Homogeneity and local boundedness in pairs of geometric structures

Evgueni Vassiliev

Memorial University of Newfoundland - Grenfell Campus
Corner Brook, Newfoundland and Labrador, Canada

Joint work with Alexander Berenstein

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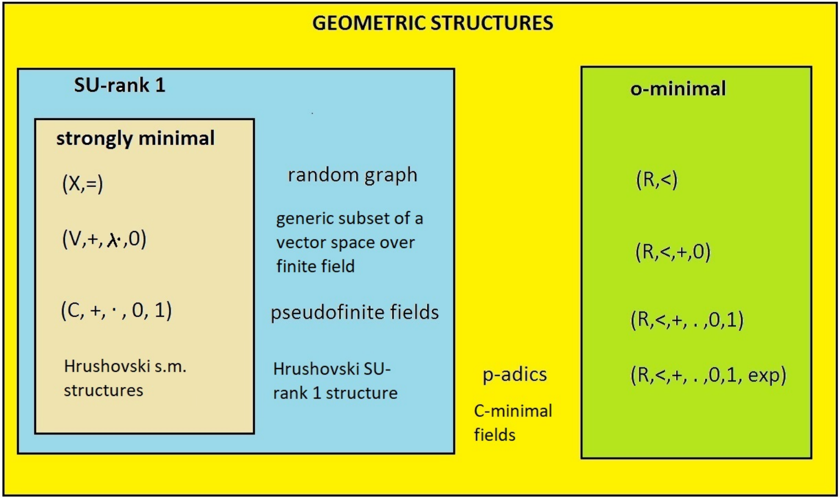


GEOMETRIC STRUCTURES

SU-rank 1

strongly minimal

$(X,=)$

$(V,+,\lambda\cdot,0)$

$(C,+,\cdot,0,1)$

Hrushovski s.m. structures

random graph

generic subset of a vector space over finite field

pseudofinite fields

Hrushovski SU-rank 1 structure

p-adics

C-minimal fields

o-minimal

$(R,<)$

$(R,<,+,0)$

$(R,<,+,.,0,1)$

$(R,<,+,.,0,1,\exp)$

Lovely (dense/codense) pair expansions

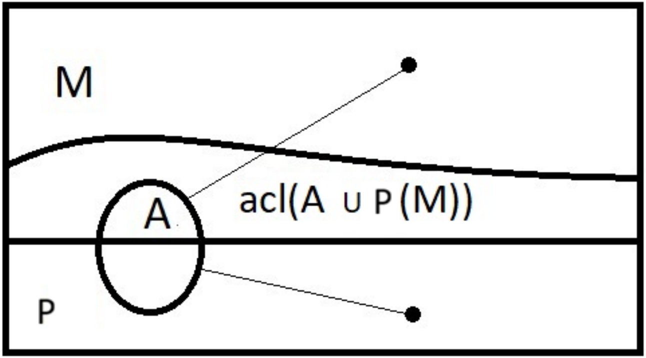
Definition (Berenstein, V.)

Let T be a geometric theory, $M \models T$.
A unary expansion (M, P) of M is a **dense/codense** (lovely) pair, if $\mathrm{acl}(P(M)) = P(M)$ and any nonalgebraic 1-type $q(x, A)$ (in T) over a finite-dimensional $A \subset M$ has realizations in

- $P(M)$ ("density" property)
- $M \setminus \mathrm{acl}(A \cup P(M))$ ("codensity" or extension property).

In this case (M, P) is an elementary pair $(P(M) \preceq M)$.
Strongly minimal case: infinite-coinfinite dimension.
O-minimal dense ordered group: density in the usual sense.

Lovely (dense/codense) pair expansions



Lovely (dense-codense) pair expansion

- We say that A is P-**independent** in (M, P), if $A \mathop{\smile\!\!\!\!\!\!\,|}_{P(A)} P(M)$.
- any expansion of $N \models T$ with an acl-closed subset $P(N)$ embeds in a lovely pair (M, P) in a P-independent way: $N \mathop{\smile\!\!\!\!\!\!\,|}_{P(N)} M$.
- Quantifier free type of P-independent tuples determines their L_P-type.
- All lovely pairs of T are elementarily equivalent.
- A saturated model of the (complete) theory T_P of all lovely pairs is again a lovely pair.

Geometry of scl

Let (M, P) be a sufficiently saturated lovely pair.

Small closure of $A \subset M$ is given by $\mathrm{scl}(A) = \mathrm{acl}(A \cup P(M))$ (localization of *acl* at the predicate).

Let $G(M/P) = (M^*, scl^*)$ be the geometry associated to *scl* (we can refer to it as **quotient geometry**).

Points of $G(M/P)$ are of the form $a^* = \mathrm{scl}(a)$ where $a \in M \backslash P(M)$, and scl^* is the closure operator induced by scl.

Geometry of scl

What kind of geometry is $G(M/P)$?

- infinite dimensional (by codensity property)
- the *acl*-geometry of any small model of T embeds in $G(M/P)$
- if *acl*-geometry is nontrivial, the relation $a^* \sim b^*$ given by

  $$a^* = b^* \text{ or } |scl^*(a^*, b^*)| \geq 3$$

  is an equivalence relation on $G(M/P)$ separating it into infinite dimensional classes with no interaction between classes
- each $\sim$-class is a geometry where lines (2-dimensional closed subsets) have ≥ 3 points

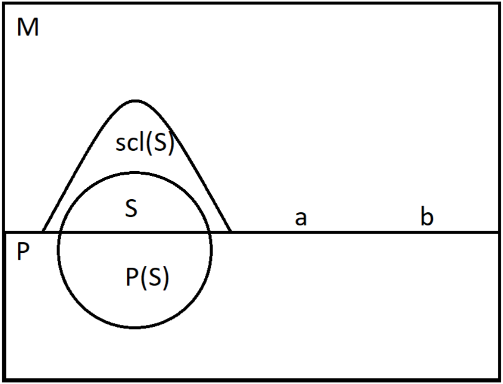
Homogeneity

Strongly minimal structures such as vector spaces and algebraically closed fields have **homogeneous** (pre)geometries in the following sense (see, e.g., *W. Hodges, Model Theory, Section 4.6*):

- A (pre)geometry (X, cl) is **homogeneous** if for any finite set S and $a, b \notin cl(S)$ there exists an automorphism $f : X \to X$ fixing $cl(S)$ pointwise such that $f(a) = b$.
- An equivalent (a priori, stronger) version (*W. Hodges, Model Theory, Ex. 4.6.9*): for any tuples $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$, each independent over S, there exists an automorphism $f : X \to X$ fixing $cl(S)$ pointwise so that $f(a_i) = b_i$ for all $i \leq n$.
- Example of non-homogeneous SU-rank 1 structure: generic subset of a vector space over a finite field (it is also ω-categorical, 1-based and not locally modular).

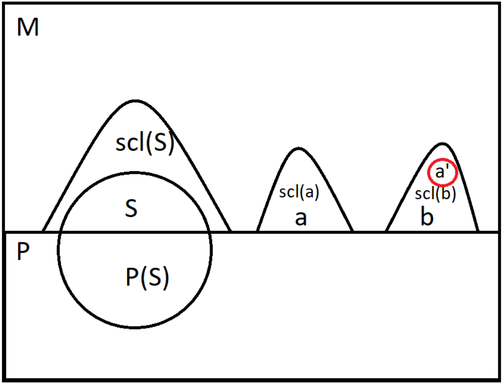
Homogeneity in $G(M/P)$

Suppose for simplicity, that there is a unique $\sim$-class in $G(M/P)$. Consider a (small) P-independent set $S = \mathrm{acl}(S) \subset M$ and $a, b \notin \mathrm{scl}(S)$.



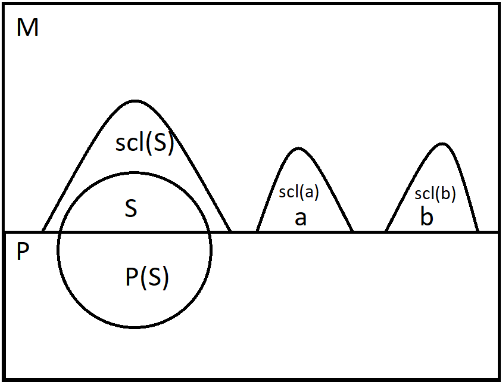
Homogeneity in $G(M/P)$

Then there exists a' such that $\mathrm{scl}(a') = \mathrm{scl}(b)$ and $\mathrm{tp}_P(a'/S) = \mathrm{tp}_P(a/S)$.



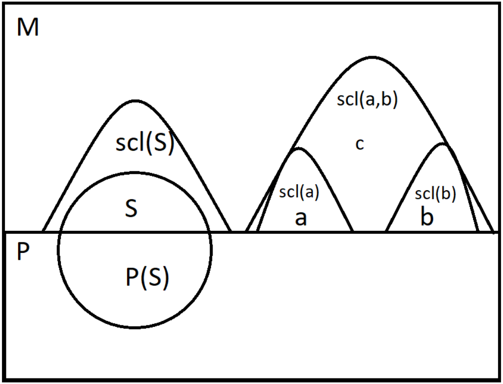
Homogeneity in $G(M/P)$: idea of proof

We may assume $a \notin \mathrm{scl}(bS)$
(by codensity and characterization of L_P-types).



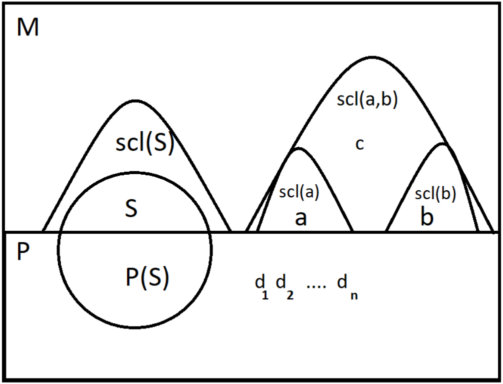
Homogeneity in $G(M/P)$: idea of proof

There exists $c \in \mathrm{scl}(a, b) \setminus \mathrm{scl}(a) \cup \mathrm{scl}(b)$.



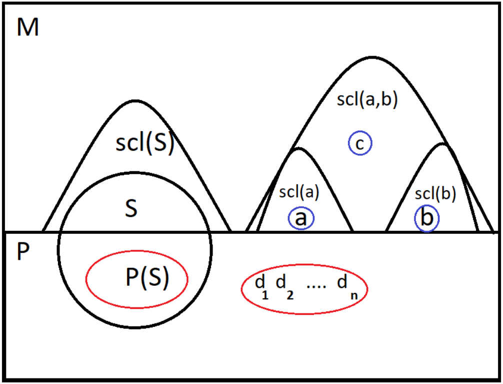
Homogeneity in $G(M/P)$: idea of proof

Let $d_1, \ldots, d_n \in P(M)$ be minimal so that $c \in \operatorname{acl}(ab\vec{d}P(S))$.



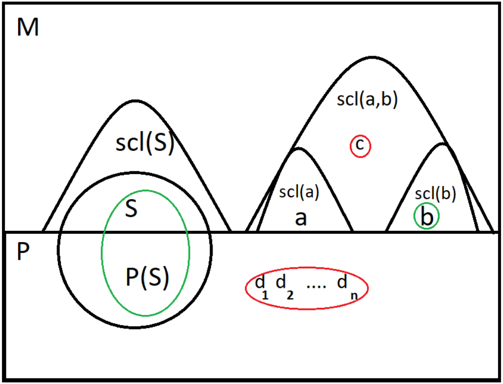
Homogeneity in $G(M/P)$: idea of proof

Note that $\vec{d}$ is acl-independent over $P(S)$, and hence, also over S.



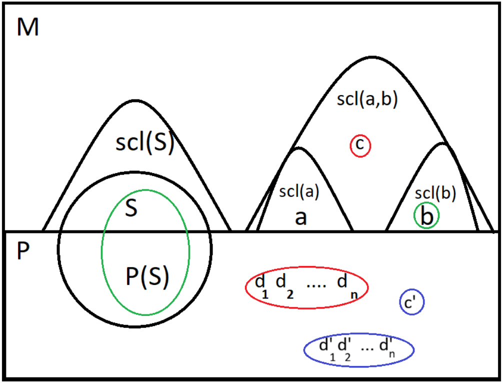
Homogeneity in $G(M/P)$: idea of proof

Next, $b \notin \mathrm{acl}(\vec{d}S)$, since $b \notin \mathrm{scl}(S)$. Finally, $c \notin \mathrm{acl}(b\vec{d}S)$, since otherwise, $a \in \mathrm{scl}(Sb)$, a contradiction. Thus, $cb\vec{d}$ is an acl-independent tuple over S.



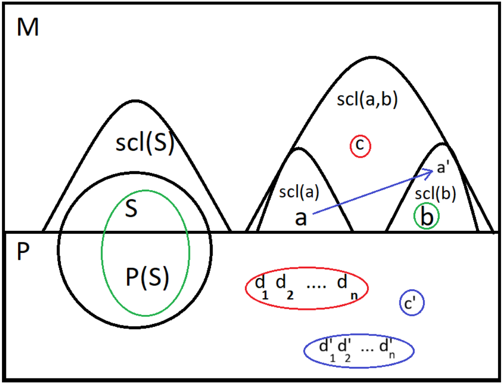
Homogeneity in $G(M/P)$: idea of proof

By the density property, there exist $c', d'_1, \ldots, d'_n \in P(M)$ realizing $\mathrm{tp}(c\vec{d}/Sb)$.



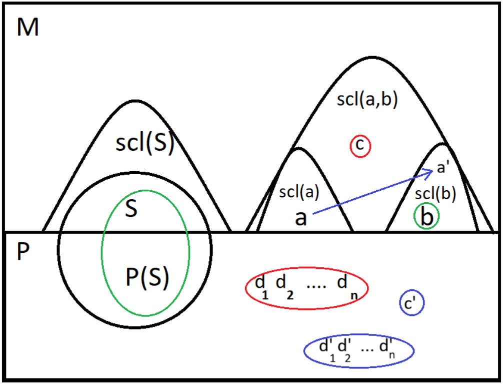
Homogeneity in $G(M/P)$: idea of proof

Let $a' \in M$ be such that $\mathrm{tp}(a'c'\vec{d}'/Sb) = \mathrm{tp}(ac\vec{d}/Sb)$. Then $a' \in \mathrm{acl}(c'b\vec{d}'P(S)) \subset \mathrm{scl}(b)$ and $\mathrm{tp}(a'/S) = \mathrm{tp}(a/S)$.



Homogeneity in $G(M/P)$: idea of proof

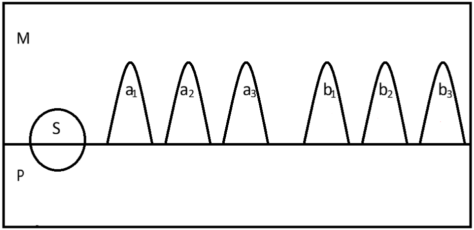
Note that $a' \notin P(M)$, since $b \in \operatorname{acl}(a'c'\vec{d}'S)$ and $b \notin \operatorname{scl}(S)$. Thus, $\operatorname{scl}(a') = \operatorname{scl}(b)$. By the quantifier elimination for P-independent tuples,$\operatorname{tp}_P(a'/S) = \operatorname{tp}_P(a/S)$.



Homogeneity in $G(M/P)$

More generally, restricting to a particular disjoint component ($\sim$-class) of $G(M/P)$, we have:

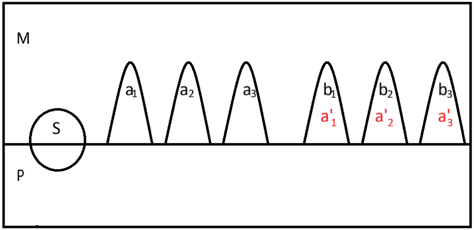
For any small P-independent $S = \mathrm{acl}(S) \subset M$ and two tuples $a_1, \ldots, a_n$ and $b_1 \ldots b_n$, each scl-independent over S, there are $a'_1, \ldots, a'_n$ such that $\mathrm{scl}(a'_i) = \mathrm{scl}(b_i)$ and $\mathrm{tp}_P(\vec{a}\,'/S) = \mathrm{tp}_P(\vec{a}/S)$.



Homogeneity in $G(M/P)$

More generally, restricting to a particular disjoint component ($\sim$-class) of $G(M/P)$, we have:

For any small P-independent $S = \mathrm{acl}(S) \subset M$ and two tuples $a_1, \ldots, a_n$ and $b_1 \ldots b_n$, each scl-independent over S, there are $a_1', \ldots, a_n'$ such that $\mathrm{scl}(a_i') = \mathrm{scl}(b_i)$ and $\mathrm{tp}_P(\vec{a}\,'/S) = \mathrm{tp}_P(\vec{a}/S)$.



Homogeneity in $G(M/P)$

In terms of the geometry $(G(M/P), \mathrm{scl}^*)$:

Assuming (M, P) is a large saturated model, for any small $\Sigma \subset G(M/P)$ and tuples $a_1^*, \ldots, a_n^*$ and $b_1^* \ldots b_n^*$ scl^*-independent over Σ there is a closure-preserving bijection of $G(M/P)$ mapping $\vec{a}^*$ to $\vec{b}^*$ and fixing Σ pointwise.

Note that we cannot require $\Sigma \subset G(M/P)$ to be scl^*-closed since fixing S pointwise does not imply that we fix $\mathrm{scl}^*(S)$ pointwise, and we do not know if $\mathrm{scl}^*(S)$ **is still small**.

This observation motivates the following question ...

Is scl^* *locally bounded*?

Given a finite $\Sigma \subset G(M/P)$, what can we say about the cardinality of $\mathrm{scl}^*(\Sigma)$?

In other words, given a finite $A \subset M$, in how many classes does the relation $\mathrm{scl}(x) = \mathrm{scl}(y)$ split the set $\mathrm{scl}(A)$?

Another motivation for this question: $G(M/P)$ as **possible example of a quasiminimal structure.**

We will say that $G(M/P)$ is **locally bounded** if for any finite $\Sigma \subset G(M/P)$ we have $|\,\mathrm{scl}^*(\Sigma)| \leq |\,T|$.

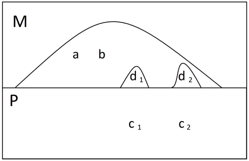
If T is the theory of a vector space $(V, +, \lambda\cdot)_{\lambda\in\mathbb{F}}$ over a division ring $\mathbb{F}$ then $|\,\mathrm{scl}^*(\Sigma)| \leq \max(\aleph_0, |\mathbb{F}|)$: $G(M/P)$ is simply a projective space over the same division ring. So, in this case, $G(M/P)$ is locally bounded.

Non-local boundedness in the field case

K is a (geometric) field $\Rightarrow$ $G(K/P)$ is not locally bounded:

Let $a, b \in K \backslash P(K)$ be algebraically independent over $P(K)$.

Let $c_1 \neq c_2 \in P(K)$, $d_i = a + bc_i$. Then $d_i \in \mathrm{scl}(a, b)$.



Since $b = \frac{d_1 - d_2}{c_1 - c_2}$ and $a = d_1 - bc_1$, we have $a, b \in \mathrm{scl}(d_1, d_2)$.

Since $\mathrm{scl}(a) \neq \mathrm{scl}(b)$, we have $\mathrm{scl}(d_1) \neq \mathrm{scl}(d_2)$.

If follows that in $G(K/P)$, closure of two points has the same cardinality as $P(K)$ (or K).

Question: Is local boundedness of $G(M/P)$ equivalent to "linearity" of M?

Theorem (Berenstein, V. 2012) [1]

The following are equivalent for any geometric theory T:

1. T is **generically linear:** any almost normal definable family of plane curves has dim ≤ 1.
2. T is **weakly locally modular:** for any A and B there exists $C \mathop{\smile\hskip-0.9em ^| \ } AB$ such that $A \mathop{\smile\hskip-0.9em ^| \ }_{\operatorname{acl}(AC) \cap \operatorname{acl}(BC)} B$.
3. T is **weakly 1-based:** for any $\vec{a}, B$ there is $\vec{a}' \equiv_B a$ with $\vec{a} \mathop{\smile\hskip-0.9em ^| \ }_B \vec{a}'$ and $\vec{a} \mathop{\smile\hskip-0.9em ^| \ }_{\vec{a}'} \vec{B}$.
4. T has no complete type definable **almost quasidesign** (a pseudoplane-like configuration).
5. $\operatorname{acl}_P = \operatorname{acl}$ in any $(M, P) \models T_P$.
6. scl is modular in any $(M, P) \models T_P$.

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[1] A. Berenstein, E. V., Weakly one-based geometric theories, J. Symb. Logic, vol. 77 (2) (2012), 392-422.

Geometry of scl: linear case

In the linear nontrivial case, $G(M/P)$ is modular.

Each $\sim$-class is infinite-dimensional modular geometry where each line has at least 3 points, and therefore, by a classical result, is a projective geometry over a division ring.

Question: Is the division ring above determined by the theory? In particular, is there a bound on its cardinality?

Note that the ω-categorical linear case, T_P is again ω-categorical, and if T is nontrivial then $G(M/P)$ splits into projective geometries over finite fields and therefore T_P interprets infinite vector spaces.

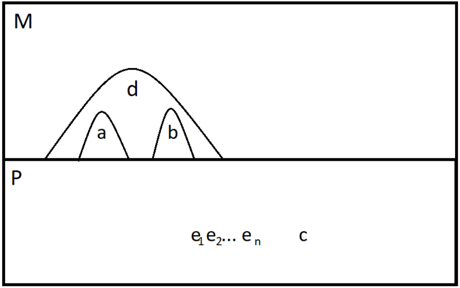
Non-linearity: complete type definable almost quasidesign

Non-linearity of a geometric theory is equivalent to the following condition:

There exist a, b, c, d and a tuple $\vec{e}$ such that any three elements of the set $\{a, b, c, d\}$ are independent over $\vec{e}$, $\dim(a, b, c, d/\vec{e}) = 3$ and if $r(xy, zt, \vec{e}) = \mathrm{tp}(ab, cd, \vec{e})$, for any $c'd' \models \mathrm{tp}(cd/\vec{e})$ such that $\mathrm{acl}(cd\vec{e}) \neq \mathrm{acl}(c'd'\vec{e})$ we have that $r(xy, cd, \vec{e}) \wedge r(xy, c'd', \vec{e})$ has finitely many realizations.

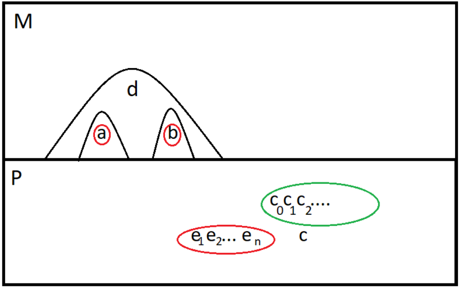
Non-linearity implies non-local boundedness

Let $a, b, c, d, \vec{e}$ be as above. By the density and codensity properties, we may choose $c, \vec{e} \in P(M)$, and a and b independent over $P(M)$, so $\dim(a, b/P(M)) = 2$, in particular $\operatorname{scl}(a) \neq \operatorname{scl}(b)$. Note that $d \in \operatorname{scl}(a, b) \setminus \operatorname{scl}(a) \cup \operatorname{scl}(b)$.



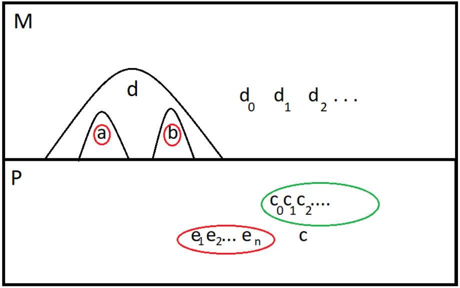
Non-linearity implies non-local boundedness

Let $(c_i : i \in I)$ be a sequence in $\mathrm{tp}(c/ab\vec{e})$ independent over $ab\vec{e}$. By the density property, we may assume that $c_i \in P(M)$ for each i.



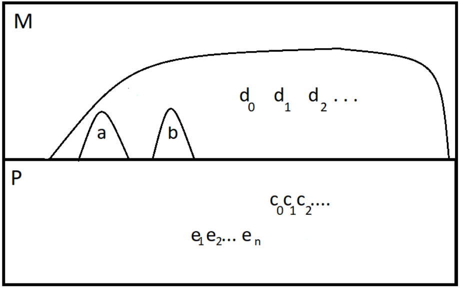
Non-linearity implies non-local boundedness

Choose d_i such that $c_i, d_i \equiv_{ab\vec{e}} cd$ for all $i \in I$. Note that, since $\dim(a, b/P(M)) = 2$ and $c, \vec{e} \in P(M)$, we must have $d_i \notin P(M)$ for all $i \in I$.



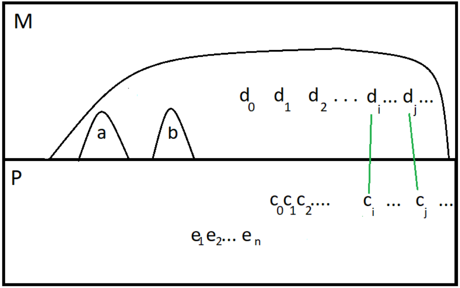
Non-linearity implies non-local boundedness

Also note that $d_i \in \mathrm{scl}(a, b)$.



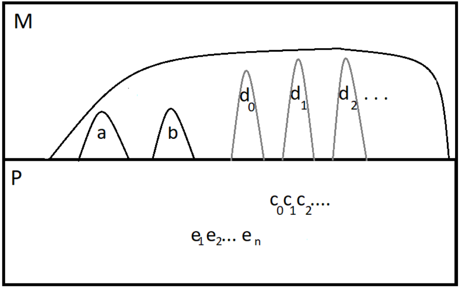
Non-linearity implies non-local boundedness

For any $i \neq j$ we have $c_i \notin \mathrm{acl}(c_j\vec{e})$, and by the exchange property and the fact that $d_j \notin P(M)$, we have $c_i \notin \mathrm{acl}(c_j d_j \vec{e})$, and, thus, $\mathrm{acl}(c_i d_i \vec{e}) \neq \mathrm{acl}(c_j d_j \vec{e})$.
By the definition of quasidesign, $(a, b) \in acl(c_i d_i c_j d_j \vec{e})$. Since $c_i, c_j, \vec{e} \in P(M)$, we have $\mathrm{scl}(a, b) = \mathrm{scl}(d_i, d_j)$ for $i \neq j$.



Non-linearity implies non-local boundedness

Since $\dim(a, b/P(M)) = 2 = \dim(d_i d_j/P(M))$, we get $\operatorname{scl}(d_i) \neq \operatorname{scl}(d_j)$ and since we can make the index set I as large as $|P(M)|$, then $G(M/P(M))$ is not locally bounded.



Does linearity imply local boundedness?

- true in the strongly minimal case: linear (locally modular) s.m. structures are "essentially" vector spaces over division rings.
- true in the o-minimal case: linear o-minimal structures are locally trivial or given (definably) by vector space intervals; there is a bound on the cardinality of the division rings involved in these vector spaces.
- we also show that it is true in the SU-rank 1 case.

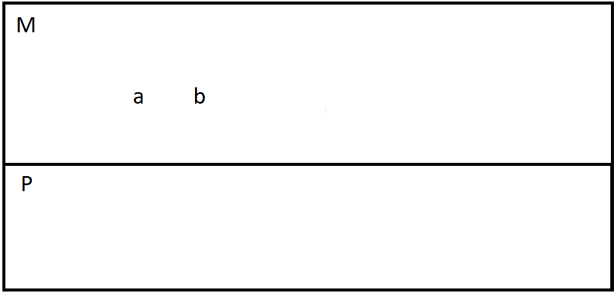
Linearity implies local boundedness in SU-rank 1 case

In the SU-rank 1 case, linearity is equivalent to 1-basedness:
$\vec{a} \equiv_B \vec{a}', \vec{a} \mathop{\smile\!\!\!\!\!\!\;|}_B \vec{a}' \Rightarrow \vec{a} \mathop{\smile\!\!\!\!\!\!\;|}_{\vec{a}'} B$.

Due to the modularity of scl, an induction argument reduces the question to showing $|\operatorname{scl}^*(a, b)| \leq |T|$.

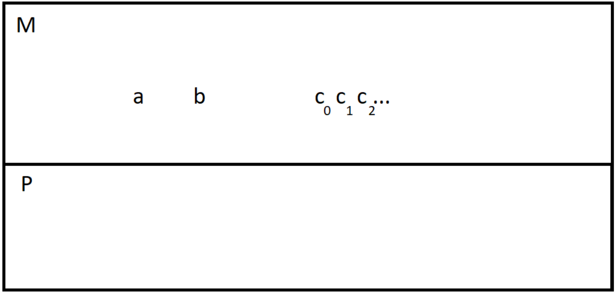
Linearity implies local boundedness in SU-rank 1 case

Let $a, b \in M$ be independent over $P(M)$. Let $\phi(x, y, z)$ be of the form $\exists t_1 \dots t_k \in P\ \psi(x, y, z, \vec{t})$ where $\psi(x, y, z, \vec{t})$ is a L-formula that witnesses $x \in \mathrm{acl}(y, z, \vec{t})$.



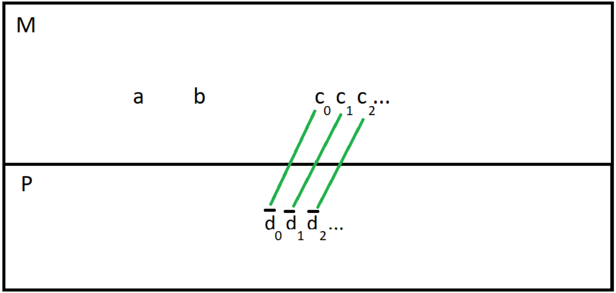
Linearity implies local boundedness in SU-rank 1 case

Suppose there exist $c_i \in \mathrm{scl}(a,b) \setminus \mathrm{scl}(a) \cup \mathrm{scl}(b)$ satisfying $\phi(x,a,b)$ such that $\mathrm{scl}(c_i) \neq \mathrm{scl}(c_j)$ for all $i \neq j < \omega$.
We may assume that this sequence is L_P-indiscernible over ab.



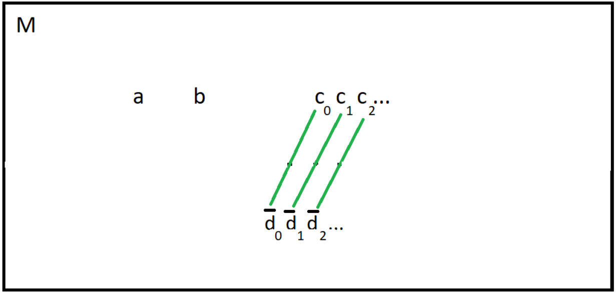
Linearity implies local boundedness in SU-rank 1 case

For each i choose $\vec{d}_i \in P$ such that $\mathrm{tp}_P(c_i\vec{d}_i/ab) = \mathrm{tp}_P(c_0\vec{d}_0/ab)$ and $\models \psi(c_i, a, b, \vec{d}_i)$. Furthermore, we may assume that the sequence $(c_i\vec{d}_i : i < \omega)$ is again L_P-indiscernible over ab.



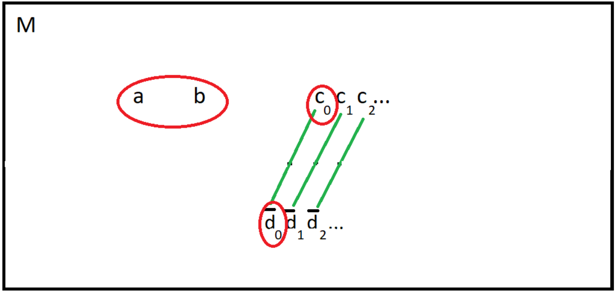
Linearity implies local boundedness in SU-rank 1 case

Now, consider the structure M (without the predicate P). Note that $(c_i\vec{d}_i : i < \omega)$ is L-indiscernible over ab.



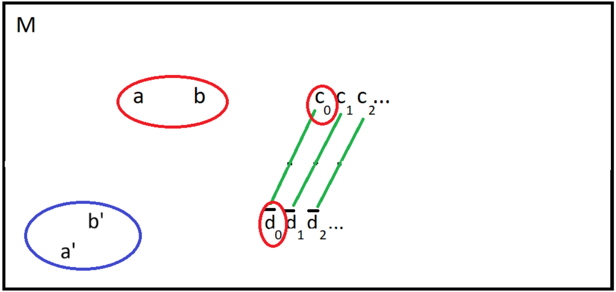
Linearity implies local boundedness in SU-rank 1 case

Let $a', b' \in M$ be such that $a'b' \models \mathrm{tp}(ab/c_0\vec{d}_0)$, $a'b' \mathop{\smile\!\!\!\!\!\!\mid}_{c_0\vec{d}_0} ab$. By 1-basedness of T, $a'b' \mathop{\smile\!\!\!\!\!\!\mid}_{ab} c_0\vec{d}_0$. Also note that $a', b' \notin \mathrm{acl}(abc_0\vec{d}_0)$.



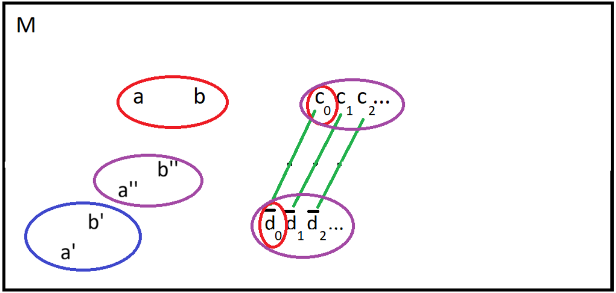
Linearity implies local boundedness in SU-rank 1 case

Let $a', b' \in M$ be such that $a'b' \models \mathrm{tp}(ab/c_0\vec{d}_0)$, $a'b' \downarrow_{c_0\vec{d}_0} ab$. By 1-basedness of T, $a'b' \downarrow_{ab} c_0\vec{d}_0$. Also note that $a', b' \notin \mathrm{acl}(abc_0\vec{d}_0)$.



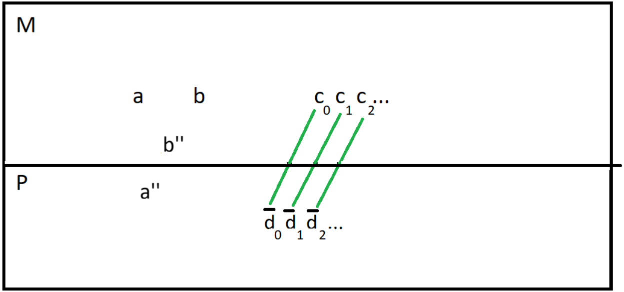
Linearity implies local boundedness in SU-rank 1 case

Using properties of forking in simple theories, we can find $a''b'' \equiv_{abc_0d_0} a'b'$ such that $a''b'' \mathop{\smile\!\!\!\!\!\!|}_{ab} (c_i\vec{d}_i : i < \omega)$.
In particular, $a'' \notin \mathrm{acl}(\{a, b\} \cup \{c_i\vec{d}_i : i < \omega\})$.



Linearity implies local boundedness in SU-rank 1 case

By the density property, we may assume that $a'' \in P$. But then $b'' \notin P$ (otherwise, $c_0 \in P$) and $c_i \in \mathrm{scl}(b'')$ for all $i < \omega$, a contradiction to $\mathrm{scl}(c_i) \neq \mathrm{scl}(c_j)$ for $i \neq j$.



Linearity implies local boundedness in SU-rank 1 case

By compactness, for any $\phi(x,y,z)$ witnessing $x \in \mathrm{scl}(y,z)$ there exists $k < \omega$ such that for any $a, b \in M \backslash P(M)$ such that $\mathrm{scl}(a) \neq \mathrm{scl}(b)$ $\phi(x,a,b)$ has at most k realizations independent over $P(M)$. It follows that for any $a^* \neq b^* \in G(M/P)$ we have $|\mathrm{scl}^*(a^*, b^*)| \leq |T|$.

$G(M/P)$ as a quasiminimal (pre)geometric structure

Assume that the relation $\sim$ on $G(M/P)$ has a unique class and scl^* of any finite set is countable. Thus, T is weakly 1-based and $G(M/P)$ is modular. Moreover, $G(M/P)$ an infinite-dimensional projective geometry over a countable division ring.

It is known (see A. Pillay, Geometric Stability Theory, Ch.V, Proposition 2.2) that there does not exist a strongly minimal structure whose acl-pregeometry is a projective geometry over an infinite division ring.

We conjecture that the following "natural" language L^* makes $(G(M/P), \mathrm{scl}^*)$ a quasiminimal structure.

$G(M/P)$ as a quasiminimal (pre)geometric structure

Let $L^* = \{R_q(x_1, \ldots, x_n) \ : \ q \in S_n(T_P)\}$, where

for any L_P-type $q(x_1, \ldots, x_n)$

$R_q(a_1^*, \ldots, a_n^*) \iff$ there exist representatives $a_1', \ldots, a_n'$ of $a_1^*, \ldots, a_n^*$ such that $(M, P) \models q(a_1', \ldots, a_n')$.

The question reduces to proving (QM5). Using modularity of scl^* it is further reduced:
$\mathrm{qftp}_{L^*}(a_1^*, a_2^*, \ldots) = \mathrm{qftp}_{L^*}(b_1^*, b_2^*, \ldots)$, $a^* \in \mathrm{scl}^*(a_1^*, a_2^*) \Rightarrow$ there
there exists $b^* \in \mathrm{scl}^*(a_1^*, a_2^*)$ such that
$\mathrm{qftp}_{L^*}(a^*, a_1^*, a_2^*, \ldots) = \mathrm{qftp}_{L^*}(b^*, b_1^*, b_2^*, \ldots)$.

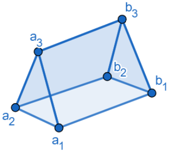
Confirmed in the case of a pure vector space and o-minimal structure with global addition.

Further properties of $G(M/P)$

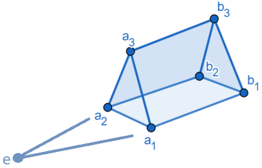
$G(M/P)$ shares some nice features with projective and algebraic combinatorial geometries.

Ingleton-Main Lemma (A. W. Ingleton, R.A. Main, Non-algebraic matroids exist, Bull. London Math. Soc. 7 (1975), 144-146)

Let $(\mathbb{F}, +, \cdot, 0, 1)$ be any field, and let $\mathbb{K}$ be an a. c. extension of $\mathbb{F}$. Then whenever we have the following configuration in $G(\mathbb{K}/\mathbb{F})$



$cl(a_1, b_1) \cap cl(a_2, b_2) = \{e\}$ for some e.



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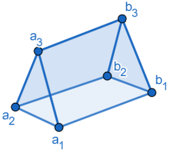
Let $(\mathbb{F}, +, \cdot, 0, 1)$ be any field, and let K be an a.c. extension of $\mathbb{F}$. Then whenever we have the following configuration in $(K, cl_{\mathbb{F}})$



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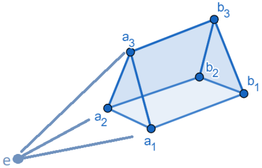
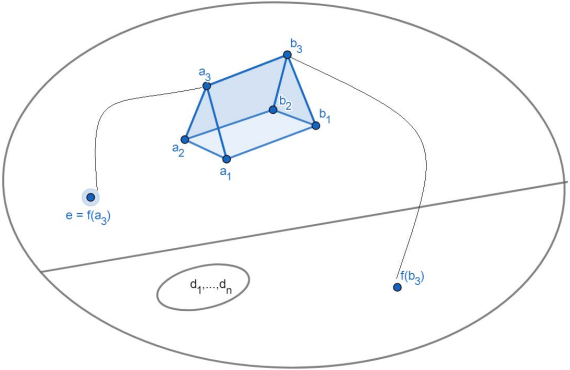


It then follows that all three lines meet at a point.
(so, any three noncoplanar pairwise coplanar lines meet at a point).

Ingleton-Main lemma for $G(M/P)$

Theorem (Mukhopadhyay, V. 2019) [2]
For any dense pair of geometric structures (M, P), Ingleton-Main lemma holds for $G(M/P)$.



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[2] M. Mukhopadhyay, E.V., On the Vámos matroid, homogeneous pregeometries and dense pairs, Australasian J. Combinatorics, vol. 75(1) (2019), 158-170.

Ingleton-Main lemma for $G(M/P)$

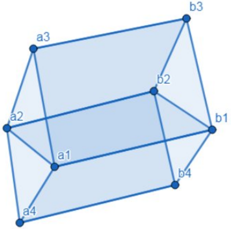
Note that IM Lemma may not hold for the acl-geometry of M itself:

Examples:

- One example is affine space (can be fixed by localizing at a point).
- Another example: the generic subset of a vector space over a finite field (cannot be fixed by localizing at a small subset).

Application of IM Lemma: the Vámos matroid

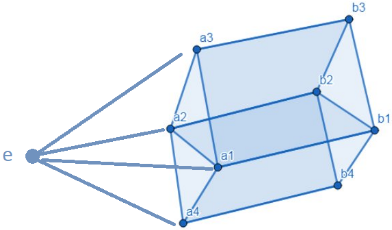
Smallest (in terms of size) non-algebraic matroid:
the Vámos matroid.



Rank 4 matroid where each face represents a rank 3 closed subset.
Any 3 points are independent.

Application of IM Lemma: the Vámos matroid

IM Lemma prevents representability of the Vámos matroid:



The top and bottom lines would have to meet at a point forcing them to be coplanar.

Application of IM Lemma: the Vámos matroid

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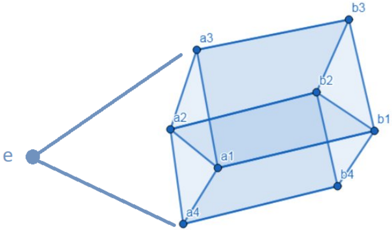

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Application of IM Lemma: the Vámos matroid

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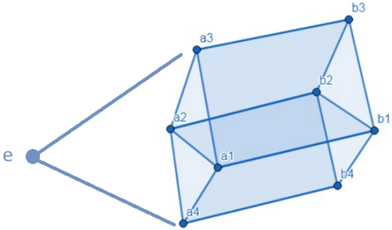
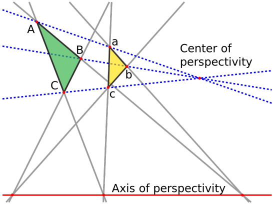
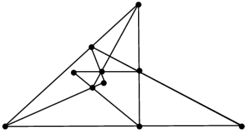


Thus, the Vámos matroid is not only non-algebraic,
but also not representable in any geometric structure.

(Converse) Desargues Theorem: algebraic case

Fact (Lindström, 1985)[3]: If a geometry satisfies the IM lemma and the following property:
(*) for any plane π_1 and a line $\ell \subset \pi_1$ there exists another plane π_2 such that $\pi_1 \cap \pi_2 = \ell$ and an incidence preserving isomorphism $\phi : \pi_1 \to \pi_2$ fixing ℓ
then the converse of Desarguesian Theorem holds for the geometry: any two triples in a plane that are axially perspective are centrally perspective.





Corollary: non-Desargues matroid is not algebraic.

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[3] B. Lindström, A Desarguesian Theorem for algebraic combinatorial geometries, Combinatorica 5 (3) (1985) 237 - 239.

Property (*)'

We know that IM lemma holds in $G(M/P)$.

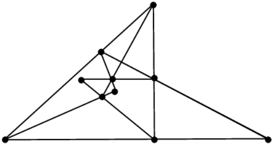
Homogeneity result implies a weaker version of (*) where we only work with a finite subset of the plane and fix finitely many points on the line:

(*)': Given finite sets $\Lambda \subset \Pi \subset G(M/P)$ such that $\dim^*(\Lambda) = 2$ and $\dim^*(\Pi) = 3$, we can find $\Pi' \subset G(M/P)$ such that $\Lambda \subset \Pi'$, $\mathrm{scl}^*(\Pi) \cap \mathrm{scl}^*(\Pi') = \mathrm{scl}^*(\Lambda)$, and a closure-preserving bijection $\phi : \Pi \to \Pi'$ such that ϕ fixes Λ pointwise.

(Converse) Desargues Theorem for $G(M/P)$

Following Lindström's proofs we obtain that the (converse) Desargues Theorem holds in $G(M/P)$.

It follows that the non-Desargues matroid

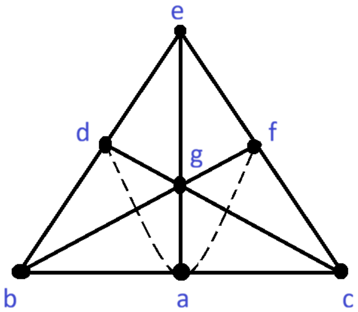


is not representable in $G(M/P)$, and, therefore, is not representable in any geometric structure.

Question: Is every (finite) matroid representable in a geometric structure algebraic?

Another application of IM Lemma: harmonic conjugate in $G(M/P)$

We say that a geometry is *harmonic* if whenever we have a configuration given below (d, a, f may or may not be dependent)



Another application of IM Lemma: harmonic conjugate in $G(M/P)$

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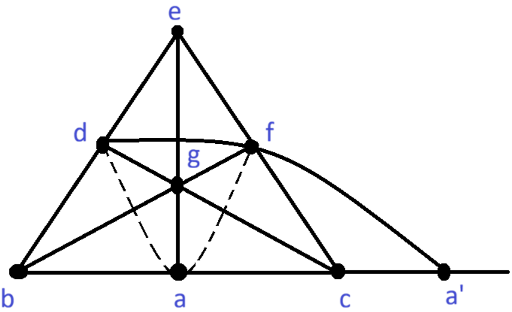

$cl(d,f) \cap cl(b,c) = \{a'\}$ where a' depends only on a, b and c (*harmonic conjugate* of a with respect to b and c)[4].

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[4] R. Flórez, Harmonic conjugation in harmonic matroids,Discr. Math. 309 (2009) 2365-2372

Another application of IM Lemma: harmonic conjugate in $G(M/P)$

Projective geometries of rank ≥ 4 and Desarguesian projective planes are harmonic. So is the full algebraic matroid (Lindström, 1986)[5]. Lindström's proof again uses property (*) and IM Lemma. We generalize it using property (*)' to prove that $G(M/P)$ is harmonic.

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[5] B. Lindström, On harmonic conjugates in full algebraic combinatorial geometries, Eur. J. Combinatorics, 7 (1986), 259-262.

Thank you!