

The model theory of large fields

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Conventions/Background

All rings are commutative with unit.

K is a field and K^{alg} is the algebraic closure of K .

V, W, V', W' are K -varieties. $V \rightarrow W$ is a K -variety morphism.

$V(K)$ is the set of K -points of V .

\mathbb{A}^n is n -dimensional affine space over K , so $\mathbb{A}^n(K) = K^n$.

Roughly speaking:

V is defined by a finite system of polynomial equations and inequations with coefficients from K . $V \rightarrow W$ is a polynomial map.

$V(K)$ is the solution set of the system in K .

We are primarily interested in $V(K)$ – basically quantifier free definable sets in K .

Even more roughly:

V “is” $V(K^{\text{alg}})$, K -varieties are sets definable in K^{alg} with parameters from K .

Suppose W is smooth.

$f: V \rightarrow W$ is **étale** if V is smooth and $T_p V \rightarrow T_{f(p)} W$ is an iso. for any $p \in V$.

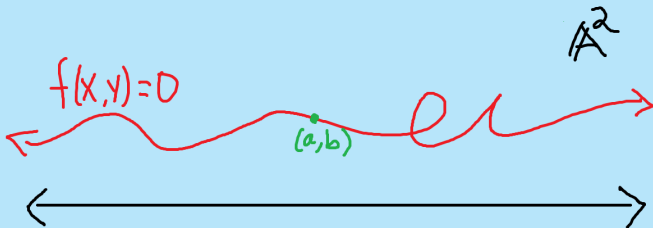
Largeness

K is **large** if it satisfies one of the following equivalent conditions:

- A smooth 1-dim K -variety with a K -point has infinitely many K -points.
- If $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

then f has infinitely many zeros in K . (Large fields form a $\forall \exists$ elem. class.)



Pop ('96) introduced largeness, proved inverse Galois theorem over $K(t)$, K large.

“the ‘right class’ of fields over which one can do a lot of interesting mathematics.”

–Pop, “Little survey on large fields”

Non-large fields

K is **large** if whenever $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

then f has infinitely many zeros in K .

Finite fields are not large.

$$f(x, y) = x^4 + y^4 - 1.$$

$$f(0, 1) = 0 \neq 4 = \partial f / \partial y(0, 1)$$

Fermat: Only zeros of f in \mathbb{Q}^2 are $(\pm 1, 0)$, $(0, \pm 1)$.

Faltings: f has only finitely many zeros in any number field.

Similar (hard) results show that function fields are not large.

function field = f.g. extension of a field F that is not algebraic over F , e.g. $F(t)$.

Most other fields you have heard of are large.

All logically tame infinite fields known before 2022 are large.

Large fields

K is **large** if whenever $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

then f has infinitely many zeros in K .

K algebraically closed \implies non-constant $f \in K[x, y]$ has infinitely many zeros.

More generally, separably closed fields are large.

Implicit function theorem $\implies \mathbb{R}$ is large \implies real closed fields are large.

Polynomial IFT \implies henselian valued fields are large.

$\mathbb{Q}_p, K((t))$ are henselian valued. Local fields are large, global fields are not.

K is **pseudofinite** if K infinite and satisfies $\text{Th}(\text{finite fields})$.

Weil conj. for curves \implies Pseudofinite is PAC \implies Pseudofinite is large.

Infinite algebraic extensions of finite fields are also PAC.

(PAC means **p**seudo **a**lgebraically **c**losed.)

More large fields

Hasse Principle: A \mathbb{Q} -variety with a point in \mathbb{R} and in each \mathbb{Q}_p has a \mathbb{Q} -point.

Often fails.

K satisfies a **local-global principle** if it satisfies a form of the Hasse principle.

(For more precision see “Little survey on large fields” by Pop.)

K satisfies a local-global principle $\implies K$ is large.

$\alpha \in \mathbb{Q}^{\text{alg}}$ is **totally real** if all roots of its min. poly. are in \mathbb{R} .

\mathbb{Q}^{tr} = field of totally real algebraic numbers.

An abs. irred. \mathbb{Q}^{tr} -variety with a point in any completion of \mathbb{Q}^{tr} has a \mathbb{Q}^{tr} -point.

Equivalently: An absolutely irreducible \mathbb{Q}^{tr} -variety with a point in every real closure of \mathbb{Q}^{tr} with respect to a field order has a \mathbb{Q}^{tr} -point.

Hence \mathbb{Q}^{tr} is large.

The field of totally p -adic algebraic numbers is also large.

More generally pseudo real closed and pseudo p -adically closed fields are large.

Possibly large fields

\mathbb{Q}_{ab} = max. abelian extension of \mathbb{Q} = extension of \mathbb{Q} by all roots of unity.

Open Question: Is \mathbb{Q}_{ab} large?

Large fields are closed under algebraic extensions.

\mathbb{Q}_{solv} = max. solvable extension of \mathbb{Q} = smallest root-closed subfield of \mathbb{Q}^{alg} .

Famous Conjecture: \mathbb{Q}_{solv} is PAC, hence large.

Conjecture (Koenigsmann):

K has $< \infty$ separable extensions of any given degree $\implies K$ is large.

Colliot-Thélène, Jarden: $\text{Gal}(K^{\text{alg}}/K)$ is pro- $p \implies K$ is large.

The étale-open topology

An **étale-image** is a subset of $W(K)$ of the form $f(V(K))$ for étale $f: V \rightarrow W$.

Étale-images are closed under finite intersections and unions.

Étale-images form a basis for the **étale-open topology** on $W(K)$.

Also call it the \mathcal{E}_K -topology.

Some Properties:

- Refines the Zariski topology.
 - $V(K) \rightarrow W(K)$ is continuous for any $V \rightarrow W$.
 - $V(K) \rightarrow W(K)$ is an open map for $V \rightarrow W$ étale.
 - Topology on K^n refines, but may not agree, with the product topology.
 - Topology on $K^n =$ product topology $\iff \mathcal{E}_K$ induced by a field top. on K .
-

Theorem (Johnson, Tran, W, Ye):

The \mathcal{E}_K -topology on K is discrete $\iff K$ is not large.

K is **large** if whenever $f \in K[x, y]$ and $(a, b) \in K^2$ satisfy

$$f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)$$

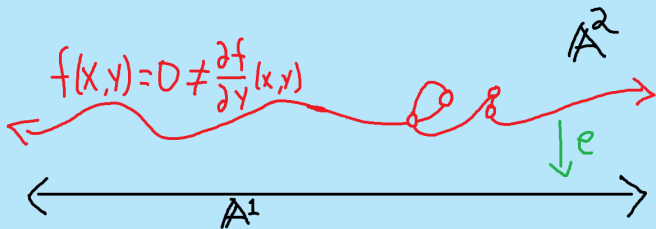
then f has infinitely many zeros in K .

An **étale-image** is a subset of $W(K)$ of the form $f(V(K))$ for étale $f: V \rightarrow W$.

Étale-images form a basis for the **étale-open** (\mathcal{E}_K -) **topology** on $W(K)$.

Theorem (JTWY): The \mathcal{E}_K -topology on K is discrete $\iff K$ is not large.

Proof: Any étale $V \rightarrow \mathbb{A}^1$ is (Zariski) locally isomorphic to a morphism e :



Sets of the form $\{a \in K : (\exists b \in K) f(a, b) = 0 \neq \frac{\partial f}{\partial y}(a, b)\}$ form a basis on K .

étale vs étale

The étale-open topology is **not** the étale topology!!!!!!

étale-open topology

a topology

defined on $V(K)$

not like the analytic topology

étale topology

not a topology

defined on V

like the analytic topology

Examples

Theorem (JTWY):

K separably closed	\implies	\mathcal{E}_K is the Zariski topology
K real closed	\implies	\mathcal{E}_K is the order topology
K henselian valued & not sep. closed	\implies	\mathcal{E}_K is the valuation topology
K is PAC and not sep. closed	\implies	\mathcal{E}_K is something new

K is PAC, $\text{Char}(K) \neq 2 \implies U - U$ cofinite for any nonempty open $U \subseteq K$.

With respect to the étale-open topology:

K is HD	\iff	K is not sep. closed
K is zero-dimensional	\iff	K is not sep. closed or \mathbb{R} .
K is loc. compact HD	\iff	K is a local field other than \mathbb{C} .
\mathcal{E}_K given by abs. value/valuation on K	\iff	K is not sep. closed and we have $K \equiv K^*$ for henselian K^* .

Our old theorem

The stable fields conjecture: An infinite field is stable iff separably closed.

True when “stable” is replaced by stronger stability-theoretic conditions.

$\mathbb{C}(t)$ might be stable??????????

Theorem (JTWY): A large field is stable iff separably closed.

Uses that the topology is HD, non-discrete, and has a basis of def. sets.

Our new theorem

R is a local (i.e. unique maximal ideal \mathfrak{m}) integral domain that is not a field.

$\text{Quot}(R)$ is its quotient field.

R is **henselian** if any simple root of $f \in R[x]$ in R/\mathfrak{m} lifts to a root of f in R .

simple root: $f(\alpha) = 0 \neq f'(\alpha)$.

Examples: Val. ring of henselian valuation, $K[[x_1, \dots, x_n]]$, complete local rings

it was generally believed that the above fields $K = k((x, y))$, and more general $K = \text{Quot}(R)$ with R complete Noetherian local and $\text{Krull.dim}(R) > 1$, were not large fields. Note that these fields are definitely *not* Henselian valued fields!

–Pop, “Henselian implies large”, 2010

$K((x, y)) = \text{Quot}(K[[x, y]])$.

Pop shows $\text{Quot}(R)$ is large when R henselian in that paper.

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2.3. For which fields K is it true that every curve with a smooth K -point contains a Zariski dense set of K -points? Characterizations of this property are given in [Po, 1.1].

The following are some interesting classes of such fields:

1. Fields complete with respect to a discrete valuation (This, in particular, includes the finite extensions of the p -adic fields \mathbb{Q}_p),
2. More generally, quotient fields of local Henselian domains,
3. \mathbb{R} and all real closed fields,
4. Infinite algebraic extensions of finite fields and, more generally, pseudo algebraically closed fields (cf. [Fr-Ja, Chap. 10]).

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THEOREM 2.1. *Let k be Henselian or real closed. A k -variety X is solid if and only if X_k contains a nonsingular point of X .*

–Efroymsen, “Henselian fields and solid k -varieties II”, 1972

Here:

Henselian = $\text{Quot}(R)$ for henselian R

variety = irreducible variety

solid = k -points are Zariski dense

X_k = $X(k)$

Our new theorem

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Theorem (JTWY): K is large $\iff K \equiv \text{Quot}(R)$ for R henselian.

\mathbb{R} is not the quotient field of a henselian local domain.

Problem: Find logically tame henselian R with $\text{Quot}(R)$ pseudofinite.

Can't be noetherian.

Other perspective: Def. of largeness axiomatizes the theory of quotient fields of henselian local domains.

Proof of our new theorem

Theorem (JTWY): K is large $\iff K \equiv \text{Quot}(R)$ for R henselian.

\Leftarrow Pop + large fields are an elementary class.

Separably closed case of \implies follows by standard valuation theory.

Main tool in non-separably closed case: Polynomial inverse function theorem.

Theorem (JTWY): If K is not separably closed then $V(K) \rightarrow W(K)$ is a local homeomorphism for étale $V \rightarrow W$.

Corollary (not used for new theorem):

V smooth irreducible $\implies V(K)$ locally homeomorphic to $K^{\dim V}$.

$V(K)$ is a “manifold”.

Proof of our new theorem: Nash functions

$U \subseteq \mathbb{R}^n$ open and definable, $s: U \rightarrow \mathbb{R}$.

s is a **Nash function** if s is C^∞ and definable. (Equiv: analytic and algebraic.)

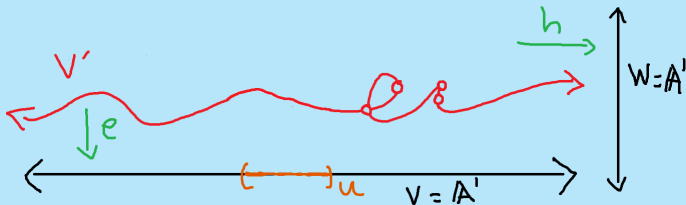
Artin-Mazur: s is Nash \iff s is the composition of the inverse of an étale map with a polynomial map.

Theorem (JTWY): If K is not separably closed then $V(K) \rightarrow W(K)$ is a local homeomorphism for étale $V \rightarrow W$.

Suppose K is large and not separably closed. $U \subseteq V(K)$ étale-open.

$s: U \rightarrow W(K)$ is **Nash** if $s = h \circ (e|_P)^{-1}$ for $e: V' \rightarrow V$ étale,

$h: V' \rightarrow W$ a morphism, $P \subseteq V'(K)$ open, e gives a homeo. $P \rightarrow U$.



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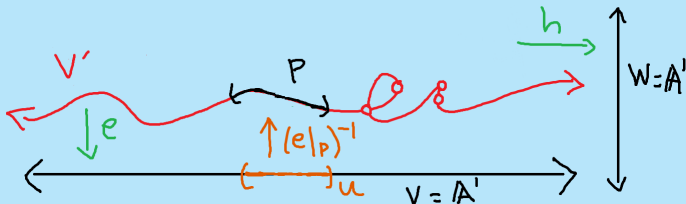
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Suppose K is large and not separably closed. $O \subseteq V(K)$ étale-open.

$f: O \rightarrow W(K)$ is **Nash** if $f = h \circ (e|_P)^{-1}$ for

$e: V' \rightarrow V$ étale, $h: V' \rightarrow W$ a morphism, e gives a homeo. $P \rightarrow O$.

Example: $\sqrt[n]{x}$ on a nbhd of 1 when $\text{Char}(K) \nmid n$.

Nash maps are closed under compositions.

Nash maps $O \rightarrow K$ form a ring.

Theorem (JTWY): The ring of germs of Nash maps to K on K^n at the origin is isomorphic to the ring of algebraic formal power series over K in n variables. (Real closed case well-known.)

$f \in K[[t_1, \dots, t_n]]$ is **algebraic** if algebraic over $K[t_1, \dots, t_n]$.

$K[[t_1, \dots, t_n]]^{\text{alg}}$ is a henselian local domain.

We can evaluate $f \in K[[t_1, \dots, t_n]]^{\text{alg}}$ on sufficiently small elements of K^n .

\mathbb{K} is a $|K|^+$ -saturated elementary extension of K .

Let $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \mathbb{K}^n$ be K -**infinitesimal**, i.e. in every nbhd of 0 def. over K .

$\text{Eval}_\varepsilon: K[[t_1, \dots, t_n]]^{\text{alg}} \rightarrow \mathbb{K}$, $\text{Eval}_\varepsilon(f) = f(\varepsilon_1, \dots, \varepsilon_n)$. A ring morphism.

Image of Eval_ε is a henselian local domain (they are closed under quotients).

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Image of Eval_ε is a henselian local domain (they are closed under quotients).

$K[[t_i]_{i < \kappa}]^{\text{alg}}$ is the ring of algebraic power series in κ variables.

Still a henselian local domain.

$\varepsilon = (\varepsilon_i)_{i < \kappa} \in \mathbb{K}^\kappa$ is **K -infinitesimal** if every finite subtuple is.

Can still define $\text{Eval}_\varepsilon: K[[t_i]_{i < \kappa}]^{\text{alg}} \rightarrow \mathbb{K}$, each series depends on only finitely many t_i .

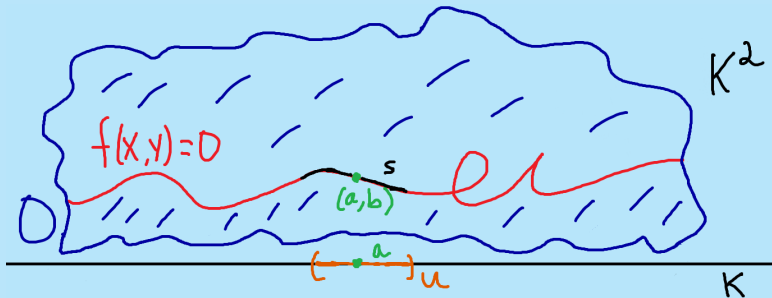
Final Lemma: \exists K -infinitesimal $\varepsilon = (\varepsilon_i)_{i < \kappa}$ s.t. the quotient field of the image of Eval_ε is an elementary extension of K .

IFT for Nash maps

Suppose K is large and not separably closed.

Theorem (W): Nash maps over K satisfy the inverse and implicit function theorems with respect to the étale-open topology.

Ex: $O \subseteq K^2$ open, $f: O \rightarrow K$ Nash, $(a, b) \in O$ s.t. $f(a, b) = 0 \neq \partial f / \partial y(a, b)$.



\exists nbhd $a \in U$, unique Nash $s: U \rightarrow K$ s.t. $s(a) = b$ and $f(a^*, s(a^*)) = 0 \forall a^* \in U$.

Roughly:

Large \iff inverse/implicit func. thm holds $\iff \equiv \text{Quot}(R)$ for R henselian

Thank you.

“Large implies Henselian”, on arxiv.