifier elimination and let $\mathcal{A} \models \Gamma^{e}_{T_a}(T^*_\delta)$. Given an $\mathcal{L}_\delta$-formula φ with $\varphi(A)$ a closed set, there is an $\mathcal{L}$-formula ψ equivalent to φ.

Corollary

$\Gamma^{e}_{T_a}(T_P)$ has $\mathcal{L}$-open core.

E.I.

Theorem (Cubides-P., 2022)

Suppose that T admits elimination of imaginaries (e.i.) in an expansion $\mathcal{L}^{\mathcal{G}}$ of $\mathcal{L}$. Then the theory T_δ^* admits elimination of imaginaries in $\mathcal{L}_\delta^{\mathcal{G}}$.

Corollary

Assume that T admits e.i. in an expansion $\mathcal{L}^{\mathcal{G}}$ of $\mathcal{L}$ and let $\tilde{T}$ be either the theory of atomic boolean algebras or the theory of atomless boolean algebras, then the theory $\Gamma^{\mathrm{e}}_{\tilde{T}}(T_\delta^*)$ admits elimination of imaginaries in $\mathcal{L}_\delta^{\mathcal{G}}$.

As a corollary of a result of Fratarcangeli (2005), we have:

Corollary

Let T_P be the theory of dense pairs of real-closed fields, then the theory $\Gamma^{\mathrm{e}}_{\tilde{T}}(T_P)$ admits elimination of imaginaries in $\mathcal{L}_P$.

Application to some theories of dense pairs of models of T

where

T is the $\mathcal{L}$-theory of real-closed fields, algebraically closed valued fields, p-adically closed fields of rank 1 (rank d), real-closed valued fields,....

A. Robinson, A. Macintyre, F. Delon, and recently put in a general context of geometric fields by Cubidès- Estrada-Pérez, Rincón.

Let $(K, P(K)) \models T_D$, then T_D admits q.e. in $\mathcal{L}$ together with:

(Delon language:) for each $n \geq 2$ one adds n-ary relation symbols: $\ell_n(x_1, \ldots, x_n)$ which means that $x_1, \ldots, x_n$ are linearly independent over $P(K)$ and component functions $\lambda_{n,i}$, $1 \leq i \leq n$, picking the coefficient of x_i in the linear combination expressing y as an element of the $P(K)$ vector-space generated by $x_1, \ldots, x_n$.

So, $\Gamma^e_{T_a}(T_D)$ is model-complete.