

δ -Topology and Curve Selection on CODF

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Summary

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Closed Ordered Differential Fields

Let R be an ordered ring. A **derivation** on R is an additive map $\delta: R \rightarrow R$ satisfying the Leibniz rule, $\delta(xy) = x\delta(y) + y\delta(x)$. An ordered ring equipped with such a derivation is called an **ordered differential ring**.

The theory of ordered differential integral domains possesses a model completion, known as the theory of **closed ordered differential fields** (CODF). Moreover, this theory also admits quantifier elimination in the language of ordered differential rings $\mathcal{L}^\delta = \{0, 1, +, \cdot, <, \delta\}$ [Singer 1978].

An important notion associated with the ordered differential fields is the **n -jet map**.

Definition (n -jet map)

Let (\mathcal{R}, δ) be an ordered differential field. For any $n \in \mathbb{N}$, the **n -jet map** $\text{Jet}_n: R \rightarrow R^{n+1}$ is defined by:

$$\text{Jet}_n(x) := (x, \delta(x), \delta^2(x), \dots, \delta^{(n)}(x))$$

For a subset $X \subseteq R$, the **n -jet set** of X is the image $\text{Jet}_n(X)$.

This map extends naturally to the multivariable case.

δ -Topology

Fix $(\mathcal{R}, \delta) \models \text{CODF}$. For each $n \in \mathbb{N}$, define the map $d_n: R \times R \rightarrow R$ by

$$d_n(x, y) = \max_{0 \leq i \leq n} |\delta^i x - \delta^i y|.$$

Each d_n induces an R -metric topology, denoted by \mathcal{T}_n . It is clear that $\mathcal{T}_n \subseteq \mathcal{T}_{n+1}$ for all n . The δ -**topology** on R is the union $\mathcal{T}_\delta = \bigcup_{n \in \mathbb{N}} \mathcal{T}_n$.

The δ -topology is the coarsest topology extending the Euclidean topology such that the derivation $\delta: R \rightarrow R$ is continuous., so it is considered as the natural topology associated with CODF. However, it is not \mathcal{L}^δ -definable.

Our ultimate goal is to determine whether the δ -closure of an \mathcal{L}^δ -definable set remains \mathcal{L}^δ -definable. As a first step toward this problem, we present a version of the curve selection lemma for this setting.

Geometry of \mathcal{L}^δ -definable Sets

Any \mathcal{L}^δ -definable set is exactly the projection of the intersection of $\text{Jet}_n(M)$ and a semialgebraic set $X \subseteq R^{n+1}$ for some $n \in \mathbb{N}$ (see [Fornasiero and Kaplan 2021]).

Let $A \subseteq R$ be \mathcal{L}^δ -definable. Then there exists a quantifier-free \mathcal{L} -formula $\varphi(x_0, \dots, x_n)$ such that $\varphi(x, \delta x, \dots, \delta^n x)$ defines A . Let $\Pi_k : R^N \rightarrow R^k$ be the projection map to the first k -coordinates for $N \geq k$. If we denote $A_\varphi \subseteq R^{n+1}$ the semialgebraic set defined by φ , then

$$A = \Pi_1(A_\varphi \cap \text{Jet}_n(M)).$$

Back to the R -metrics d_n , it is then easy to see that if we let $d_n^{\mathcal{L}} : R^{n+1} \times R^{n+1} \rightarrow R$ be

$$d_n^{\mathcal{L}}(x, y) = \max_{0 \leq i \leq n} |x_i - y_i|,$$

then (R, d_n) is isometric to $(\text{Jet}_n(R), d_n^{\mathcal{L}}|_{\text{Jet}_n(R)}) \subseteq (R^{n+1}, d_n^{\mathcal{L}})$ via the map Jet_n .

Lifting

The following fact is a special case from Fornasiero and Kaplan 2021. Let $k \in \mathbb{N}$ and $V \subseteq R^k$ be Nash and $g : U \rightarrow R$ be a Nash $\mathcal{L}(R)$ -definable function with variables $x^{(0)}, \dots, x^{(k)}$, and let \mathcal{F}_k denote the set of all such $\mathcal{L}(R)$ -definable functions. Define the maps $\mathcal{S} : \mathcal{F}_k \rightarrow \mathcal{F}_{k+1}$ by

$$\begin{aligned} & \mathcal{S}(g)(x^{(0)}, \dots, x^{(k+1)}) \\ &= \left(g^{[\delta]}(x^{(0)}, \dots, x^{(k)}) + \mathbf{J}_g(x^{(0)}, \dots, x^{(k)}) \cdot \begin{pmatrix} x^{(1)} \\ \vdots \\ x^{(k+1)} \end{pmatrix} \right) \Big|_{\Gamma(g)}, \end{aligned}$$

where $g^{[\delta]} : V \rightarrow R$ is a unique Nash function associated with δ such that $\mathcal{S}(g)(x, \delta x, \dots, \delta^{k+1}x) = \delta(g(x, \delta x, \dots, \delta^k x))$ for every $x \in R$ with $g(x, \delta x, \dots, \delta^k x) = \delta^{k+1}x$. Set $\mathcal{S}^{\circ 1} = \mathcal{S}$. Suppose that $\mathcal{S}^{\circ i}$ is defined. Define $\mathcal{S}^{\circ i+1} = \mathcal{S} \circ \mathcal{S}^{\circ i}$. Note that $\{\mathcal{S}^{\circ k}(g) | k \geq 1\}$ are diffeomorphic via the projection map Π^{k+1} .

δ -Cells and δ -Curves

From now on, we work in \mathbb{R} and we only consider an \mathcal{L}^δ -definable set $X \subseteq \mathbb{R}$. By a cell decomposition process (for some insights on how to achieve this, see Brihaye, Michaux, and Rivière 2009 and Cubides Kovacsics and Point 2023), we may assume that there exists a Nash function $f : U \rightarrow \mathbb{R}$ over an open Nash cell $U \subseteq \mathbb{R}^n$ such that $\text{Jet}_n(X)$ is Euclidean dense in $\Gamma(f)$.

For each $k \geq n$, define $X_{n-1} = U$, $X_n = \Gamma(f)$ and for each $k > n$, $X_k = \Gamma(\mathcal{S}^{\circ k-n}(f))$. By the previous discussion, we have the following.

Fact

For fixed $k \geq n - 1$, $a \in \text{cl}_k(X)$ if and only if $\text{Jet}_k(a) \in \text{cl}(X_k)$. Thus, $a \in \text{cl}_\delta(X)$ if and only if $\text{Jet}_k(a) \in \text{cl}(X_k)$ for all $k \geq n - 1$.

Since X_k 's are semialgebraic, we have that $a \in \text{cl}_k(X)$ if and only if there exists a Nash curve $\gamma : (0, 1) \rightarrow X_k$ such that $\lim_{t \rightarrow 0} \gamma(t) = \text{Jet}_k(a)$. Moreover, since the X_k 's are diffeomorphic, for any $k \geq n - 1$, a Nash curve in X_k is uniquely determined by a Nash curve in U . This observation motivates using Nash curves in U to give a form a sensible curve selection lemma, even though the curve $\gamma : (0, 1) \rightarrow X_n$ does not necessarily intersect with the jet space $\text{Jet}_n(\mathbb{R})$.

Setting up the Curve Selection

Without loss of generality, after translation, we may assume that $a = 0$. For each $i = 1, \dots, n$, let x_i be the i^{th} coordinate function. Define $i_n = f$ and for each $k > n$, such that i_k is defined, define $i_{k+1} = \mathcal{S}^{k-n}(f) \circ (x_1, \dots, x_n, i_n, \dots, i_k)$, and for each $k \geq n$, define the function $f_k : U \rightarrow X_k$ by setting $f_k = (x_1, \dots, x_n, i_n, \dots, i_k)$. Essentially, f_k is the inverse of $\Pi_n|_{X_k}$. The a Nash curve $\gamma : (0, 1) \rightarrow U$ uniquely determines a sequence of curves $(f_k \circ \gamma)_{k \in \mathbb{N}}$. The goal is find a “nice” curve $\gamma : (0, 1) \rightarrow U$ such that $\lim_{t \rightarrow 0} f_k \circ \gamma(t) = \bar{0}$ for all $k \geq 0$.

As we have discussed previously, for fixed $k \geq n - 1$, there exists a Nash curve $\gamma : (0, 1) \rightarrow X_k$ such that for all $n < j \leq k + 1$, we have $\lim_{t \rightarrow 0} \Pi_j \circ \gamma(t) = \bar{0}$. However, there is no reason that $\lim_{t \rightarrow 0} \mathcal{S}^{\circ(k+1)}(f) \circ \gamma(t) = \bar{0}$ because the lifting map \mathcal{S} does not guaranteed any continuity at the boundary of X_k , and thus distinct paths in X_k could potentially result in distinct limits in X_{k+1} .

Germ

Let K be the Hardy field of a semi-algebraic functions. Define \mathcal{G} to be the set of germs of semialgebraic curves approaching the origin in \mathbb{R}^n . Define an K -metric $d_{\mathcal{G}} : \mathcal{G} \times \mathcal{G} \rightarrow K$ by

$$d_{\mathcal{G}}([\gamma_1], [\gamma_2]) = [f(t)],$$

where $f(t) = D(\text{cl}(\gamma_1 \cap B(\bar{0}, t)), \text{cl}(\gamma_2 \cap B(\bar{0}, t)))$ and D is the Hausdorff distance function. Let

$$\mathcal{G}_k := \left\{ [\gamma] \in \mathcal{G} \mid \gamma \subseteq U, \lim_{t \rightarrow 0} (f_k \circ \gamma)(t) = \bar{0} \right\}.$$

Then for each $k \geq n - 1$, $\mathcal{G}_{k+1} \subseteq \mathcal{G}_k \subseteq \mathcal{G}$.

More on Germs

Assume that $n > 1$. Consider K^n with the Euclidean topology. Consider the quotient topological space K^n / \sim where $\gamma = (\gamma_1, \dots, \gamma_k) \sim (\sigma_1, \dots, \sigma_k) = \sigma$ if there exists $i = 1, \dots, k$ such that $\gamma_i \neq 0$, $\sigma_i \neq 0$ and $(\gamma_i^{-1} \circ \gamma_1, \dots, \gamma_i^{-1} \circ \gamma_k) = (\sigma_i^{-1} \circ \sigma_1, \dots, \sigma_i^{-1} \circ \sigma_k)$. It is clear that $[\gamma] = [\sigma]$ in \mathcal{G} if and only if $\gamma \sim \sigma$.

For each $1 \leq i \leq n$, let $H_i^+ = \{x_i > 0\}$ and $H_i^- = \{x_i < 0\}$ be the half spaces. Then $\{H_i^+, H_i^- | i = 1, \dots, n\}$ is a covering of K^n , and one of H_i^\pm intersects all but finitely many \mathcal{G}_i 's.

Without loss of generality, assume that for each $k \geq n - 1$, for all $\gamma \in \text{cl}_{d_{\mathcal{G}}}(\mathcal{G}_k)$, $\gamma \subseteq H_n^+$. Then each $(\mathcal{G}_k, d_{\mathcal{G}})$ can be identified as a subspace of (K^{n-1}, d) .

Even More on Germs

Note that $[\gamma] \in \mathcal{G}_k$ if and only if for sufficiently small $\varepsilon > 0$, $\gamma|_{(0,\varepsilon)} \subseteq U$ and $\text{st}([f_k \circ \gamma]) = \bar{0}$. Thus, \mathcal{G}_k is definable in the structure (K, V) , where V is the convex hull of \mathbb{R} . Since $(\mathcal{G}_k)_{k \geq n-1}$ is nonincreasing and has finite intersection property, there exists an \aleph_1 -saturated structure $R^* > K$ and $\gamma^* \in R$ such that γ^* realizes $\bigcap_{k \geq n-1} \mathcal{G}_k$.

Proposition

If $a \in \text{cl}_\delta(X)$, then there exists γ^ realizing $\bigcap_{k \geq n-1} \mathcal{G}_k$ such that for any $k \geq n-1$ and any semialgebraic neighbourhood V of γ^* that is definable in \mathcal{G} , there exists $[\gamma] \in V$ such that $\lim_{t \rightarrow 0} f_k \circ \gamma(t) = \bar{0}$.*

Explicit Form

The ordered ring of Puiseux series over \mathbb{R} is the union of former Laurent series over \mathbb{R} in variable $t^{1/k}$ where k ranges in positive integers. That is

$$\mathbb{R}\{\{t\}\} = \bigcup_{k=1}^{\infty} \mathbb{R}((t^{1/k})).$$

By Newton-Puiseux Theorem, we have the following.

Fact

K embeds into the ordered field of Puiseux series $\mathbb{R}\{\{t\}\}$. Furthermore, $\mathbb{R} < K < \mathbb{R}\{\{t\}\}$, and the embedding is honest in the sense that if $[\gamma]$ is mapped to $f(x) \in \mathbb{R}\{\{t\}\}$, then $[f(x)] = [\gamma]$. In particular, $K = \mathbb{R}\{\{t\}\}^{alg}$, the field of algebraic Puiseux series.

There is a sequence of honest embedding.

$$\mathbb{R} \hookrightarrow K \hookrightarrow \mathbb{R}\{\{t\}\} \hookrightarrow \mathbb{R}[[t^{\mathbb{R}}]] \hookrightarrow \mathbb{R}[[t^{\Gamma}]],$$

where $\mathbb{R}[[t^{\mathbb{R}}]]$ is the Hahn field over \mathbb{R} with valued group \mathbb{R} and Γ is a sufficiently saturated ODAG extending \mathbb{R} .





Realizing Cuts

Each coordinate of a realization $\gamma^* \in \mathbb{R}[[t^\Gamma]]$ of $\bigcap_{k=n-1}^{\infty} \mathcal{G}_k$ has one of the following form:

- (i) $\sum_{k \in \omega} a_k t^{q_k} \in \mathbb{R}[[t^{\mathbb{Q}}]]$, so $(q_k)_{k \in \mathbb{N}}$ is a strictly increasing sequence in \mathbb{Q} ;
- (ii) $P(t^{\frac{1}{k}}) + t^r$ for some positive integer k , $P(X) \in \mathbb{R}[X]$, $r \in \Gamma \setminus \mathbb{Q}$ such that $r > \text{ord}(P)/k$.

Note that $r > 0$. If r is positively infinite, then t^r decays faster than any Puiseux series, so we may set $t^r = e^{-\frac{1}{t}}$; if r is a positive infinitesimal, then t^r decays slower than any Puiseux series, then replace t^r by $-\frac{1}{\ln(t)}$; if r is bounded and not an infinitesimal, then set $\gamma^* = t^{\text{st}(r) \pm t}$. Thus, γ^* has an interpretation definable in $\mathbb{R}_{\text{an}^*, \text{exp}}$.

Reference

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