Hrushovski constructions in ordered fields

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Introduction

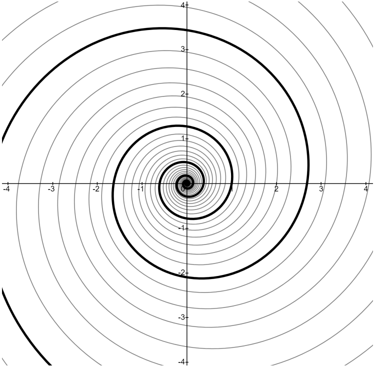
- Construct the theory of the real field with dense logarithmic spirals (Green points).
- Construct the theory of the real field with power functions on the unit circle (Raising to powers).
- Both expansions preserve the open core.

Green points – model

Let $(\mathbb{R}, G)$ be the expansion of the real field by:

$$G = \exp(\epsilon\mathbb{R} + \mathbb{Q}) = \{e^{\epsilon t+s} \in \mathbb{R}^2 \mid t \in \mathbb{R}, s \in \mathbb{Q}\},$$

where $\epsilon = 1 + \beta i$ for some $\beta \in \mathbb{R} \setminus \{0\}$.



Green points – history

Theorem ([Poizat, 2001])

There is a rank $\omega \cdot 2$ expansion of ACF by a divisible multiplicative subgroup, called green points.

Theorem ([Zilber, 2004])

Assume SC. Then $(\mathbb{C}, G)$ is a model of Poizat's theory of green points.

Theorem ([Baudisch et al., 2009])

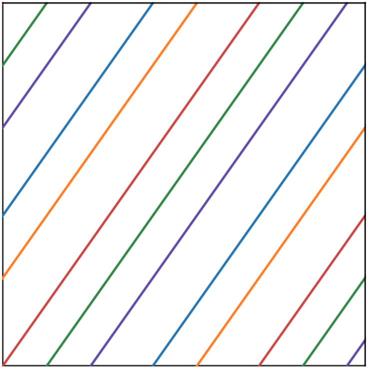
There is a rank 2 expansion of ACF by a divisible multiplicative subgroup which has rank 1.

Power functions

Consider $(\mathbb{R}, \{\Gamma_r\}_{r\in\mathbb{R}})$ where

$$\Gamma_r := \{(e^{it}, e^{irt}) \in S^1 \times S^1 \mid t \in \mathbb{R}\}.$$

Definable open sets in $(\mathbb{R}, \{\Gamma_r\}_{r\in\mathbb{R}})$?



Raising to powers – model

Let K be a subfield of $\mathbb{R}$.
The previous structure $(\mathbb{R}, \{\Gamma_r\}_{r \in K})$ is a reduct of the following two-sorted structure:

$$\mathbb{R}^K := \left(\mathbb{R}_{K-\mathsf{VS}} \xrightarrow{\exp} S^1 \subseteq \mathbb{R}^2 \right),$$

where the first sort is $\mathbb{R}$ being a K-vector space, the second sort is the real ordered field $\mathbb{R}$, and $\exp(t) = e^{it}$.

Raising to powers – history

$$\mathbb{C}^K := \left(\mathbb{C}_{K-\mathrm{VS}} \xrightarrow{\exp} \mathbb{C}_{\mathrm{field}}\right).$$

Theorem ([Zilber, 2003])

There is a superstable expansion of ACF by raising to powers, axiomatized by $\mathrm{SC}_K \cup \mathrm{EC}_\mathrm{p}$.

Theorem ([Bays and Kirby, 2018])

If $\mathbb{C}_{\exp}$ *satisfies* EC_e, *then it is quasiminimal.*

Theorem ([Gallinaro and Kirby, 2024])

$\mathbb{C}^K$ *satisfies* EC_p. *As a result, it is quasiminimal.*

Hrushovski construction

- L : some language
- $\mathcal{C}$: some class of L-structures
- δ : some predimension function defined on structures in $\mathcal{C}$

Definition

Let $X, Y \in \mathcal{C}$. An embedding $i : X \to Y$ is a *strong* embedding (denoted by $i : X \leq Y$) if

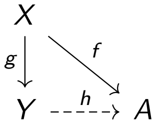
$$\forall \mathbf{y} \subseteq Y, \, \delta(\mathbf{y}/X) \geq 0.$$

Richness

- $\mathcal{C}^{\text{fin}} := \{X \in \mathcal{C} \mid X \text{ is finitely generated}\}$

Definition

A structure $A \in \mathcal{C}$ is *rich* if for any $X, Y \in \mathcal{C}^{\text{fin}}$ and $f : X \leq A$, $g : X \leq Y$, there is $h : Y \leq A$ such that $h \circ g = f$.



If $\leq$ is the usual L-embedding, then we get the Fraïssé limit.

Green points – results

Let $\mathcal{C}_g$ be the class of (R, G) where

- $R \models \mathrm{RCF}$,
- G is a divisible multiplicative subgroup of $R^2 \models \mathrm{ACF}$,
- $\delta(\mathbf{x}) = \mathrm{trd}(\mathbf{x}) - \mathrm{md}(\mathbf{x} \cap G)$, where $\mathbf{x} \subseteq R^2$,
- $\delta(\mathbf{x}) \geq 0$ for all $\mathbf{x} \subseteq R^2$.

Theorem ([Zhang, 2026])

Rich structures in $\mathcal{C}_g$ are exactly ω-saturated models of T_g.

Lemma ([Zhang, 2026])

In every model of T_g, every definable open set is semialgebraic.

Theorem ([Zhang, 2026])

Assume SC_K holds for $K = \mathbb{Q}(\beta i)$. Then $(\mathbb{R}, G)$ is a model of T_g.

Open core

Corollary ([Boxall and Hieronymi, 2012, Corollary 3.1])

Let $\mathcal{M}^*$ be an expansion of $\mathcal{M}$ by language. Let $C \subseteq M$ be a small set. Suppose for every $n \in \mathbb{N}$ there is a set $D_n \subseteq M^n$ such that the following conditions hold:

1. *D_n is dense in M^n,*
2. *for any $\mathbf{x} \in D_n$ and any open set $U \in M^n$, if $\mathrm{tp}_{\mathcal{M}}(\mathbf{x}|C)$ is realized in U then it is realized in $U \cap D_n$,*
3. *for any $\mathbf{x} \in D_n$, the conjunction of $\mathrm{tp}_{\mathcal{M}}(\mathbf{x}|C)$ and $\mathbf{x} \in D_n$ implies $\mathrm{tp}_{\mathcal{M}^*}(\mathbf{x}|C)$.*

Then every open set definable over C in $\mathcal{M}^*$ is definable over C in $\mathcal{M}$.

Powered fields

Definition

A *K-powered field* is a tuple (V, R, ex) consisting of:

- a real closed field R,
- its unit circle $S = \{(x, y) \in R^2 \mid x^2 + y^2 = 1\}$ as a multiplicative subgroup of R^2,
- a K-vector space V,
- a surjective group homomorphism $\mathrm{ex} : V \to S$ whose kernel kex being a $\mathbb{Z}$-group.

- $\delta(\mathbf{x}) := \mathrm{ld}^K(\mathbf{x}) + \mathrm{trd}(\mathrm{ex}(\mathbf{x})) - \mathrm{ld}^{\mathbb{Q}}(\mathbf{x})$, where $\mathbf{x} \subseteq V$,
- Let $\mathcal{C}_0$ be the class of all K-powered fields (V, R, ex) such that $\delta(\mathbf{x}) \geq 0$ for all $\mathbf{x} \subseteq V$

Axiomatizating $\mathcal{C}_0$

Lemma

There is a set of axioms T_0 *such that* $(V, R, \mathrm{ex}) \models T_0$ *if and only if* $\delta(\mathbf{x}) \geq 0$ *for all* $\mathbf{x} \subseteq V$.

Axiomatizing kernel

Let $f_{r_1,\ldots,r_n}$ be the following map:

$$\begin{aligned} V &\longrightarrow S^n \\ x &\longmapsto (\mathrm{ex}(r_1 x), \ldots, \mathrm{ex}(r_n x)). \end{aligned}$$

Let KN be the set of axioms stating that, for each $(r_1, \ldots, r_n) \in K$ such that $(1, r_1, \ldots, r_n)$ is $\mathbb{Q}$-linearly independent, $f_{r_1,\ldots,r_n}(i + k \cdot \mathrm{kex})$ is dense in S^n for all natural numbers $0 \le i < k$.

Fact

Let $a_1, \ldots, a_n \in \mathbb{R}$ such that $(1, a_1, \ldots, a_n)$ is $\mathbb{Q}$-linearly independent. Let f be the following map:

$$\begin{aligned} \mathbb{R} &\longrightarrow (\mathbb{R}/\mathbb{Z})^n \\ x &\longmapsto ([a_1 x], \ldots, [a_n x]) \end{aligned}$$

Then $f(\mathbb{Z})$ is dense in $(\mathbb{R}/\mathbb{Z})^n$.

Axiomatizing richness

Let D be a K-special subspace of V^n, and let W be a subvariety of S^n.

Definition

We say $D \times W$ is

- *free* if both $\mathbf{a}(D)$ and $W^{\mathbf{a}}$ have dimension 1 for every $\mathbf{a} \in \mathbb{Z}^n \setminus \{0\}$,
- *rotund* if
$$\dim W^M + \dim M(D) \geq k$$
for every $M \in \mathbb{Z}^{k \times n}$ of rank k.

Let EC be the set of axioms stating that, for each rotund and free pair $D \times W \subseteq V^n \times S^n$, the image $\mathrm{ex}(D)$ is dense in W.

Lemma

$\mathbb{R}^K$ *satisfies* EC.

Raising to powers – results

Let $T_1 = T_0 \cup \mathrm{KN} \cup \mathrm{EC}$.

Theorem

Rich structures in $\mathcal{C}_0$ are ω-saturated models of T_1, and ω^+-saturated models of T_1 are rich structures in $\mathcal{C}_0$.

Lemma

Let $(V, R, \mathrm{ex}) \models T_1$, every definable open subset of R^n is semialgebraic.

Theorem

Assume SC_K. Then $\mathbb{R}^K$ is a model of T_1.

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