Structure theorems and applications in semi-bounded and tame pairs

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Panteleimon Eleftheriou

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Part 0

Résumé

RÉSUMÉ

The common topic of the six works presented in this volume is the analysis of definable sets in certain well-behaved model-theoretic pairs of first-order structures. Articles [1-3] concern pairs $\langle \mathcal{M}, P \rangle$, where $\mathcal{M}$ is a linear o-minimal structure and P is an o-minimal expansion of a real closed field defined on a bounded interval in M. Such a pair is called *semi-bounded*. Article [1] establishes a structure theorem for all definable sets in a semi-bounded pair. Articles [2, 3] provide an application of this structure theorem to definable groups. Namely, every group definable in $\langle \mathcal{M}, P \rangle$ is analysed in terms of semi-linear groups (definable in $\mathcal{M}$) and groups definable in P. Article [4] proves a special case of a conjecture that arose in [2]. Articles [5, 6] concern recent work in pairs $\langle \mathcal{M}, P \rangle$, where $\mathcal{M}$ is an o-minimal structure and P is a dense subset of M, such that certain tameness conditions hold. Such a pair is called a *tame pair*. Article [5] establishes a structure theorem for all definable sets in a tame pair, generalizing the well-known cell decomposition theorem for o-minimal structures. Article [6] provides an application of this structure theorem to point counting theorems. Namely, it extends the influential Pila-Wilkie theorem from the o-minimal setting to the general tame setting.

Background

Definable groups have always been at the core of model theory, largely because of their prominent role in important applications of the subject, such as Hrushovski's proof of the function field Mordell-Lang conjecture in all characteristics ([Hr]). Examples include algebraic groups (which are definable in algebraically closed fields) and real Lie groups (which are definable in o-minimal structures). An indispensable tool in their analysis has been a structure theorem for the definable sets and types: analyzability of types and the existence of a rank in the stable category, and a cell decomposition theorem and the associated topological dimension in the o-minimal setting. In this habilitation, we establish structure theorems for definable sets and functions in semi-bounded and tame pairs. In the first case, we use the structure theorem to solve the compact domination conjecture in expansions of ordered groups. In the second case, we use the structure theorem to extend the Pila-Wilkie theorem to the general tame setting.

O-minimal structures were introduced and first studied by van den Dries [Dries1] and Knight-Pillay-Steinhorn [KPS, PS] and have since provided a rigid framework to study real algebraic and analytic geometry. The starting point for the study of definable groups was Pillay's theorem in [Pillay] that every such group admits a definable manifold topology that makes it into a topological group. Since then, an increasing number of theorems have reinforced the resemblance of o-minimal groups with real Lie groups, culminating in the solution of Pillay's Conjecture (PC) and Compact Domination Conjecture (CDC) in recent years. (PC) can be viewed as a non-standard analogue of Hilbert's fifth problem. In its simplified form, it asserts that every definably connected, definably compact group G admits a surjective homomorphism π onto a real Lie group, whose dimension (as a Lie group) is equal to the o-minimal dimension of G. (CDC) carries this connection further and implies that π induces a unique left-invariant *Keisler measure* on the collection of all definable subsets of G.

The standard setting for studying these conjectures has been that of an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0 \ldots \rangle$ of an ordered group (although, in [EPR] and [EMPRT], the assumption of the ambient group structure is removed.) Prior to the work presented in this volume, the two conjectures were established in the case $\mathcal{M}$ is a pure ordered vector space over an ordered division ring ([ES], [El]), henceforth called a 'linear o-minimal structure', and in the case $\mathcal{M}$ expands a real closed field ([HPP], [HP]). Peterzil [Pet] combined the two settings and settled (PC) for a general $\mathcal{M}$ as above. The work in the linear case actually yields a stronger characterization of semi-linear groups as quotients of a $\bigvee$-definable subgroup $\mathcal{U} \leqslant \langle M^n, + \rangle$ by a lattice. Articles [1-3] generalize this strong characterization to groups definable in a semi-bounded pair $\langle \mathcal{M}, P \rangle$ and obtain from it the solution of (CDC) in the general case.

Tame expansions of o-minimal structures have been developed as a context that escapes the o-minimal, locally finite one, yet preserves the tame geometric behavior on the class of all definable sets. An important category of such structures are those where every open definable set is already definable in the o-minimal reduct. The primary example is that of the real field expanded by the subfield of real algebraic numbers, studied by A. Robinson in his classical paper [Ro], where the decidability of its theory was proven. Forty years later, van den Dries [Dries2] extended Robinson's results to arbitrary dense pairs of o-minimal structures, and a stream of further developments in the subject followed ([BZ, BEG, BH, DMS1, DMS2, DG, GH, MS]). Besides dense pairs, examples of structures in this category now include pairs of the form $\langle \mathcal{M}, P \rangle$, where $\mathcal{M}$ is an o-minimal expansion of an ordered group, and P is a dense multiplicative subgroup with the Mann property, or a dense subgroup of the unit circle or of an elliptic curve, or it is a dense independent set. In [5], we establish a cone decomposition theorem and develop the associated dimension function in a general setting that includes the above pairs, extending the known cell decomposition theorem from o-minimal structures and the usual o-minimal dimension. A local analysis for definable groups is also obtained. In [6], we give an independent, new application of this structure theorem; namely, we extend the Pila-Wilkie theorem to the general tame setting.

Description of articles

1. Local analysis for semi-bounded groups. In this article, we establish a structure theorem for definable sets in a semi-bounded pair $\langle \mathcal{M}, P \rangle$. Following [Pet], a definable set $X \subseteq M^n$ is called *short* if it is in definable bijection with a set definable in P. Otherwise, it is called *large*. Previous work by Edmundo and Peterzil provided cone decomposition theorems for definable sets with respect to the dichotomy 'bounded versus unbounded'. Peterzil [Pet] conjectured a refined cone decomposition with respect to the dichotomy 'short versus long'. This article proves Peterzil's conjecture. The notions of a *long cone* and *almost linear* are a bit technical and we postpone them until the actual article.

Theorem 0.1. *Every A-definable set $X \subseteq M^n$ is a finite union of A-definable long cones. Furthermore, for every A-definable function $f : X \subseteq M^n \to M$, there is a finite collection $\mathcal{C}$ of A-definable long cones, whose union is X and such that f is almost linear with respect to each long cone in $\mathcal{C}$.*

We also introduce a new closure operator that defines a pregeometry and gives rise to the refined notions of 'long dimension' and 'long-generic' elements. Those are in turn used in a local analysis for semi-bounded groups:

Theorem 0.2. *Let $G = \langle G, \oplus \rangle$ be a definable group of long dimension k. Then every long-generic element a in G is contained in a k-long cone $C \subseteq G$, such that for every $x, y \in C$,*

$$x \ominus a \oplus y = x - a + y.$$

In particular, on C, G is locally isomorphic to $\langle M^k, + \rangle$.

2. Definable quotients of locally definable groups. Here we work in an arbitrary o-minimal expansion $\mathcal{M}$ of an ordered group, and study those $\bigvee$-definable groups which arise as covers of definable groups. Earlier, in [EE], it was shown that for every abelian, definably connected definable group G, there is a connected, divisible, torsion-free $\bigvee$-definable group $\mathcal{U}$ and a surjective $\bigvee$-definable homomorphism $\phi : \mathcal{U} \to G$, whose kernel has dimension 0. Such a $\mathcal{U}$ is called the *universal cover* of G. In [2], we introduced the following notion.

Definition 0.3. Given a $\bigvee$-definable group $\mathcal{U}$ and a normal subgroup $L \subseteq \mathcal{U}$, the quotient group $\mathcal{U}/L$ is called *definable* if there is a definable group K and a surjective $\bigvee$-definable homomorphism $\phi : \mathcal{U} \to K$ whose kernel is L. We write $K = \mathcal{U}/L$.

The main theorem of the article is the following. It is crucially used in the subsequent article [3], as we sketch in the proof of Theorem 0.5 below.

Theorem 0.4. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group, which is definably generated. Then the following are equivalent:*

(1) $\mathcal{U}$ contains a definable generic set; namely, a set whose boundedly many translates cover $\mathcal{U}$.
(2) $\mathcal{U}^{00}$ exists; that is, $\mathcal{U}$ contains a smallest type-definable subgroup $\mathcal{U}^{00}$ of bounded index.
(3) $\mathcal{U}$ contains a normal subgroup L of dimension 0, such that $\mathcal{U}/L$ is definable.

Moreover, the above conditions imply that both $\mathcal{U}$ and $\mathcal{U}^{00}$ are divisible.

We also stated the following conjecture, which stands open till now.

Conjecture 0.5. *For $\mathcal{U}$ as above, conditions (1)-(3) hold.*

3. Definable groups as homomorphic images of semilinear and field-definable groups. In this article, we complete the analysis of groups definable in semi-bounded pairs. The main theorem is as follows:

Theorem 0.6. *Assume $\mathcal{N} = \langle \mathcal{M}, P \rangle$ is semi-bounded. Let G be an abelian, definably connected, definably compact definable group of long dimension k. Denote by $\phi : \mathcal{U} \to G$ its universal cover. Then there is a short exact sequence of $\bigvee$-definable groups*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \longrightarrow & \mathcal{U} & \xrightarrow{\sigma} & \mathcal{K} & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \phi} & & & & \\ & & & & G & & & & \end{array}$$

where $\mathcal{H}$ is definably generated in $\mathcal{M}$, and $\mathcal{K}$ is definably generated in P.

A corollary of this theorem is the solution to (CDC) in $\mathcal{N} = \langle \mathcal{M}, P \rangle$, and hence in any o-minimal expansion of an ordered group.

Theorem 0.7. *The Compact Domination Conjecture holds in semi-bounded pairs.*

Sketch of the proof. First, reduce the conjecture to the universal cover $\mathcal{U}$ of G and then to the $\bigvee$-definable groups $\mathcal{H}$ and $\mathcal{K}$. For $\mathcal{H}$, methods from the semi-linear case apply and we can easily see that $\mathcal{H}$ is compactly dominated. On the other hand, unless we know that $\mathcal{K}$ is a cover of a definable group in P, we cannot conclude it is compactly dominated. We see that $\mathcal{K}$ is such a cover by applying Theorem 0.4 twice, as follows. Since $\mathcal{U}$ is a cover of a definable group in $\mathcal{N}$, it contains a definable generic set X. Then $\sigma(X)$ is generic in $\mathcal{K}$, and hence $\mathcal{K}$ is a cover of a group definable in P. □

4. Lattices in locally definable subgroups of $\langle R^n, +\rangle$. In this article, we solve Conjecture 0.1 in the case $\mathcal{U}$ is a $\bigvee$-definable subgroup of some cartesian power of $\langle M, +\rangle$. Moreover, we reduce the conjecture to a simple statement that we describe next. First, we introduce the notion of '$\bigvee$-dimension' for $\mathcal{U}$, which assists us in doing some inductive arguments. The $\bigvee$-dimension intends to count how 'non-definable' $\mathcal{U}$ is. Let us call a $\bigvee$-definable set $A \subseteq \mathcal{U}$ *compatible in* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap A$ is a definable set.

Definition 0.8. The $\bigvee$*-dimension* of $\mathcal{U}$, denoted by vdim($\mathcal{U}$), is the maximum k such that $\mathcal{U}$ contains a compatible subgroup isomorphic to $\mathbb{Z}^k$, if such k exists, and ∞, otherwise.

Theorem 0.9. *Conjecture 0.1 is true if and only if, for every $\mathcal{U}$ that satisfies the assumptions of the conjecture, the following hold:*

(1) If $\mathcal{U}$ is not definable, then vdim($\mathcal{U}$) > 0.
(2) vdim($\mathcal{U}$) $\leq \dim(\mathcal{U})$. *(In particular,* vdim($\mathcal{U}$) *is finite.)*

In [BEM], property (2) was established, so Conjecture 0.1 reduces to property (1).

5. Structure theorems in tame expansions of o-minimal structures by a dense set. In this article we study pairs $\mathcal{N} = \langle \mathcal{M}, P\rangle$, where $P \subseteq M^n$ and three tameness conditions hold. Let use call a definable set $X \subseteq M^n$ *large* if there is an $\mathcal{L}$-definable map $f : M^{nk} \to M$ such that $f(X^k)$ contains an open interval. Otherwise, it is called *small*. We impose the following three conditions on $\mathcal{N}$:

(I) P is small,
(II) $Th(\mathcal{N})$ is near-model complete, and
(III) every open definable open set is definable in $\mathcal{M}$.

Under these assumptions, it turns out that a definable set is small if and only if it is internal to P. It is shown in [5] that the following examples fall into this category: (a) dense pairs, (b) expansions of the real field by a multiplicative subgroup with the Mann property, or by a dense subgroup of the unit circle or of an elliptic curve, (c) expansions by a dense independent set. The main result of this article is a structure theorem for definable sets and functions in $\mathcal{N}$. This theorem is inspired by the cone decomposition theorem (Theorem 0.1) in semi-bounded pairs. Again, the notions of a *cone* and *fiber $\mathcal{M}$-definable* are a bit technical and we postpone them until the actual article.

Theorem 0.10. *Every A-definable set $X \subseteq M^n$ is a finite union of A-definable cones. Furthermore, for every A-definable function $f : X \to M$, there is a finite collection $\mathcal{C}$ of A-definable cones, whose union is X and such that f is fiber $\mathcal{M}_A$-definable with respect to each cone in $\mathcal{C}$.*

We also introduce and analyze the relevant notions of dimension (called 'large dimension') and generics (called 'large-generics'), and establish a local theorem for definable groups in this setting:

Theorem 0.11. *Let $G = \langle G, *\rangle$ be a definable group of large dimension k. Then for every large-generic element a in G, there is a $2k$-cone $C \subseteq G \times G$, whose topological closure contains (a, a), and on which the operation*

$$(x, y) \mapsto x * a^{-1} * y$$

is given by an $\mathcal{L}$-definable map.

6. Counting algebraic points in expansions of o-minimal structures by a dense set. Pila's recent solution of certain cases of the André-Oort Conjecture ([Pila]) makes a beautiful use of o-minimal geometry, together with some number theoretic and functional transcendence results around Ax-Schanuel. Let us roughly state Pila-Zannier's approach to arithmetical problems, in the case of the Manin-Mumford Conjecture. The goal is to prove that if an algebraic subvariety X (of a given abelian variety V) contains many torsion-points of V (that is, those should be Zariski dense in X), then X contains an abelian variety. The Pila-Zannier approach begins with applying the Pila-Wilkie theorem ([PW]), which is a counting theorem from o-minimal geometry, to prove that such an X contains an infinite semi-algebraic set A. The Ax-Schanuel results are then employed to prove that a maximal such A is actually an abelian variety contained in X.

In [6], we extend the Pila-Wilkie theorem to the general tame setting, as follows. The Pila-Wilkie theorem states that if a set $X \subseteq \mathbb{R}$ is definable in an o-minimal structure $\mathcal{R}$ and contains 'many' rational points, then it contains an infinite semialgebraic set. Let $\mathcal{N} = \langle \mathcal{M}, P\rangle$ be a tame pair, where P is either a dense elementary substructure of $\mathcal{R}$, or a dense independent set. We show that if X is definable in $\mathcal{N}$ and contains many rational points, then it is dense in an infinite semialgebraic set. Moreover, it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$, where $\overline{\mathbb{R}}$ is the real field. Along the way, we introduce the notion of the *algebraic trace part* of any set $X \subseteq \mathbb{R}^n$, generalizing the usual notion of the algebraic part of X.

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Part 1

Local analysis for semi-bounded groups

LOCAL ANALYSIS FOR SEMI-BOUNDED GROUPS

PANTELIS E. ELEFTHERIOU

Abstract. An o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group is called *semi-bounded* if it does not expand a real closed field. Possibly, it defines a real closed field with bounded domain $I \subseteq M$. Let us call a definable set *short* if it is in definable bijection with a definable subset of some I^n, and *long* otherwise. Previous work by Edmundo and Peterzil provided structure theorems for definable sets with respect to the dichotomy 'bounded versus unbounded'. In [Pet3], Peterzil conjectured a refined structure theorem with respect to the dichotomy 'short versus long'. In this paper, we prove Peterzil's conjecture. In particular, we obtain a quantifier elimination result down to suitable existential formulas in the spirit of [vdD1]. Furthermore, we introduce a new closure operator that defines a pregeometry and gives rise to the refined notions of 'long dimension' and 'long-generic' elements. Those are in turn used in a local analysis for a semi-bounded group G, yielding the following result: on a long direction around each long-generic element of G the group operation is locally isomorphic to $\langle M^k, + \rangle$.

1. Introduction

For an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group, there are naturally three possibilities: $\mathcal{M}$ is either (a) linear, (b) semi-bounded (and non-linear), or (c) it expands a real closed field. Let us define the first two.

Definition 1.1. Let Λ be the set of all partial $\emptyset$-definable endomorphisms of $\langle M, <, +, 0 \rangle$, and $\mathcal{B}$ the collection of all bounded definable sets. Then $\mathcal{M}$ is called *linear* ([LP]) if every definable set is already definable in $\langle M, <, +, 0, \{\lambda\}_{\lambda \in \Lambda} \rangle$, and it is called *semi-bounded* ([Ed, Pet1]) if every definable set is already definable in $\langle M, <, +, 0, \{\lambda\}_{\lambda \in \Lambda}, \{B\}_{B \in \mathcal{B}} \rangle$.

Obviously, if $\mathcal{M}$ is linear then it is semi-bounded. By [PeSt], $\mathcal{M}$ is not linear if and only if there is a real closed field defined on some bounded interval. By [Ed], $\mathcal{M}$ is not semi-bounded if and only if $\mathcal{M}$ expands a real closed field if and only if for any two intervals there is a definable bijection between them.

An important example of a semi-bounded non-linear structure is the expansion of the ordered vector space $\langle \mathbb{R}; <, +, 0, x \mapsto \lambda x \rangle_{\lambda \in \mathbb{R}}$ by all bounded semialgebraic sets.

It is largely evident from the literature that among the three cases, (a) and (c) have provided the most accommodating settings for studying general mathematics. For example, the definable sets in a real closed field are the main objects of study

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in semialgebraic geometry (a classical reference is [DK]). Moreover, o-minimal linear topology naturally extends the classical subject of piecewise linear topology and has the potential to tackle problems that arise in the study of algebraically closed valued fields (see, for example, [HL]). From an internal aspect, the study of definable groups in both of these two settings has been rather successful (see further comments below).

On the other hand, the middle case (b) remains as elusive as interesting from a classification point of view. Although a local field may be definable, and thus the definable structure can get quite rich, there is no global field, and hence many known technics do not apply. In particular, little is known with respect to structure theorems of definable groups in this setting. In this paper, we set forth a project of analyzing semi-bounded groups, mainly motivated by two conjectures asked by Peterzil in [Pet3]. Let us describe our project.

For the rest of the paper, we fix a semi-bounded o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ of an ordered group, which is not linear. We fix an element $1 > 0$ such that a real closed field, whose universe is $(0,1)$ and whose order agrees with $<$, is definable in $\mathcal{M}$.

Let $\mathcal{L}$ denote the underlying language of $\mathcal{M}$. By 'definable' we mean 'definable in $\mathcal{M}$' possibly with parameters. A group G is said to be definable if both its domain and its group operation are definable. Definable sets and groups in this setting are also referred to as *semi-bounded*. If they are defined in the *linear reduct* $\mathcal{M}_{lin} = \langle M, <, +, 0, \{\lambda\}_{\lambda\in\Lambda}\rangle$ of $\mathcal{M}$, we call them *semi-linear*. The underlying language of $\mathcal{M}_{lin}$ is denoted by $\mathcal{L}_{lin}$.

Following [Pet3], an interval $I \subseteq M$ is called *short* if there is a definable bijection between I and $(0,1)$; otherwise, it is called *long*. Equivalently, an interval $I \subseteq M$ is short if a real closed field whose domain is I is definable. An element $a \in M$ is called *short* if either $a = 0$ or $(0, |a|)$ is a short interval; otherwise, it is called *tall*. A tuple $a \in M^n$ is called *short* if $|a| := |a_1| + \cdots + |a_n|$ is short, and *tall* otherwise. A definable set $X \subseteq M^n$ (or its defining formula) is called *short* if it is in definable bijection with a subset of $(0,1)^n$; otherwise, it is called *long*. Notice that this is compatible, for $n = 1$, with the notion of a short interval.

In [Pet1] and [Ed] the authors proved structure theorems about definable sets and functions. (See also [Bel] for an analysis of semi-bounded sets in a different context.) The gist of those theorems was that the definable sets can be decomposed into 'cones', which are bounded sets 'stretched' along some unbounded directions. Conjecture 1 from [Pet3] asks if we can replace 'bounded' by 'short', and 'unbounded' by 'long', in the definition of a cone and still obtain a structure theorem. We answer this affirmatively (the precise terminology to be given in Section 2 below).

Theorem 3.8 (Refined Structure Theorem). *Every A-definable set $X \subseteq M^n$ is a finite union of A-definable long cones. (In particular, a short set is a 0-long cone.) Furthermore, for every A-definable function $f : X \subseteq R^n \to R$, there is a finite collection $\mathcal{C}$ of A-definable long cones, whose union is X and such that f is almost linear with respect to each long cone in $\mathcal{C}$.*

As noted in Remark 3.9 below, it is not always possible to achieve *disjoint* unions in our theorem.

This theorem implies, in particular, a quantifier elimination result down to suitable existential formulas in the spirit of [vdD1] (see Corollary 3.10). The proof of the Refined Structure Theorem involves an induction on the 'long dimension' of definable sets, which is a refinement of the notion of 'linear dimension' from [Ed].

We then turn our attention to semi-bounded groups. Groups definable in o-minimal structures have been a central object of study in model theory. The climax of that study was the work around Pillay's Conjecture (PC) and Compact Domination Conjecture (CDC), stated in [Pi3] and [HPP1], respectively. In the linear case, (PC) was solved in [ElSt] and (CDC) in [El]. The proofs involved a structure theorem for semi-linear groups from [ElSt] that states that every such group is a quotient of a suitable convex subgroup of $\langle M^n, +\rangle$ by a lattice. In the field case, (PC) was solved in [HPP1] and (CDC) in [HP, HPP2] (see also [Ot] for an overview of all preceding work). In the case of semi-bounded groups, (PC) was solved in [Pet3] after developing enough theory to allow the combination of the linear and the field cases. The (CDC) for semi-bounded groups remains open. Conjecture 2 from [Pet3] asks if we can prove a structure theorem for semi-bounded groups in the spirit of [ElSt]. In the second part of this paper, we prove a local theorem for semi-bounded groups which we see as a first step towards Conjecture 2 from [Pet3].

The proof of the local theorem involves a new notion of a closure operator in $\mathcal{M}$, the 'short closure operator' *scl*, which makes $(\mathcal{M}, scl)$ into a pregeometry. The rising notion of dimension coincides with the long dimension (Corollary 5.10). This allows us to make use of desirable properties of 'long-generic' elements and 'long-large' sets, by virtue of Claim 5.13 below. The local theorem is the following:

Theorem 6.3 *Let $G = \langle G, \oplus\rangle$ be a definable group of long dimension k. Then every long-generic element a in G is contained in a k-long cone $C \subseteq G$, such that for every $x, y \in C$,*

$$x \ominus a \oplus y = x - a + y.$$

In particular, on C, G is locally isomorphic to $\langle M^k, +\rangle$.

We expect that Theorem 6.3 will be the start point in subsequent work for analyzing semi-bounded groups globally.

Structure of the paper and a few words for the proofs. Section 2 contains basic definitions and preparatory lemmas about the main objects we are dealing with in this paper: the set Λ, long cones and long dimension.

Section 3 contains the proof of three main statements: Lemma on Subcones 3.1, Lemma 3.6(v) on long dimension of unions, and the Refined Structure Theorem 3.8. These statements refine the corresponding ones from [Ed], and so do their proofs. A new phenomenon, however, is that the relative position of two long cones can now range over a bigger range of possibilities. This is because long cones are not necessarily unbounded (which was the case with the cones used in [Ed]). The Lemma on Subcones, as well as Lemma 2.16 from Section 2, provide two main tools for controlling this situation.

Some difficulties that are incorporated in handling the long dimension are worked out in Section 4, and they are the following: although it is fairly easy to see that a definable set X which is the cartesian product of two definable sets with long dimensions l and m has long dimension $l + m$ (Lemma 3.6(iv)), it is not a priori clear why if a definable set X is the union of a definable family of fibers each of long dimension m over a set of long dimension n, then X has long dimension $n+m$. We establish this in Lemma 4.2.

Section 5 deals with the new pregeometry coming from the 'short closure operator'.

In Section 6 we prove the local theorem for semi-bounded groups.

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2. Basic notions and lemmas

We assume familiarity with the basic notions from o-minimality, such as the inductive definition of cells either as graphs or 'cylinders' of definable continuous functions, the cell decomposition theorem, dimension, generic elements, definable closure, etc. The reader may consult [vdD2] or [Pi2] for these notions.

Lemma 2.1. *Let $f : I \to M$ be a definable function, where I is a long interval. If $f(I)$ is short, then f is piecewise constant except for a finite collection of short subintervals of I.*

Proof. The function f is piecewise strictly monotone or constant. If it were strictly monotone on a long subinterval of I, then on that subinterval f would be a definable bijection between a long interval and a short set. □

Lemma 2.2. *Let $f : X \subseteq M^n \to M$ be a definable function. For every $i = 1, \dots, n$, and $\bar{x}^i := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n) \in M^{n-1}$, let*

$$X_{\bar{x}^i} = \{x_i \in M : (x_1, \dots, x_n) \in X\}$$

be the fiber of X above $\bar{x}^i$ and $f_{\bar{x}^i} : X_{\bar{x}^i} \to M$ the map $f_{\bar{x}^i}(x_i) = f(\bar{x})$. Consider the set

$$A = \{\bar{a} \in X : \forall i \in \{1, \dots, n\},\ f_{\bar{a}^i} \text{ is monotone in an interval containing } a_i\}.$$

Then $\dim(X \setminus A) < \dim(X)$.

Proof. We may assume that f and X are $\emptyset$-definable. The set A is then also $\emptyset$-definable and it clearly contains every generic element of X. □

2.1. Properties of Λ.

The definition of a long cone in the next subsection requires the notion of M-independence for elements of Λ^n. We define this notion and elaborate on it sufficiently in this subsection. Let us first fix some of our standard terminology and notation.

By a *partial endomorphism* of $\langle M, <, +, 0\rangle$ we mean a map $f : (a, b) \to M$ such that for every $x, y, x+t, y+t \in (a, b)$,

$$f(x+t) - f(x) = f(y+t) - f(y).$$

As we said in the introduction, Λ denotes the set of all $\emptyset$-definable partial endomorphisms. A definable function $f : A \subseteq M^n \to M$ is called *affine on A* if it has form

$$f(x_1, \dots, x_n) = \lambda_1 x_1 + \dots + \lambda_n x_n + a,$$

for some fixed $\lambda_i \in \Lambda$ and $a \in M$. For every $i = 1, \dots, n$, we denote by

$$e_i = (0, \dots, 0, 1, 0, \dots, 0)$$

the standard i-th unit vector from Λ^n, where $1 : M \to M$ is the identity map. For $v \in \Lambda$, we denote by $\operatorname{dom}(v)$ and $\operatorname{ran}(v)$ the domain and range of v, respectively. We write vt for $v(t)$. Following [Pet3], we consider the equivalence relation $\sim$ on Λ where two $\lambda, \mu \in \Lambda$ are said to be *$\sim$-equivalent* if there is $\epsilon > 0$, $a \in \operatorname{dom}(\lambda)$ and $b \in \operatorname{dom}(\mu)$, such that the restrictions of the maps $\lambda(a+x)-\lambda(a)$ and $\mu(b+x)-\mu(b)$ on $(-\epsilon, \epsilon)$ are the same. (That is, those last maps have the same germ at 0). It is observed in [Pet3, Section 6], that Λ modulo $\sim$ can be given the structure of an ordered field with multiplication given by composition. This implies in particular that

(1) for every $\lambda, \mu \in \Lambda$ and $x \in \operatorname{dom}(\lambda\mu) \cap \operatorname{dom}(\mu\lambda)$, $\lambda\mu(x) = \mu\lambda(x)$.

We also recall from [LP, Proposition 4.1] that

(2) if two partial endomorphisms agree at some non-zero point of their domain then they agree at any other point of their common domain.

It is a standard practice in this paper that whenever we write an expression of the form 'vt', with $v \in \Lambda$ and $t \in M$, we mean in particular that $t \in \operatorname{dom}(v)$. Sometimes, however, we say explicitly that $t \in \operatorname{dom}(v)$. For a matrix $A = (a_{ij})$ with entries from Λ, the *rank* of A is the rank of the matrix $\overline{A} = (\bar{a}_{ij})$, where $\bar{a}_{ij}$ is the $\sim$-equivalence class of a_{ij}. It is then a routine to check, using notes (1) and (2) above, that various classical results from linear algebra hold for matrices with entries from Λ. For example, a $n \times n$ linear system with coefficients from Λ has a unique solution if and only if the coefficient matrix has rank n. We freely use such results in this paper.

We now proceed to the notion of M-independence.

Definition 2.3. If $v = (v_1, \dots, v_n) \in \Lambda^n$ and $t \in M$, we denote $vt := (v_1t, \dots, v_nt)$ and $\operatorname{dom}(v) := \cap_{i=1}^n \operatorname{dom}(v_i)$. We say that $v_1, \dots, v_k \in \Lambda^n$ are *M-independent* if for all $t_1, \dots, t_k \in M$ with $t_i \in \operatorname{dom}(v_i)$,

$$v_1t_1 + \cdots + v_kt_k = 0 \text{ implies } t_1 = \cdots = t_k = 0.$$

If $v = (v_1, \dots, v_n) \in \Lambda^n$ and $\mu \in \Lambda$, we denote $\mu v := (\mu v_1, \dots, \mu v_n)$. We say that $v_1, \dots, v_k \in \Lambda^n$ are *Λ-independent* if for all $\mu_1, \dots, \mu_k$ in Λ, with $\operatorname{ran}(v_i) \subseteq \operatorname{dom}(\mu_i)$,

$$\mu_1v_1 + \cdots + \mu_kv_k = 0 \text{ implies } \mu_1 = \cdots = \mu_k = 0.$$

The proofs of the following two lemmas are straightforward computations but we include them anyway for completeness.

Lemma 2.4. *For $v_1, \dots, v_l \in \Lambda^n$ with common domain $(-a, a) \subseteq M$, the following are equivalent:*

(i) $v_1, \dots, v_l$ are M-independent
(ii) $v_1, \dots, v_l$ are Λ-independent
(iii) the set

$$H = \{v_1t_1 + \cdots + v_lt_l : -a < t_i < a\}$$

has dimension l. (This was called 'open l-parallelogram' in [ElSt]*.)*

Proof. (i)⇒(ii). This is essentially a straightforward application of (1) and (2) above, but we include the complete proof in the interests of completeness. If $v_1, \dots, v_l$ are Λ-dependent, then there are $\mu_1, \dots \mu_l \in \Lambda$ with $\operatorname{ran}(v_i) \subseteq \operatorname{dom}(\mu_i)$, not all 0, such that $\mu_1v_1 + \cdots + \mu_lv_l = 0$. In particular, the domain of each μ_i contains some interval containing 0 (because so does the range of v_i). So we can restrict μ_i so that its range contains that interval and is contained in the domain of v_i.

We claim that for any $t \neq 0$ in the domain of all μ_i's, we have $v_1(\mu_1 t) + \cdots + v_l(\mu_l t) = 0$, which will show that the v_i's are M-dependent. To prove the claim we will need to use commutativity between elements of Λ. We argue how this is precisely done.

By restricting μ_i even more, we can assume that the domain of μ_i is also contained in the domain of v_i. Let us call that new restriction μ_i'. We want to argue that for some $t \neq 0$ in the domain of μ_i', we have

$$v_i \mu_i'(t) = \mu_i v_i(t), \tag{3}$$

where now all arguments make sense.

If we look at the germs of μ_i and μ_i', they are the same. Hence the germs of the maps $v_i\mu_i'$ and $\mu_i v_i$ are also the same. Hence the maps $v_i\mu_i'$ and $\mu_i v_i$ are equal at any t that lies in both of their domains, by (2) above. This finishes the proof of (3).

We conclude that there is $t \neq 0$ so that

$$v_1(\mu_1' t) + \cdots + v_l(\mu_l' t) = (\mu_1 v_1 + \cdots + \mu_l v_l)(t) = 0.$$

(ii)⇒(iii). Since $v_i = \begin{pmatrix} v_i^1 \\ \vdots \\ v_i^n \end{pmatrix}$, $i = 1, \ldots, l$, are Λ-independent, the matrix

$$A = \begin{pmatrix} v_1^1 & \ldots & v_l^1 \\ \vdots & \ldots & \vdots \\ v_1^n & \ldots & v_l^n \end{pmatrix}$$

has rank l. Clearly, it is enough to prove that:

the map $f : (-a, a)^l \to M^n$ with $x \mapsto Ax$ is injective and onto H.

This claim can be proved by induction on n. For the base step, if A is 1×1 matrix, we observe that if λ is not the zero endomorphism, then it must be non-zero at any non-zero point, by (2). So the kernel is 0.

The inductive step is a straightforward argument, which we omit.

(iii)⇒(ii). This is an easy adaptation of the proof of [El, Corollary 2.5]. □

Lemma 2.5. *Let $v_1, \ldots, v_l \in \Lambda^n$ be Λ-independent with $\cap_{i=1}^l \text{dom}(v_i) \neq \emptyset$ and denote by $\pi : \Lambda^n \to \Lambda^{n-1}$ the usual projection. The following are equivalent:*

(i) There are $\lambda_1, \ldots, \lambda_{n-1} \in \Lambda$, such that for all $t_1, \ldots, t_l \in M$ with $t_i \in \text{dom}(v_i)$, $v_1 t_1 + \cdots + v_l t_l$ has form:

$$v_1 t_1 + \cdots + v_l t_l = (a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1}).$$

(ii) $\pi(v_1), \ldots, \pi(v_l)$ are Λ-independent.

Proof. (i)⇒(ii). The assertion from (i) says that the last coordinate of $v_1t_1+\cdots+v_lt_l$ is a function of the first $n-1$ coordinates. Therefore the projections under π of any two distinct elements from the set $\{v_1 t_1 + \cdots + v_l t_l : t_i \in \text{dom}(v_i)\}$ are distinct. We claim that the projections $\pi(v_1), \ldots, \pi(v_l)$ are Λ-independent. Indeed, if they are not, then one of them, say $\pi(v_l)$, can be written as linear combination $\mu_1 \pi(v_1) + \cdots + \mu_{l-1}\pi(v_{l-1})$. But then, for any $a \in \cap_{i=1}^l \text{dom}(v_i)$, the elements $v_l a$ and $(\mu_1 v_1 + \cdots + \mu_{l-1} v_{l-1})a$ would have the same projection, a contradiction.

(ii)⇒(i). We need to compute the λ_i's and a_i's. Assume $v_i = (v_i^1, \ldots, v_i^n)$. Since $\pi(v_1), \ldots, \pi(v_l)$ are Λ-independent, the system

$$\begin{aligned} v_1^n &= \lambda_1 v_1^1 + \cdots + \lambda_{n-1} v_1^{n-1} \\ &\vdots \\ v_l^n &= \lambda_1 v_l^1 + \cdots + \lambda_{n-1} v_l^{n-1} \end{aligned}$$

has a unique solution for $\lambda_1, \ldots, \lambda_{n-1}$. The above equations imply that

$$v_1^n t_1 + \cdots + v_l^n t_l = \lambda_1(v_1^1 t_1 + \cdots + v_l^1 t_l) + \cdots + \lambda_{n-1}(v_1^{n-1} t_1 + \cdots + v_l^{n-1} t_l)$$

and, hence,

$$v_1 t_1 + \cdots + v_l t_l = (a_1, \ldots, a_{n-1}, \lambda_1 a_1 + \cdots + \lambda_{n-1} a_{n-1}).$$

where $a_i = v_1^i t_1 + \cdots + v_l^i t_l$, for $i = 1, \ldots, n-1$. □

Here is another lemma.

Lemma 2.6. *Let $v_1, \ldots, v_l \in \Lambda^n$ be M-independent. Then, for every $t_1, \ldots, t_l \in M$ with $t_i \in \mathrm{dom}(v_i)$,*

$$v_1 t_1 + \cdots + v_l t_l \text{ is short} \Rightarrow t_1, \ldots, t_l \text{ are short.}$$

Proof. Since $v_i = \begin{pmatrix} v_i^1 \\ \vdots \\ v_i^n \end{pmatrix}$, $i = 1, \ldots, l$, are Λ-independent, the matrix

$$A = \begin{pmatrix} v_1^1 & \ldots & v_l^1 \\ \vdots & \ldots & \vdots \\ v_1^n & \ldots & v_l^n \end{pmatrix}$$

has rank l. Let B be an $l \times l$ submatrix of A of rank l. Then $B\begin{pmatrix} t_1 \\ \vdots \\ t_l \end{pmatrix} = \begin{pmatrix} s_1 \\ \vdots \\ s_l \end{pmatrix}$, for some short $s_1, \ldots, s_l \in M$. Hence $\begin{pmatrix} t_1 \\ \vdots \\ t_l \end{pmatrix} = B^{-1}\begin{pmatrix} s_1 \\ \vdots \\ s_l \end{pmatrix}$ and each row of the last matrix consists of a short element. □

The following two lemmas will be used in the proof of the Lemma on Subcones 3.1 below.

Lemma 2.7. *Let $w, v_1, \ldots, v_m \in \Lambda^n$, with $\mathrm{dom}(w) = (0, a)$ and $\mathrm{dom}(v_i) = (-a_i, a_i)$, for some positive $a, a_i \in M$. Assume that*

$$wt = v_1 t_1 + \cdots + v_m t_m$$

for some $t, t_1, \ldots, t_m \in M$, with $t \in \mathrm{dom}(w)$ and $t_i \in \mathrm{dom}(v_i)$. Then for every $s \in \mathrm{dom}(w)$ with $s < t$, there are $s_1, \ldots, s_m \in M$ with $|s_i| < |t_i|$ such that

$$ws = vs_1 + \cdots + v_m s_m.$$

Moreover, s_i has the same sign as t_i.

Proof. This follows from [ElSt, Lemma 3.4], whose proof used only the fact that $\mathcal{M}$ is an o-minimal expansion of an ordered group. Indeed, since $s < t$, then by convexity of the set $A = \{v_1 x_1 + \cdots + v_m x_m : x_i \in \mathrm{dom}(v_i)\}$ and the aforementioned lemma, $ws \in A$. □

Lemma 2.8. *Let $w_1, \dots, w_n \in \Lambda^n$ be M-independent and $\lambda_1, \dots, \lambda_n \in \Lambda^n$. Let $t_1, \dots, t_n \in M$ be non-zero elements. Assume that:*

$$\begin{aligned} w_1 t_1 &= \lambda_1 s_1^1 + \cdots + \lambda_n s_1^n \\ &\vdots \\ w_n t_n &= \lambda_1 s_n^1 + \cdots + \lambda_n s_n^n \end{aligned}$$

for some $s_i^j \in M$. Then there non-zero $a_1, \dots, a_n \in M$ and $b_i^j \in M$, $i, j = 1, \dots, n$, such that:

$$\begin{aligned} \lambda_1 a_1 &= w_1 b_1^1 + \cdots + w_n b_1^n \\ &\vdots \\ \lambda_n a_n &= w_1 b_n^1 + \cdots + w_n b_n^n \end{aligned}$$

Proof. In the Appendix. □

2.2. **Long cones.** Here we refine the notion of a 'cone' from [Ed].

Definition 2.9. Let $k \in \mathbb{N}$. A *k-long cone* $C \subseteq M^n$ is a definable set of the form

$$\left\{ b + \sum_{i=1}^{k} v_i t_i : b \in B,\, t_i \in J_i \right\},$$

where $B \subseteq M^n$ is a short cell, $v_1, \dots, v_k \in \Lambda^n$ are M-independent and $J_1, \dots, J_k$ are long intervals each of the form $(0, a_i)$, $a_i \in M^{>0} \cup \{\infty\}$, with $J_i \subseteq \mathrm{dom}(v_i)$. So a 0-long cone is just a short cell. A *long cone* is a k-long cone, for some $k \in \mathbb{N}$. We say that the long cone C is *normalized* if for each $x \in C$ there are unique $b \in B$ and $t_1 \in J_1, \dots, t_k \in J_k$ such that $x = b + \sum_{i=1}^{k} v_i t_i$. In this case, we write:

$$C = B + \sum_{i=1}^{k} v_i t_i | J_i.$$

In what follows, all long cones are assumed to be normalized, and we thus drop the word 'normalized'. We also often refer to $\bar{v} = (v_1, \dots, v_k) \in \Lambda^{kn}$ as the *direction* of the long cone C. If we want to distinguish some v_j, say v_k, from the rest of the v_i's, we write:

$$C = B + \sum_{i=1}^{k-1} v_i t_i | J_i + v_k | J_k.$$

By a *subcone of* C we simply mean a long cone contained in C.

Remark 2.10. By Lemma 2.4, a (normalized) k-long cone $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ has dimension k if and only if B is finite. In fact, $\dim(C) = \dim(B) + k$.

Definition 2.11. Let $C = B + \sum_{i=1}^{k} v_i t_i | J_i$ be a k-long cone and $f : C \to M$ a definable continuous function. We say that f *is almost linear with respect to* C if there are $\mu_1, \dots, \mu_k \in \Lambda$ and an extension $\tilde{f}$ of f to $\{b + \sum_{i=1}^{k} v_i t_i : b \in B,\, t_i \in \{0\} \cup J_i\}$, such that

$$\forall b \in B,\, t_1 \in \{0\} \cup J_1, \dots, t_k \in \{0\} \cup J_k,\ \tilde{f}\left(b + \sum_{i=1}^{k} v_i t_i\right) = \tilde{f}(b) + \sum_{i=1}^{k} \mu_i t_i. \tag{4}$$

Remark 2.12. Let $C = B + \sum_{i=1}^k v_i t_i | J_i$ be a k-long cone.

(i) If $f : C \to M$ is almost linear with respect to C, then, since C is normalized, the $\mu_1, \dots, \mu_k$ and $\tilde{f}$ as above are unique. In particular, $\tilde{f}$ is continuous. For this reason, we often abuse notation and write f for $\tilde{f}$. Indeed, we simply denote (4) by

$$f\left(b + \sum_{i=1}^k v_i t_i\right) = f(b) + \sum_{i=1}^k \mu_i t_i.$$

(ii) If $B = \{b\}$ and $f : C \to M$ is a definable function, then f is almost linear with respect to C if and only if f is affine on C. More generally, f is almost linear with respect to $B + \sum_{i=1}^k v_i t_i | J_i$ if and only if there are $\mu_1, \dots, \mu_k \in \Lambda$ such that for every $b \in B$ and $s_i, s_i + t_i \in J_i$, we have

$$f\left(b + \sum_{i=1}^k v_i (s_i - t_i)\right) - f\left(b + \sum_{i=1}^k v_i s_i\right) = \sum_{i=1}^k \mu_i t_i.$$

(iii) If $f : C \to M$ is almost linear with respect to C, then the graph of f is also k-long cone, with the short cell being $\{(b, f(b)) : b \in B\}$:

$$Graph(f) = \left\{(b, f(b)) + \sum_{i=1}^k (v_i, \mu_i) t_i : b \in B, t \in J_i\right\},$$

(iv) Let $j \in \{1, \dots, k\}$ and assume $J_j = (0, a_j)$ with $a_j \in M$. Then

$$C = B + v_j a_j + \sum_{i=1}^k v'_i t_i | J_i,$$

where $v'_j = -v_j$ and for $i \neq j$, $v'_i = v_i$. Indeed, if $x = b + \sum_{i=1}^k v_i t_i$ is in C, then for $s_j = a_j - t_j \in J_j$ we have $x = b + v_j a_j - v_j s_j + \sum_{i \neq j} v_i t_i$.

If, moreover, $f : C \to M$ is almost linear with respect to C and of the form

$$f\left(b + \sum_{i=1}^k v_i t_i\right) = f(b) + \sum_{i=1}^k \mu_i t_i,$$

then

$$f\left(b + v_j a_j + \sum_{i=1}^k v'_i t_i\right) = f(b + v_j a_j) + \sum_{i=1}^k \mu'_i t_i,$$

where $\mu'_j = -\mu_j$ and for $i \neq j$, $\mu'_i = \mu_i$.

Corollary 2.13. *If $D = b + \sum_{i=1}^l v_i t_i | J_i \subseteq M^n$ is an l-long cone, then some projection $\pi : M^n \to M^l$, restricted to D, is a bijection onto an l-long cone.*

Proof. By Lemmas 2.4 and 2.5. □

Notation. If $J = (0, a)$, we denote $\pm J := (-a, a)$. Let $C = B + \sum_{i=1}^m v_i t_i | J_i$ be an m-long cone. We set:

$$\langle C \rangle := \left\{\sum_{i=1}^m v_i t_i : t_i \in \pm J_i\right\}.$$

Corollary 2.14. *Let $C = b + \sum_{i=1}^k v_i t_i | J_i$ be a k-long cone. Let $\lambda \in \Lambda^k$ be such that for some positive $t \in M$, $\lambda t \in \langle C \rangle$. Then there is a tall $b \in M$ such that $\lambda b \in \langle C \rangle$.*

Proof. Fix i. Let $a = \sup\{x \in M : \lambda x \in \langle C \rangle\}$. It is easy to see that $a = v_1t_1 + \cdots + v_kt_k$, with at least one of $t_1, \dots, t_k$, say t_i, equal to $\pm|J_i|$. Hence, by Lemma 2.6, a is tall. Take $b = \frac{1}{2}a$ (since a is not in $\langle C \rangle$). □

2.3. **Long dimension.** Here we refine the notion of 'linear dimension' from [Ed].

Definition 2.15. Let $Z \subseteq M^n$ be a definable set. Then the *long dimension of* Z is defined to be

$$\mathrm{lgdim}(Z) = \max\{k : Z \text{ contains a } k\text{-long cone}\}.$$

Equivalently, the long dimension of Z is the maximum k such that Z contains a definable homeomorphic image of J^k, for some long interval J. Indeed, this follows from the proof of Lemma 2.4, (ii)⇒(iii).

Some main properties of long dimension will be proved in Section 3.2 below, after proving the Lemma on Subcones in Section 3.1. For the moment, we state a lemma which says that given a cone we can always find subcones of suitable direction. An analogous statement fails in the context of [Ed], where all cones were unbounded.

Lemma 2.16. *Let $C = b + \sum_{i=1}^k v_it_i|J_i$ be a k-long cone. Let $w_1, \dots, w_k \in \Lambda^n$ be M-independent such that for every i, there is a positive $s_i \in M$, $w_is_i \in \langle C \rangle$. Then there is a k-long subcone $C' \subseteq C$ of the form $C' = c + \sum_{i=1}^k w_it_i|(0, \kappa_i)$, for some tall $\kappa_i \in M$.*

Proof. By Corollary 2.14, we may assume that each s_i is tall. Assume $J_i = (0, a_i)$. Let $c = b + \sum_{i=1}^k \frac{1}{2}v_ia_i$ and for each i, let $\kappa_i = \frac{1}{2k}|s_i|$. Using Lemma 2.7, one can easily check that $C' = c + \sum_{i=1}^k w_it_i|(0, \kappa_i) \subseteq C$. □

The following lemma will be used in the proof of the Refined Structure Theorem.

Lemma 2.17. *Let $X = (f, g)_{\pi(X)}$ be a cylinder in M^{n+1} such that $\pi(X)$ is a k-long cone and f and g are almost linear with respect to $\pi(X)$. If there is an $x \in \pi(X)$ such that $\pi^{-1}(x)$ is long, then* $\mathrm{lgdim}(X) = k + 1$.

Proof. If $k = 0$, then there is an 1-long cone $\pi^{-1}(x) \subseteq X$. Now assume $k > 0$ and that for some $x \in \pi(X)$, $\pi^{-1}(x) = \big(f(x), g(x)\big)$ is long. Since f, g are almost linear on $\pi(X)$, there is clearly a k-long cone $C_x = x + \sum_{i=1}^k v_it_i|(0, a_i) \subseteq \pi(X)$ such that for each element $y \in C_x$, $g(y) - f(y)$ must be tall. Let $\alpha = \inf\{g(y) - f(y) : y \in C_x\}$. Since f is affine,

$$\forall t_1 \in J_1, \dots, t_k \in J_k,\ f\left(x + \sum_{i=1}^k v_it_i\right) = f(x) + \sum_{i=1}^k \mu_it_i,$$

for some $\mu_1, \dots, \mu_k \in \Lambda^n$. Then clearly the $(k + 1)$-long cone

$$\big(x, f(x)\big) + \sum_{i=1}^k (v_i, \mu_i)t_i|J_i + e_{n+1}t_{k+1}|(0, \alpha)$$

is contained in X. □

3. Structure Theorem for semi-bounded sets

In this section we prove the main results for semi-bounded sets.

3.1. **Generalizing the Lemma on Subcones** [Ed, Lemma 3.4]**.** The Lemma on Subcones can be viewed as a kind of converse to Lemma 2.16. Recall from Section 2 that if $C = B + \sum_{i=1}^{m} v_i t_i|J_i$ is an m-long cone, we denote $\langle C\rangle = \{\sum_{i=1}^{m} v_i t_i : t_i \in \pm J_i\}$.

Lemma 3.1 (Lemma on subcones)**.** *If $C' = B' + \sum_{i=1}^{m'} w_i t_i|J'_i$ and $C = B + \sum_{i=1}^{m} v_i t_i|J_i$ are two long cones such that $C' \subseteq C \subseteq M^n$, then $\langle C'\rangle \subseteq \langle C\rangle$ (and hence $m' \leq m$).*

Proof. Clearly, we may assume that B' is a singleton. Moreover, we can translate both C' and C, so that C' gets the form $C' = \sum_{i=1}^{m'} w_i t_i|J'_i$. Let $j \in \{1, \dots, m'\}$, and denote for convenience $J := J'_j$. Then $\forall u \in J, w_j u \in C' \subseteq C$, so there exist a unique $b \in B$ and, for each $i \in \{1, \dots, m\}$, a unique $t_i \in J_i$ such that $w_j u = b + \sum_{i=1}^{m} v_i t_i$. This yields the following definable functions:

- $\beta : J \to B$, with $u \mapsto \beta(u)$
- for each $i \in \{1, \dots, m\}$, $\tau_i : J \to J_i$, with $u \mapsto \tau_i(u)$,

where
$$w_j u = \beta(u) + \sum_{i=1}^{m} v_i(\tau_i(u)).$$
By Lemma 2.1 and o-minimality, there are long subintervals $I_1, \dots I_l \subseteq J$ such that $J \setminus (I_1 \cup \dots \cup I_l)$ is short and on each of them $\beta(u)$ is constant. Let $I = (p, q)$ be an interval with maximum length among the I_i's, and assume that on I the map $\beta(u)$ is equal to b. Now let $u_1 < u_2$ in I, with u_1 close enough to p and u_2 close enough to q, so that, if $u := u_2 - u_1$, then for some $k \in \mathbb{N}$, $J \subseteq (0, ku)$ (this is possible by the choice of I). We have:
$$w_j u = w_j(u_2 - u_1) = \sum_{i=1}^{m} v_i(\tau_i(u_2) - \tau_i(u_1)).$$
If we denote $t_i = \tau_i(u_2) - \tau_i(u_1)$, then
$$w_j u = \sum_{i=1}^{m} v_i t_i. \tag{5}$$
Hence the condition of Lemma 2.7 is satisfied for $w = w_j$.

Now pick any $t \in J$. We have to show that $w_j t \in \langle C\rangle$. We split two cases.

CASE I. $t \leq u$. By Lemma 2.7, we have $w_j t = \sum_{i=1}^{m} v_i s_i$, for some $0 < |s_i| \leq |t_i|$, and we are done.

CASE II. $t > u$. By the choice of u, there is $k \in \mathbb{N}$, so that $t - u < ku$. Hence, by Lemma 2.7 again, we have $\frac{1}{k} w_j(t - u) = \sum_{i=1}^{m} v_i s_i$, for some $0 < |s_i| \leq |t_i|$, and s_i having the same sign as t_i. Equivalently,
$$w_j(t - u) = \sum_{i=1}^{m} v_i k s_i. \tag{6}$$
By (5) and (6), we obtain
$$w_j t = \sum_{i=1}^{m} v_i(t_i + k s_i), \tag{7}$$
so it remains to show that $-a_i < t_i + k s_i < a_i$, where $J_i = (0, a_i)$. We split two subcases:

SUBCASE II(a). $t_i > 0$. We observe that, since $C' \subseteq C$, we have

$$w_j t = b' + \sum_{i=1}^{m} v_i r_i,$$

for some $r_i \in J_i$ and $b' \in B$. Together with (7),

$$\sum_{i=1}^{m} v_i(t_i + ks_i) = b' + \sum_{i=1}^{m} v_i r_i.$$

If $t_i + ks_i > r_i$, then we would have $b' = \sum_{i=1}^{m} v_i z_i$, for some positive $z_i < t_i + ks_i$. By (7), this would imply that $b' = w_j s$ for some $0 < s < t$. In particular, $b' \in C'$, a contradiction. So $0 < t_i + ks_i \leq r_i < a_i$, as required.

SUBCASE II(b). $t_i < 0$. Then also $s_i < 0$. Since $0 \in C'$, we have

$$0 = b' + \sum_{i=1}^{m} v_i r_i,$$

for some $r_i \in J_i$ and $b' \in B$. Together with (7),

$$w_j t = b' + \sum_{i=1}^{m} v_i(r_i + t_i + ks_i).$$

Hence, $0 < r_i + t_i + ks_i$ and, therefore, $-a_i < -r_i < t_i + ks_i < 0 < a_i$, as required.

Finally, the fact that $m' \leq m$ is now a consequence of Lemma 2.4. □

Remark 3.2. Observe that it is not always possible to get $w_j t \in \langle C \rangle^{>0} := \{\sum_{i=1}^{m} v_i t_i : t_i \in J_i\}$, as in the corresponding conclusion of [Ed, Lemma 3.4].

We can now characterize exactly the subcones of a given long cone C.

Corollary 3.3. *The subcones of a long cone C are exactly those cones whose direction $\bar{v} = (v_1, \dots, v_k)$ satisfies the following property: for every $i = 1, \dots, k$, there is a positive $s \in M$, such that $v_i s \in \langle C \rangle$.*

Proof. By Corollary 2.16 and Lemma on Subcones. □

Lemma 3.4. *Let $C' = B' + \sum_{i=1}^{k'} v'_i t_i | J'_i \subseteq C = B + \sum_{i=1}^{k} v_i t_i | J_i$ be two long cones and $f : C \to M$ a definable function which is almost linear with respect to C. Then f is almost linear with respect to C'.*

Proof. By the Lemma on Subcones, for each $i = 1, \dots, k'$ and $t \in J'_i$, we have $v'_i t \in \langle C \rangle$. It is then an easy exercise to check that f is affine in each v'_i, uniformly on $b' \in B'$; that is, there are $\mu_1, \dots, \mu_{k'} \in \Lambda$ such that for every $b' \in B'$ and $s_i, s_i + t_i \in J'_i$, we have

$$f\left(b' + \sum_{i=1}^{k} v_i(s_i - t_i)\right) - f\left(b + \sum_{i=1}^{k} v_i s_i\right) = \sum_{i=1}^{k} \mu_i t_i.$$

This exactly means (Remark 2.12(ii)) that f is almost linear with respect to C'. □

Corollary 3.5. *Let $C \subseteq C'$ be two k-long cones and let $\bar{v}$ be the direction of C'. Then there is a k-long cone of direction $\bar{v}$ contained in C.*

Proof. By the Lemma on Subcones, Lemma 2.8 and Corollary 2.14. □

3.2. **Properties of long dimension.**

Lemma 3.6. *Let $X, Y, X_1, \dots, X_k$ be definable sets. Then:*

(i) $\operatorname{lgdim}(X) \leq \dim(X)$.
(ii) $X \subseteq Y \subseteq M^n \Rightarrow \operatorname{lgdim}(X) \leq \operatorname{lgdim}(Y) \leq n$.
(iii) If C is a n-long cone, then $\operatorname{lgdim}(C) = n$.
(iv) $\operatorname{lgdim}(X \times Y) = \operatorname{lgdim}(X) + \operatorname{lgdim}(Y)$.
(v) $\operatorname{lgdim}(X_1 \cup \dots \cup X_k) = \max\{\operatorname{lgdim}(X_1), \dots, \operatorname{lgdim}(X_k)\}$.

Proof. (i) is by Lemma 2.4, and (ii) is clear. Item (iii) follows from the Lemma on Subcones 3.1. The proof of (iv) is word-by-word the same with the proof of [EdEl, Fact 2.2(3)] after replacing 'ldim' by 'lgdim' and the notion of a cone by that of a long cone we have here.

For (v), we prove by parallel induction on $n \geq 1$ the following two statements.

$(1)_n$ *For all definable X_1, X_2 such that* $\operatorname{lgdim}(X_1 \cup X_2) = n$, *either* $\operatorname{lgdim}(X_1) = n$ *or* $\operatorname{lgdim}(X_2) = n$.

$(2)_n$ *Let $C \subseteq M^n$ be an n-long cone. For any definable set $X \subseteq C$ with* $\dim(X) \leq n - 1$ *we have* $\operatorname{lgdim}(C \setminus X) = n$.

Statement (v) then clearly follows from $(1)_n$ by induction on k.

STEP I: $(2)_1$ follows from [Pet3, Lemma 3.4(2)].

STEP II: $(1)_{n-1}$ and $(2)_l$ for $l \leq n-1$ imply $(2)_n$, for $n \geq 2$. Assume $(1)_{n-1}$ and $(2)_l$ for all $l \leq n-1$. We perform a sub-induction on $\dim(X)$. Observe that after some suitable linear transformation we may assume that C has form

$$C = \sum_{i=1}^{n} e_i t_i | J_i,$$

where the e_i's are the standard basis vectors.

If $\dim(X) = 0$, then X is finite and, without loss of generality, we may assume that X contains only one point a. Then it is easy to see that $C \setminus \{a\}$ contains 2^n disjoint long cones of the form $a + \sum_{i=1}^{n} e_i t_i | J'_i$ such that, for at least one of them, all J'_i's are long.

Suppose the result holds for all X with $\dim(X) \leq l < n - 1$, and assume now that $\dim(X) = l + 1$. If $l + 1 < n - 1$, then $\dim(\pi(X)) \leq n - 2$ and by $(2)_{n-1}$, $\operatorname{lgdim}(\pi(C) \setminus \pi(X)) = n - 1$, which implies that $\operatorname{lgdim}(C \setminus X) = n$, by (iv).

So now assume that $\dim(X) = n - 1$. By cell decomposition and by the Sub-Inductive Hypothesis, we may assume that X is a finite union of cells $X_1, \dots, X_k$, each of dimension $n - 1$. We perform a second sub-induction on k.

Base Step: suppose $k = 1$. If X_1 is not the graph of a function or $\operatorname{lgdim}(X_1) < n-1$, then by $(2)_{n-1}$ or $(1)_{n-1}$, respectively, we have $\operatorname{lgdim}(\pi(C) \setminus \pi(X_1)) = n-1$, which implies $\operatorname{lgdim}(C \setminus X_1) = n$, by (iv). Thus it remains to examine the case where X_1 is the graph of a function $f : \pi(X_1) \to M$ and $\operatorname{lgdim}(X_1) = n - 1$. In this case, $\operatorname{lgdim}(\pi(X_1)) = \operatorname{lgdim}(X_1) = n - 1$, where the first equality is by Lemma 2.5. Let $D \subseteq \pi(X_1)$ be a $(n-1)$-long cone. Let

$$A = \{\bar{a} \in D : \forall i \in \{1, \dots, n-1\},\ f_{\bar{a}^i} \text{ is monotone around } a_i\}.$$

according to the notation of Lemma 2.2. By that lemma,

$$\dim(D \setminus A) < \dim(D) = n - 1.$$

Hence, by $(2)_{n-1}$, A contains an $(n-1)$-long cone E, and by Lemma 2.16, we may assume that $E = b + \sum_{i=1}^{n-1} e_i t_i|(0,\kappa)$, for some tall κ. Let $\bar{a} = b + \sum_{i=1}^{n-1} e_i \frac{1}{2}\kappa$. Since f is continuous on E, each $f_{\bar{x}^i}$ is monotone on its domain $(0,\kappa)$. Without loss of generality, we may assume that $\forall i \in \{1,\dots,n-1\}$, $f_{\bar{x}^i}$ is increasing on $(0,\kappa)$. We split into two cases:

Case 1: $f(\bar{a})$ is short. Then the n-long cone

$$E_1 = (b, f(\bar{a})) + \sum_{i=1}^{n-1} e_i t_i|(0,\kappa/2) + e_n t_n|J_n/2$$

is contained in X_1.

Case 2: $f(\bar{a})$ is tall. Then the n-long cone

$$E_2 = (\bar{a}, 0) + \sum_{i=1}^{n-1} e_i t_i|(0,\kappa/2) + e_n t_n|J_n/2$$

is contained in X_1. This completes the case $k = 1$.

Inductive Step: suppose the result holds for any X which is a union of less than k cells of dimension $n-1$, and assume now that X is the union of the cells $X_1,\dots,X_k$, each of dimension $n-1$. By Second Sub-Inductive Hypothesis, there is an n-long cone F contained in $C \setminus (X_1 \cup \cdots \cup X_{k-1})$. Now, we reduce to the Base Step for C equal to F and X_1 equal to X_k. This completes the proof of the second sub-induction, as well as that of Step II of the original induction.

STEP III: $(2)_n \Rightarrow (1)_n$. Without loss of generality, we may assume that X_1 and X_2 are disjoint. Since $\mathrm{lgdim}(X_1 \cup X_2) = n$, we may also assume that $X_1 \cup X_2$ is an n-long cone C of dimension n. If $X = bd(X_1) \cup bd(X_2)$, then $\dim(X) \leq n-1$. By $(2)_n$, we conclude that either X_1 or X_2 contains an n-long cone. □

The following corollary will not be used until Section 6.

Corollary 3.7. *Let $X \subseteq M^n$ be a definable set of long dimension k. If $C \subseteq X \times X$ is a $2k$-long cone, then there are k-long cones $C_1, C_2 \subseteq X$, such that $C_1 \times C_2 \subseteq C$.*

Proof. We may assume that $C = b + \sum_{i=1}^{2k} v_i t_i|J_i$. Let $\pi : M^{2n} \to M^{2k}$ be the projection given by Corollary 2.13, whose restriction $\pi_{\restriction C}$ is a bijection onto the $2k$-long cone $\pi(C)$. Moreover, as it can easily be checked, its inverse $(\pi_{\restriction C})^{-1}$ can be written as $\pi_{\restriction C} = (f_1,\dots,f_{2n})$ for some affine maps $f_j : M^{2k} \to M$. By Remark 2.12(ii) and (iii), the graph of $\pi_{\restriction C}^{-1}$ on a k-long cone contained in $\pi(C)$ is a k-long cone, contained in C.

Now let $p_1 : M^{2k} \to M^k$ and $p_2 : M^{2k} \to M^k$ be the suitable projections, so that $\pi(C) \subseteq p_1\pi(C) \times p_2\pi(C)$. Since $\pi(C)$ has long dimension k, by the Lemma on Subcones and 3.6(iv), each of $p_1\pi(C)$ and $p_2\pi(C)$ must have long dimension k. In particular, for each $i = 1,\dots,2k$, there is $t > 0$ with $e_i t \in \langle \pi(C) \rangle$. By Lemma 2.16, $\pi(C)$ contains a $2k$-long cone

$$C' = (b_1, b_2) + \sum_{i=1}^{2k} e_i t_i|(0,a).$$

The k-long cones

$$C_1' = b_1 + \sum_{i=1}^{k} e_i t_i|(0,a) \quad \text{and} \quad C_2' = b_2 + \sum_{i=k}^{2k} e_i t_i|(0,a)$$

are clearly contained in $p_1\pi(C)$ and $p_2\pi(C)$, respectively. By the first paragraph of this proof, the set

$$D = \pi^{-1}_{\restriction C}(C')$$

is a $2k$-long cone contained in C, and each of

$$D_1 = \pi^{-1}_{\restriction C}(C'_1 \times \{b_2\}) \quad \text{and} \quad D_2 = \pi^{-1}_{\restriction C}(\{b_1\} \times C'_2)$$

is a k-long subcone of D. If we take the projection C_1 of D_1 onto the first n coordinates, and the projection C_2 of D_2 onto the last n coordinates, then both C_1 and C_2 are k-long cones, contained in X, such that

$$C_1 \times C_2 = D \subseteq C,$$

as desired. □

3.3. **The Refined Structure Theorem.** We are now in a position to prove the first main result of this paper. For a given a definable function $f : A \times M \to M$, with $A \subseteq M^n$, let us denote

$$\Delta_t f(a, x) := f(a, x + t) - f(a, x),$$

for all $x, t \in M$ and $a \in A$.

Theorem 3.8. *(Refined Structure Theorem). Let $X \subseteq M^n$ be an A-definable set. Then*

(i) X is a finite union of A-definable long cones.

(ii) If X is the graph of an A-definable function $f : Y \to M$, for some $Y \subseteq M^{n-1}$, then there is a finite collection $\mathcal{C}$ of A-definable long cones, whose union is Y and such that f is almost linear with respect to each long cone in $\mathcal{C}$.

Proof. By cell decomposition we may assume that X is an A-definable cell. We prove (i) and (ii), along with (iii) below, by induction on $\langle n, \text{lgdim}(X)\rangle$.

(iii) In the notation from (ii), Y contains an A-definable lgdim(Y)*-long cone such that f is almost linear with respect to it.*

If $n = 1$, then (i), (ii) and (iii) are clear. Assume the Inductive Hypothesis (IH): (i), (ii) and (iii) hold for $\{\langle n, k\rangle\}_{k\leq n}$, and let $X \subseteq M^{n+1}$ with $\text{lgdim}(X) = k \leq n+1$.

Case (I): $\dim(X) < n + 1$. So, after perhaps permuting the coordinates, we may assume that X is the graph of a continuous A-definable function $f : Y \to M$.

(i) This is clear, by (IH)(ii) and Remark 2.12(iii).

(ii) By (IH)(i), we may further assume that $Y = B' + \sum_{i=1}^{k} v_i t_i|J_i$ is an A-definable k-long cone, where $k \leq n$.

Claim. *We may assume that $Y = B + \sum_{i=1}^{k} e_{n-k+i} t_i|J_i$.*

Proof. To see this, we will define a suitable affine transformation from Y into M^n. The idea is to map elements of the form $v_i t$ to $e_{n-k+i}t$. Since the v_i's are not necessarily global endomorphisms, we need to explain how this transformation works.

First extend each v_i, $1 \leq i \leq k$, to a vector u_i in Λ^n with domain $2J_i$. More precisely, if $J_i = (0, a_i)$, let $u_i : (0, 2a_i) \to M^n$ be equal to $v_i(t)$ for $t \in (0, a_i)$, and equal to $(\lim_{s\to a_i} v_i s) + v_i(t - a_i)$ for $t \in (a_i, 2a_i)$. Also, choose $u_{k+1}, \dots, u_n \in \Lambda^n$ with long domains $J_{k+1}, \dots, J_n$ so that all $u_1, \dots u_n$ are M-independent (in fact, $u_{k+1}, \dots, u_n$ can be chosen among the unit vectors in Λ^n).

Now, fix any $b \in B'$ and let $C = \sum_{i=1}^n v_i t_i | J_i$. By Lemma 2.4, $b + \langle C \rangle$ is open. We claim that $b + \langle C \rangle$ contains Y. First we observe that B' is contained in $b + \langle C \rangle$. Since B' is connected and contains b, if B' were not contained in $b + \langle C \rangle$, we would have a definable path that starts from b and ends outside $b + \langle C \rangle$. This path has short domain but long range, a contradiction.

Now we want to see that every element x in Y is contained in $b + \langle C \rangle$. Let $x = b' + \sum_{i=1}^k v_i t_i$. Since b' is in $b + \langle C \rangle$, we have $b' = b + \sum_{i=1}^k v_i s_i + \sum_{i=k+1}^n u_i s_i$. Therefore, $x = b + \sum_{i=1}^k u_i(s_i + t_i) + \sum_{i=k+1}^n u_i s_i$, that is, $x \in b + \langle C \rangle$.

Now that we know that $b + \langle C \rangle$ contains Y, we define the following transformation:

$$T : b + \langle C \rangle \to M^n, \quad T\left(b + \sum_{i=1}^n u_i t_i\right) = b + \sum_{i=1}^k e_{n-k+i} t_i + \sum_{i=k+1}^n e_{n-i+1} t_i$$

This is a bijection map onto its image. Clearly, $T(Y) = T(B') + \sum_{i=1}^k e_{n-k+i} t_i | J_i$, as the reader can verify that $T(b' + \sum_{i=1}^k v_i t_i) = T(b') + \sum_{i=1}^k e_{n-k+i} t_i$. Hence, we can let $B = T(B'$ and replace Y by $T(Y)$. □

Let $\pi : M^n \to M^{n-1}$ be the usual projection. By [Pet3, Lemma 4.10] and its proof, there are A-definable linear functions $\lambda_1, \ldots, \lambda_l$, A-definable functions $a_0, \ldots, a_m : \pi(Y) \to M$ and a short positive element $b \in dcl(A)$ of M, such that for every $x \in \pi(Y)$,

- $0 = a_0(x) \leq a_1(x) \leq \cdots \leq a_{m-1}(x) \leq a_m(x) = e_n | J_k|$
- for every i, either $|a_{i+1}(x) - a_i(x)| < b$ or the map $t \mapsto \Delta_t f(x, a_i(x))$ on $(0, a_{i+1}(x) - a_i(x))$ is the restriction of some λ_j; that is

$$f(x, a_i(x) + t) - f(x, a_i(x)) = \lambda_j(t). \tag{8}$$

For every $z = (x, y) \in Y$, let $b_z := a_{i+1}(x) - a_i(x)$, where $y \in (a_i(x), a_{i+1}(x))$. Observe that $b_z \in dcl(\emptyset)$. Set

$$Y_0 = \{z \in Y : b_z \geq b\},$$

and consider (by cell decomposition) a partition $\mathcal{C}$ of Y_0 into cells so that for every $Z \in \mathcal{C}$,

- there is some λ_j such that the restriction of f on Z satisfies (8) above,
- Z is contained in $\{(x, y) : a_i(x) \leq y \leq a_{i+1}(x)\}$.

By (IH)(ii), there is a finite collection $\mathcal{C}'$ of A-definable long cones, whose union is $\pi(Z)$ and such that each a_i is almost linear with respect to each $C \in \mathcal{C}'$. By (IH)(i), there is a finite collection $\mathcal{C}''$ of A-definable long cones, whose union is $Z \cap \pi^{-1}(C)$. Observe now that $Z \cap \pi^{-1}(C)$ is contained in some long cone W on which f is almost linear; namely, if $C = D + \sum_{i=1}^l w_i t_i | K_i$, then W is of the form

$$W = D \times \{d\} + \sum_{i=1}^l w_i t_i | K_i + e_n t_n | K_n,$$

where K_n is a long interval of length equal to $\max\{a_{i+1}(x) - a_i(x) : x \in C\}$. By Lemma 3.4, we conclude that f is almost linear with respect to each long cone in $\mathcal{C}''$.

It remains to prove (i) for $Y \setminus Y_0$. But this is given by (IH)(ii), since, in fact, $\text{lgdim}(Y \setminus Y_0) < k$: assuming not, apply (IH)(iii) to get a k-long cone $C \subseteq Y \setminus Y_0 \subseteq Y$. By Corollary 3.5, there is a tall $a \in M$ such that $e_n a \in C$. But then f is linear in x_n on some long interval contained in $Y \setminus Y_0$, a contradiction. Hence $\operatorname{lgdim}(Y \setminus Y_0) < k$.

(iii) In the above notation, for every $i \in \{0, \dots, m-1\}$, the set

$$P_i := \{\bar{x} \in \pi(Y) : a_{i+1}(\bar{x}) - a_i(\bar{x}) \geq b\}$$

is A-definable and, since J_n is long, $\pi(Y) = \bigcup_{i=0}^{m-1} P_i$. By Lemma 3.6(v), one of the P_i's, say P_j, must have long dimension $k-1$. By (IH)(iii), there is a finite collection $\mathcal{C}'$ of A-definable long cones, whose union is W_j and such that each a_j and a_{j+1} are almost linear with respect to each $C \in \mathcal{C}'$. By Lemma 2.17, there is an A-definable k-long cone $E \subseteq Y$ and, as before, f is almost linear with respect to E.

Case (II): $\dim(X) = n+1$. The argument in this case is a combination of the proofs of [ElSt, Lemma 3.6] and of [Pet1, Theorem 3.1]. So $X = (g,h)_Y$ is a cylinder. By (IH)(ii) and Lemma 3.4, we may assume that $Y = B + \sum_{i=1}^k v_i t_i | J_i$ is a long cone and that g, h are almost linear with respect to it. Assume they are of the form:

$$g\left(b + \sum_{i=1}^k v_i t_i\right) = g(b) + \sum_{i=1}^k n_i t_i \text{ and } h\left(b + \sum_{i=1}^k v_i t_i\right) = h(b) + \sum_{i=1}^k m_i t_i.$$

Since $g < h$ on Y, it follows that for every $b \in B$, $g(b) \leq h(b)$. One of the following two cases must occur:

Case (II$_a$): for all $i = 1, \dots, k$, we have $n_i = m_i$.

Case (II$_b$): for all $i = 1, \dots, k$, we have $n_i \leq m_i$, and for at least one i we have $n_i < m_i$. (We may assume so by Remark 2.12(iv): indeed, if for some i, $n_i > m_i$, then we can change B and replace n_i by $n_i' = -n_i$, and m_i by $m_i' = -m_i$, as indicated in Remark 2.12(iv). Then $n_i' < m_i'$.)

Proof of Case (II$_a$). We have

$$X = \left\{(b, y) + \sum_{i=1}^k (v_i, n_i) t_i : g(b) < y < h(b), b \in B, t_i \in J_i\right\}.$$

It is easy to check that, if $(g(b), h(b))$ is a long interval, then

$$X = \{(b, g(b)) : b \in B\} + \sum_{i=1}^k (v_i, n_i) t_i | J_i + e_{n+1} t_{n+1} | (0, h(b) - g(b))$$

is a $(k+1)$-long cone, and if $(g(b), h(b))$ is short, then

$$X = \big\{\{b\} \times (g(b), h(b)) : b \in B\big\} + \sum_{i=1}^k (v_i, n_i) t_i | J_i$$

is a k-long cone.

Proof of Case (II$_b$). We have

$$X = \left\{\left(b + \sum_{i=1}^k v_i t_i, y\right) : g(b) + \sum_{i=1}^k n_i t_i < y < h(b) + \sum_{i=1}^k m_i t_i, b \in B, t_i \in J_i\right\}.$$

Notice that if $h = +\infty$ on X (similarly, if $g = -\infty$), then we are done because

$$X = \{(b, g(b)) : b \in B\} + \sum_{i=1}^k v_i t_i | J_i + e_n t_n | (0, +\infty).$$

We partition X in the following way, going from "top" to "bottom":

$$X_1 = \left\{ \left(b + \sum_{i=1}^{k} v_i t_i, y \right) : h(b) + \sum_{i=1}^{k} n_i t_i < y < h(b) + \sum_{i=1}^{k} m_i t_i, b \in B, t_i \in J_i \right\},$$
$$X_2 = \left\{ \left(b + \sum_{i=1}^{k} v_i t_i, y \right) : y = h(b) + \sum_{i=1}^{k} n_i t_i, b \in B, t_i \in J_i \right\},$$
$$X_3 = \left\{ \left(b + \sum_{i=1}^{k} v_i t_i, y \right) : g(b) + \sum_{i=1}^{k} n_i t_i < y < h(b) + \sum_{i=1}^{k} n_i t_i, b \in B, t_i \in J_i \right\}.$$

By Remark 2.12(iii), X_2 is a k-long cone, whereas X_3 clearly satisfies the condition of Case (II$_a$). Hence we only need to account for X_1.

Let $S_{X_1} = \{i = 1, \dots, k : n_i < m_i\}$. By induction on $|S_{X_1}|$ we may assume that $|S_{X_1}| = 1$. Indeed, if, say, $n_1 < m_1$ and $n_2 < m_2$, then we can partition X_1 in the following way, going again from "top" to "bottom":

$$X_1' = \left\{ \left(b + \sum_{i=1}^{k} v_i t_i, y \right) : h(b) + n_1 t_1 + \sum_{i=2}^{k} m_i t_i < y < h(b) + \sum_{i=1}^{k} m_i t_i, b \in B, t_i \in J_i \right\},$$
$$X_1'' = \left\{ \left(b + \sum_{i=1}^{k} v_i t_i, y \right) : y = h(b) + n_1 t_1 + \sum_{i=2}^{k} m_i t_i, b \in B, t_i \in J_i \right\},$$
$$X_1''' = \left\{ \left(b + \sum_{i=1}^{k} v_i t_i, y \right) : h(b) + \sum_{i=1}^{k} n_i t_i < y < h(b) + n_1 t_1 + \sum_{i=2}^{k} m_i t_i, b \in B, t_i \in J_i \right\}.$$

Observe then that X_1'' is a k-long cone, and for X_1' and X_1''', each of the corresponding $S_{X_1'}$ and $S_{X_1'''}$ has size less than $|S_{X_1}|$.

So assume now that $|S_{X_1}| = 1$ with, say, $n_1 < m_1$ and $n_i = m_i$ for $i > 1$. Let

$$A = \left\{ \left(\sum_{i=1}^{k} v_i t_i, y \right) : \sum_{i=1}^{k} n_i t_i < y < \sum_{i=1}^{k} m_i t_i, b \in B, t_i \in J_i \right\}.$$

We show that A is the union of long cones which clearly implies that so is X_1. If $J_1 = (0, \infty)$, then

$$A = (v_1, n_1) t_1 | J_1 + \sum_{i=1}^{k} (v_i, m_i) t_i | J_i$$

is already a $(k+1)$-long cone. If $J_1 = (0, a_1)$, with $a_1 \in M$, then A is the union of the following $(k+1)$-long cones:

$$Y_1 = (v_1, n_1) t_1 | (0, \frac{a_1}{2}) + (v_1, m_1) t_1 | (0, \frac{a_1}{2}) + \sum_{i=2}^{k} (v_i, m_i) t_i | J_i,$$
$$Y_2 = (v_1, n_1) \frac{a_1}{2} + (v_1, n_1) t_1 | (0, \frac{a_1}{2}) + \sum_{i=2}^{k} (v_i, m_i) t_i | J_i + e_n t_n | (0, \frac{(m_1 - n_1) a_1}{2})$$
$$Y_3 = (v_1, n_1) \frac{a_1}{2} + (v_1, m_1) t_1 | (0, \frac{a_1}{2}) + \sum_{i=2}^{k} (v_i, m_i) t_i | J_i + e_n t_n | (0, \frac{(m_1 - n_1) a_1}{2})$$

□

Remark 3.9. As opposed to the corresponding results from [Ed] and [Pet1], it is not always possible to achieve a *disjoint* union in (i) or (ii). We leave it to the reader to verify that the following set cannot be written as a disjoint union of long cones: let X be the 'triangle' with corners the origin, the point (a, a) and the point $(0, 2a)$, for some long element a.

As a first corollary, we obtain a quantifier elimination result down to suitable existential formulas in the spirit of [vdD1].

Corollary 3.10. *Every definable subset $X \subseteq M^m$ is a boolean combination of subsets of M^m defined by*

$$\exists y_1 \dots \exists y_m \, B(y_1, \dots, y_m) \wedge \varphi(x_1, \dots, x_m, y_1, \dots, y_m),$$

where $B(y)$ is a short formula and $\varphi(x, y)$ is a quantifier-free $\mathcal{L}_{lin}$-formula. In fact, X is a finite union of such sets.

Another corollary is the following.

Corollary 3.11. *If $f : X \to M^n$ is a definable injective function, then* $\operatorname{lgdim}(X) = \operatorname{lgdim}\big(f(X)\big)$.

Proof. Assume that $X \subseteq M^k$ and that $f = (f^1, \dots, f^n)$, where $f^j : X \to M$. By the Refined Structure Theorem and Lemma 3.6(v), we may assume that X is a long cell of the form $X = b + \sum_{i=1}^k v_i t_i | J_i$ and such that each f_j is almost linear on X. Hence, for every j, there are $\mu_1^j, \dots, \mu_k^j$ so that $f^j(b + \sum_{i=1}^k v_i t_i) = f^j(b) + \sum_{i=1}^k \mu_i^j t_i$. Thus, $f(X)$ is the long cell

$$(f^1(b), \dots, f^n(b)) + \sum_{i=1}^k \mu_i t_i | J_i,$$

where each $\mu_i = (\mu_i^1, \dots, \mu_i^n) \in \Lambda^n$. □

4. On definability of long dimension

The following example shows that we lack 'definability of long dimension'.

Example 4.1. Let $a > 0$ be a tall element and let

$$X = \{(x, y) : 0 \le x \le a, 0 \le y \le x\}.$$

Denote by $\pi : M^2 \to M$ the usual projection. Then, by [Pet3, Proposition 3.6], the set

$$X_1 = \{x \in [0, a] : \pi^{-1}(x) \text{ has long dimension } 1\}$$

is not definable.

However, X_1 clearly contains a 'suitable' definable set; namely, a definable set of long dimension 1. It follows from the lemmas of this section that the set of fibers of long dimension l of a given definable set X always lies between two definable sets each of long dimension $\operatorname{lgdim}(X) - l$ (Corollary 4.4 below).

Lemma 4.2. *Let $X \subseteq M^{n+m}$ be a definable set such that the projection $\pi(X)$ onto the first n coordinates has long dimension k. Let $0 \le l \le m$. Then*

(i) $\operatorname{lgdim}(X) \le k + m$.

(ii) $\operatorname{lgdim}(X) \ge k + l$ *if and only if $\pi(X)$ contains a k-long cone C such that every fiber X_c, $c \in C$, has long dimension $\ge l$.*

Proof. (i) By Lemma 3.6(ii)&(iv), since $X \subseteq \pi(X) \times M^m$.

(ii) ($\Leftarrow$) Assume that every fiber X_c, $c \in C$, has long dimension l. We prove that $\text{lgdim}(X) \geq k + l$ by induction on k. For $k = 0$, it is clear, since any fiber above C contains a l-long cone. Now assume that it is proved for $\text{lgdim}(C) < k$, and let $\text{lgdim}(C) = k$. Clearly, we may assume that $\pi(X) = C$. For the sake of contradiction, assume $\text{lgdim}(X) < k+l$. By the Refined Structure Theorem, X can be covered by finitely many long cones $X_1, \ldots, X_s$, each with $\text{lgdim}(X_i) < k + l$. By the inductive hypothesis, each $\pi(X_i)$ has long dimension $< k$. But then $C = \pi(X_1) \cup \cdots \cup \pi(X_s)$ must have long dimension $< k$, a contradiction.

($\Rightarrow$) This is clearly equivalent to the following:

Claim. *Let*

$$X_l = \{x \in \pi(X) : \pi^{-1}(x) \text{ has long dimension} \geq l\}$$

Then there is a definable set $Y_l \subseteq X_l$, such that

$$\text{lgdim}(Y_l) = \text{lgdim}(X) - l.$$

The proof of the Claim is by induction on m.

Base Step: $m = l = 1$. By cell decomposition, X is a finite union of cells, and by the Refined Structure Theorem the domain of each cell is a finite union of long cones such that the corresponding restrictions of the defining functions of the cell are almost linear with respect to each of the long cones. If a cell is a graph of a function, or if its domain has long dimension $< k$, then clearly its long dimension is at most k. Hence X contains a cylinder $X_1 = (f, g)_{\pi(X_1)}$, where $\pi(X_1)$ is a k-long cone, such that X_1 contains a $(k + 1)$-long cone $C = b + \sum_{i=1}^{k+1} v_i t_i|(0, \alpha_i)$. We will first show that for some elements $x, y \in \overline{C}$ in the closure of C, with $\forall i = 1, \ldots, n$, $x_i = y_i$, and $(y - x)_{n+1}$ tall. This is straightforward and we only sketch its proof.

The projection $\pi(C)$ is a union of long cones whose directions are tuples with elements from the set $\{v_1, \ldots, v_{k+1}\}$. By (i) and Lemma 3.6(v), there must be subset of $\{v_1, \ldots, v_{k+1}\}$ of k elements, say $\{v_1, \ldots, v_k\}$, whose projections onto the first n-coordinates is an M-independent set. Without loss of generality, assume $A = \{v_1, \ldots, v_k\}$. It is then an easy exercise to see that there is an element $y = v_1 t_1 + \cdots + v_{k+1} t_{k+1} \in \overline{C}$, such that the element

$$x = \min\{z \in \overline{C} : \forall i \leq n, z_i = y_i\}$$

has form $x = v_1 s_1 + \cdots + v_{k+1} s_{k+1} \in \overline{C}$ such that $t_{k+1} - s_{k+1}$ is long. But then $y - x$ must be tall, by Lemma 2.6. It follows that $(y - x)_{n+1}$ must be tall.

Now, we conclude that there is $x \in \pi(X_1)$, such that $\pi^{-1}(x) = \big(f(x), g(x)\big)$ is long. Since f, g are almost linear on $\pi(X_1)$, it is easy to see that there is a k-long cone $C_x = x + \sum_{i=1}^{k} v_i t_i|(0, a_i) \subseteq \pi(X_1)$ such that for each element $y \in C_x$, $g(y) - f(y)$ is tall. We let $Y_l = C_x$. Since, by (i), $k \geq \text{lgdim}(X) - 1$, we are done.

Inductive Step: assume we know the lemma for every n and $X \subseteq M^n \times M^m$, and let $X \subseteq M^n \times M^{m+1}$. Let $q : M^{n+m+1} \to M^{n+m}$ and $r : M^n \times M^m \to M^n$ be the usual projections. Of course, $\pi = r \circ q$.

Case (I). $\text{lgdim}(q(X)) = \text{lgdim}(X)$. In this case, by the Inductive Hypothesis, the set

$$q(X)_l = \{x \in \pi(X) : \text{lgdim}(r^{-1}(x)) \geq l\}$$

contains a definable set A such that

$$\text{lgdim}(A) = \text{lgdim}(q(X)) - l = \text{lgdim}(X) - l.$$

Since, clearly, $q(X)_l \subseteq X_l$, we are done.

Case (II). $\text{lgdim}(q(X)) = \text{lgdim}(X) - 1$. Let

$$Y_1 = \{x \in q(X) : \text{lgdim}(q^{-1}(x)) = 1\}.$$

By the Base Step, Y_1 contains some definable set Y with $\text{lgdim}(Y) = \text{lgdim}(X) - 1$. By the Inductive Hypothesis, the set

$$Y_l = \{x \in r(Y) : \text{lgdim}(r^{-1}(x)) \geq l - 1\}$$

contains a definable set A with

$$\text{lgdim}(A) = \text{lgdim}(Y) - (l - 1) = \text{lgdim}(X) - l.$$

But clearly X_l contains A and hence we are done. □

On the other hand, we have the following lemma. It will not be essential until the proof of Proposition 6.4.

Lemma 4.3. *Let $X \subseteq M^{n+m}$ be a definable set and denote by $\pi : M^{n+m} \to M^n$ the usual projection. For $0 \leq l \leq m$, let*

$$X_l = \{x \in \pi(X) : \pi^{-1}(x) \text{ has long dimension} \geq l\}.$$

Then there is a definable subset $Z_l \subseteq \pi(X)$ with $X_l \subseteq Z_l$ such that

$$\text{lgdim}(Z_l) = \text{lgdim}(X) - l.$$

Proof. The proof is by induction on m. For any m, if $l = 0$, then take $Z_l = \pi(X)$, since, by Lemma 4.2(ii), $\text{lgdim}(\pi(X)) \leq \text{lgdim}(X)$.
Base Step: $m = 1$. Let $X \subseteq M^n \times M$ and $l = 1$. By cell decomposition and Lemma 3.6(v), we may assume that X is a cell. If X is the graph of a function, then let Z_l be any subset of $\pi(X)$ of long dimension $\text{lgdim}(X) - 1$. So assume X is the cylinder $(f, g)_{\pi(X)}$ between two continuous functions f and g. By the Refined Structure Theorem, we may further assume that $\pi(X)$ is a long cone such that f and g are both almost linear with respect to it. If $\text{lgdim}(\pi(X)) = \text{lgdim}(X) - 1$, then take $Z_l = \pi(X)$. If $\text{lgdim}(\pi(X)) = \text{lgdim}(X)$, then by Lemma 2.17, for every $x \in \pi(X)$, $\pi^{-1}(x)$ is short, in which case we let again Z_l be any subset of $\pi(X)$ of long dimension $\text{lgdim}(X) - 1$.
Inductive Step: assume we know the lemma for every n and $X \subseteq M^n \times M^m$, and let $X \subseteq M^n \times M^{m+1}$. Let $q : M^{n+m+1} \to M^{n+m}$ and $r : M^n \times M^m \to M^n$ be the usual projections. Let

$$Y_1 = \{x \in q(X) : \text{lgdim}(q^{-1}(x)) = 1\}.$$

By Lemma 4.2(ii), Y_1 is contains some definable set Y with $\text{lgdim}(Y) = \text{lgdim}(X) - 1$. Now, X_l is contained in the union of the following two sets:

$$A_1 = \{x \in r(Y) : \text{lgdim}(r^{-1}(x)) \geq l - 1\} \text{ and}$$
$$B_1 = \{x \in r(q(X) \setminus Y) : \text{lgdim}(r^{-1}(x)) = l\}.$$

By the Inductive Hypothesis, A_1 is contained in a definable set A with

$$\text{lgdim}(A) = \text{lgdim}(Y) - (l - 1) = \text{lgdim}(X) - l$$

and B_1 is contained in a definable set B with

$$\text{lgdim}(B) = \text{lgdim}(q(X) \setminus Y) - l \leq \text{lgdim}(X) - l.$$

Hence X_l is contained in the definable set $Z_l = A \cup B$, satisfying $\text{lgdim}(Z_l) \leq \text{lgdim}(X) - l$. By Lemma 4.2(ii), $\text{lgdim}(Z_l) = \text{lgdim}(X) - l$. □

We are finally in the position to prove the promised corollary on the definability of long dimension. Note that this corollary is not needed in the rest of the paper, but it is recorded here in the interests of completeness.

Corollary 4.4. *Let $X \subseteq M^{n+m}$ be a definable set and denote by $\pi : M^{n+m} \to M^n$ the usual projection. For $0 \le l \le m$, let*

$$l(X) = \{x \in \pi(X) : \pi^{-1}(x) \text{ has long dimension } l\}.$$

Then there are definable subsets $Y, Z \subseteq \pi(X)$ with $Y \subseteq l(X) \subseteq Z$ such that

$$\text{lgdim}(Y) = \text{lgdim}(Z) = \text{lgdim}(X) - l.$$

Proof. With the notation of the previous two lemmas, let $Y = Y_l \setminus Z_{l+1}$ and $Z = Z_{l+1}$. Since $\text{lgdim}(Y_l) = \max\{\text{lgdim}(Y), \text{lgdim}(Z_{l+1})\}$, it follows that $\text{lgdim}(Y)$ is as desired (and, clearly, so is $\text{lgdim}(Z)$). □

5. Pregeometries

In this section we develop the combinatorial counterpart of the long dimension and define the corresponding notion of 'long-genericity'. This notion is used in the application to definable groups in the next section.

Definition 5.1. A (finitary) *pregeometry* is a pair (S, cl), where S is a set and $cl : P(S) \to P(S)$ is a *closure operator* satisfying, for all $A, B \subseteq S$ and $a, b \in S$:
(i) $A \subseteq cl(A)$
(ii) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
(iii) $cl\big(cl(A)\big) = cl(A)$
(iv) $cl(A) = \cup\{cl(B) : B \subseteq A \text{ finite}\}$
(v) (Exchange) $a \in cl(bA) \setminus cl(A) \Rightarrow b \in cl(aA)$.

Definition 5.2. We define the *short closure* operator $scl : P(M) \to P(M)$ as:

$$scl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and } \phi(x, \bar{y}) \text{ from } \mathcal{L}, \text{ such that } \phi(\mathcal{M}, \bar{b}) \text{ is a short interval and } \mathcal{M} \vDash \phi(a, \bar{b})\}.$$

We say that the formula $\phi(x, \bar{y}) \in \mathcal{L}$ *witnesses* $a \in scl(\bar{b})$ if $\phi(\mathcal{M}, \bar{b})$ is a short interval and $\mathcal{M} \vDash \phi(a, \bar{b})$.

We will omit, as usually, the bar from tuples.

Remark 5.3. Given a formula $\phi(x, y) \in \mathcal{L}$ witnessing $a \in scl(b)$, we can form a formula $S^{\phi}_{a,b}(x, y) \in \mathcal{L}$ which is satisfied by the pair (a, b) and such that for every $b' \in M^n$, $S^{\phi}_{a,b}(\mathcal{M}, b')$ is short. Indeed, let $\kappa \in M$ be short such that

$$\forall z_1, z_2[\phi(z_1, b) \wedge \phi(z_2, b) \to |z_1 - z_2| < \kappa].$$

By [Pet3, Corollary 3.7(3)], κ may be taken in $dcl(\emptyset)$. We then let

$$S^{\phi}_{a,b}(x, y) : \phi(x, y) \wedge \forall z_1, z_2[\phi(z_1, y) \wedge \phi(z_2, y) \to |z_1 - z_2| < \kappa].$$

Lemma 5.4. *$a \in scl(b) \Leftrightarrow \exists a' \in dcl(b), a - a'$ is short.*

Proof. ($\Rightarrow$). Let f be a $\emptyset$-definable Skolem function for $S^{\phi}_{a,b}(x, y)$, where ϕ witnesses $a \in scl(b)$; that is, for every $c \in M$, $\models \exists z S^{\phi}_{a,b}(z, c) \to S^{\phi}_{a,b}(f(c), c)$. Let $a' = f(b)$.

($\Leftarrow$). Assume $\phi(x, y)$ witnesses $a' \in dcl(b)$. Let $\kappa \in dcl(\emptyset)$ such that $|a - a'| < \kappa$. Then a satisfies the following short formula:

$$\exists x' \phi(x', b) \wedge (|x - x'| < \kappa).$$

□

Lemma 5.5. (M, scl) *is a pregeometry.*

Proof. Properties (i), (ii) and (iv) are straightforward.

(iii). This boils down to the fact that (Lemma 4.2(ii)) a short union of short sets is short. We provide the details. Let $a \in scl(\bar{b})$, where $\bar{b} = (b_1, \ldots, b_n) \in M^n$, such that each $b_i \in scl(\bar{c})$, for some $\bar{c} \subseteq A$. Assume that $\psi(x, \bar{b})$ witnesses $a \in scl(\bar{b})$, and that for each $i = 1, \ldots, n$, $\phi_i(y_i, \bar{c})$ witnesses $b_i \in scl(\bar{c})$, where $\psi, \phi_i \in \mathcal{L}$. Denote

$$S(\bar{y}, \bar{z}) := S^{\phi_1}_{b_1,\bar{c}}(y_1, \bar{z}) \wedge \cdots \wedge S^{\phi_n}_{b_n,\bar{c}}(y_n, \bar{z})$$

Then the set X defined by the formula

$$\exists \bar{y} S(\bar{y}, \bar{c}) \wedge S^{\psi}_{a,\bar{b}}(x, \bar{y})$$

is $\bar{c}$-definable and contains a. We show that X is short. Clearly, the set

$$X' = \bigcup_{\bar{y} \in S(\mathcal{M}, \bar{c})} \{\bar{y}\} \times S^{\psi}_{a,\bar{b}}(\mathcal{M}, \bar{y})$$

has long dimension at least the long dimension of X, since the function $f : (\bar{y}, x) \mapsto x$ maps X' onto X. But X' is a short union of short sets and, by Lemma 4.2(ii), it must have long dimension 0.

(v). Without loss of generality, assume $A = \emptyset$. Let $\phi(x, y)$ be a formula witnessing $a \in scl(b)$. We assume that $b \notin scl(a)$ and show $a \in scl(\emptyset)$. Let $f(x)$ be a $\emptyset$-definable Skolem function for $S^{\phi}_{a,b}(x, y)$. Let $\kappa \in M$ be short and in $dcl(\emptyset)$ such that

$$\forall z_1, z_2[\phi(z_1, b) \wedge \phi(z_2, b) \to |z_1 - z_2| < \kappa].$$

(see Remark 5.3). Let

$$Y = \{b' \in M : |f(b') - a| < \kappa\}.$$

Then since Y is a-definable and contains b, it must be long. By Lemma 2.1, there is some interval contained in Y on which f is constant, equal say to a'. But then $a' \in dcl(\emptyset)$ and, by Lemma 5.4, $a \in scl(\emptyset)$. □

Definition 5.6. Let $A, B \subseteq M$. We say that B is *scl-independent over A* if for all $b \in B$, $b \notin scl\big(A \cup (B \setminus \{b\})\big)$. A maximal *scl*-independent subset of B over A is called *a basis for B over A*.

By the Exchange property in a pregeometric theory, any two bases for B over A have the same cardinality. This allows us to define the *rank of B over A*:

$$rank(B/A) = \text{ the cardinality of any basis of } B \text{ over } A.$$

Lemma 5.7. *If p is a partial type over $A \subseteq M$ and $a \models p$ with* $\mathrm{rank}(a/A) = m$*, then for any set $B \supseteq A$ there is $a' \models p$ (possibly in an elementary extension) such that* $\mathrm{rank}(a'/B) \geq m$.

Proof. The proof of the analogous result for the usual rank (coming from *acl*) is given, for example, in [G, page 315]. The proof of the present lemma is word-by-word the same with that one, after replacing an 'algebraic formula' by a 'short formula' in the definition of Φ^m_B ([G, Definition 2.2]) and the notion of 'algebraic independence' by that of '*scl*-independence' we have here. □

Definition 5.8. Assume $\mathcal{M}$ is sufficiently saturated. Let p be a partial type over $A \subset M$. The *short closure dimension of* p is defined as follows:

$$\text{scl-dim}(p) = \max\{\text{rank}(\bar{a}/A) : \bar{a} \subset M \text{ and } \bar{a} \models p\}.$$

Let X be a definable set. Then the *short closure dimension of* X, denoted by scl-dim(X) is the dimension of its defining formula.

In Corollary 5.10 below we establish that the scl-dimension of a definable set coincides with its long dimension we defined earlier. We note that the equivalence between the usual geometric and topological dimension was proved in [Pi1].

Lemma 5.9. *Let $\bar{a} \subseteq M$ be an n-tuple and $A \subseteq M$ a set. Then* rank$(\bar{a}/A) = n$ *if and only if $\bar{a}$ does not belong to any A-definable set with long dimension $< n$.*

Proof. ($\Leftarrow$) Assume $\bar{a} = (a_1, \dots, a_n)$ and rank$(\bar{a}/A) < n$. Then for some i, say $i = 1$, $a_1 \in scl(A \cup \{a_2, \dots, a_n\})$. Let $\phi(x, \bar{y})$ be an $\mathcal{L}(A)$-formula witnessing this fact. Recall from Remark 5.3 that the $\mathcal{L}(A)$-formula $S^{\phi}_{\bar{a}}(x, \bar{y})$ is satisfied by $\bar{a}$ and for every $b' \in M^{n-1}$, $S^{\phi}_{\bar{a}}(\mathcal{M}, b')$ is short. By Lemma 4.2(ii), $S^{\phi}_{\bar{a}}(\mathcal{M}^n)$ has long dimension $< n$.

($\Rightarrow$) We prove the statement by induction on n. For $n = 1$, it is clear. Suppose it is proved for n. Let $\bar{a} = (a_1, \dots, a_{n+1})$ be a tuple of rank, over A, equal to $n+1$ and assume, for a contradiction, that X is an A-definable set containing a with lgdim$(X) < n+1$. By cell decomposition, we may assume that X is an A-definable cell. If X is the graph of a function, then a_{n+1} is in $dcl(A \cup \{a_1, \dots, a_n\})$ and hence in $scl(A \cup \{a_1, \dots, a_n\})$, a contradiction. Now assume that X is a cylinder. By the Refined Structure Theorem, we may assume that $X = (f, g)_{\pi(X)}$ is a cylinder whose domain is an A-definable long cone such that f and g are almost linear with respect to it. Since rank$(\bar{a}/A) = n+1$, $g(a_1, \dots, a_n) - f(a_1, \dots, a_n)$ must be long. But by Inductive Hypothesis, lgdim$(\pi(X)) = n$. Hence, by Lemma 2.17, lgdim$(X) = n+1$, a contradiction. □

Corollary 5.10. *For every definable $X \subseteq M^n$,*

$$\text{lgdim}(X) = \text{scl-dim}(X).$$

Proof. It is easy to see that every A-definable k-long cone X contains a tuple a with rank$(a/A) = k$. On the other hand, by Lemmas 2.13 and 5.9, X cannot contain a tuple a with rank$(a/A) > k$. □

5.1. **Long-generics.** For a treatment of the classical notion of generic elements, corresponding to the algebraic closure *acl*, see [Pi2]. Here we introduce the corresponding notion for *scl*.

Definition 5.11. Let $X \subseteq M^n$ be an A-definable set, and let $\bar{a} \in X$. We say that $\bar{a}$ is a *long-generic element of X over A* if it does not belong to any A-definable set of long dimension $<$ lgdim(X). If $A = \emptyset$, we call $\bar{a}$ a *long-generic element of X*.

In a sufficiently saturated o-minimal structure, *long-generic elements always exist*. More precisely, every A-definable set X contains a long-generic element over A. Indeed, by Compactness and Lemma 3.6(v), the collection of all formulas which express that x belongs to X but not to any A-definable set of long dimension $<$ lgdim(X) is consistent.

A definable subset V of a definable set X is called *long-large in* X if lgdim$(X \setminus V) <$ lgdim(X). In a sufficiently saturated o-minimal structure, V is long-large in

X if and only if for every A over which V and X are defined, V contains every long-generic element a in X over A.

Two long-generics are called *independent* if one (each) of them is long-generic over the other.

Let G be a definable abelian group. Let us recall the notion of a left generic *set* (not to be confused with the notion of a generic element). A subset $X \subseteq G$ is called *left n-generic* if n left translates of X cover G. It is called *left generic* if it is left n-generic for some n. We recall from [ElSt, Lemma 3.10] (see [PeS] for the notion of definable compactness):

Fact 5.12 (Generic Lemma)**.** *Assume G is definably compact. Then, for every definable subset $X \subseteq G$, either X or $G \setminus X$ is left generic.*

The facts that (M, scl) is a pregeometry and that the scl-dim agrees with lgdim imply:

Claim 5.13. *Let $G = \langle G, \cdot \rangle$ be a definable group with* $\text{lgdim}(G) = n$*. Then*

(1) If $X \subseteq G$ long-large in G, then X is left $(n+1)$-generic in G.
(2) If a and $g \in G$ are independent long-generics, then so are a and $a \cdot g$.

Proof. The proof is standard. (1) is word-by-word the same with that of [Pet2, Fact 5.2] after replacing: a) the notion of a 'large' set by that of a 'long-large' set, b) the 'dimension' of a definable set by 'long dimension', and c) the 'dimension' of a tuple by 'rank'. (2) is straightforward using the Exchange property. □

Note that none of the notions 'generic element' and 'long-generic element' implies the other.

Lemma 5.14. *Let X, W be definable subsets of a definable group G. Assume that X is a long-large subset of W and that W is left generic in G. Then X is left generic in G.*

Proof. This is similar to the proof of [PePi, Lemma 3.4(ii)]. Since W is left generic we can write $G = g_1 W \cup \cdots \cup g_m W$. Let $Y = W \setminus X$. Then $Z = g_1 Y \cup \cdots \cup g_m Y$ has long dimension $< \text{lgdim}(G)$. So, by Claim 5.13, finitely many left translates of $G \setminus Z$ cover G. It follows then that finitely many left translates of X cover G. □

We record one more lemma, which however will not be used in this paper:

Lemma 5.15. *Let G be a definable group and X a definable subset of G. If X is left generic in G then* $\text{lgdim}(X) = \text{lgdim}(G)$*.*

Proof. Since the group conjugation is a definable bijection, the statement follows from Lemma 3.6(v) and Corollary 3.11. □

6. The local structure of semi-bounded groups

In this section, we assume that $\mathcal{M}$ is sufficiently saturated, and we fix a $\emptyset$-definable group $G = \langle G, \oplus, e_G \rangle$, with $G \subseteq M^n$ and long dimension k.

By [Pi2], we know that every group definable in an o-minimal structure can be equipped with a unique definable manifold topology that makes it into a topological group, called the t-topology. We refer the reader elsewhere for the basic facts about the t-topology (which we will not make any essential use of, anyway).

Remark 6.1. By the Refined Structure Theorem, Lemma 3.7 and Corollary 5.10, for any two independent long-generic elements a and b of G and any $\emptyset$-definable function $f : G \times G \to G$, there are k-long cones C_a and C_b in G containing a and b, respectively, such that for all $x \in C_a$ and $y \in C_b$,

$$f(x, y) = \lambda x + \mu y + d,$$

for some fixed $\lambda, \mu \in \mathbb{M}(n, \Lambda)$ and $d \in M^n$. In the case that $f(x, y) = x \oplus y$ is the group operation of G, the λ and μ have to be moreover invertible matrices (for example, setting $y = b$, $x \oplus b = \lambda x + \mu b + d$ is invertible, showing that λ is invertible).

Lemma 6.2. *For every two independent long-generics $a, b \in G$, there is a k-long cone C_a containing a, invertible λ, $\lambda' \in \mathbb{M}(n, \Lambda)$ and $c, c' \in M^n$, such that for all $x \in C_a$,*

$$x \ominus a \oplus b = \lambda x + c \quad \textit{and} \quad \ominus a \oplus b \oplus x = \lambda' x + c'.$$

Proof. By Claim 5.13, since a and b are independent long-generics of G, a and $\ominus a \oplus b$ are independent long-generics of G as well. Therefore, by Remark 6.1, there are long cones C_a of a and $C_{\ominus a \oplus b}$ of $\ominus a \oplus b$ in G, as well as invertible $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that $\forall x \in C_a$, $\forall y \in C_{\ominus a \oplus b}$,

$$x \oplus y = \lambda x + \mu y + d.$$

In particular, for all $x \in C_a$, $x \ominus a \oplus b = \lambda x + \mu(\ominus a \oplus b) + d$. Letting $c = \mu(\ominus a \oplus b) + d$ shows the first equality. The second equality can be shown similarly. □

We are now ready to prove the local theorem for semi-bounded groups.

Theorem 6.3. *Let a be a long-generic element of G. Then there is a k-long cone $C_a \subseteq G$ containing a, such that for every $x, y \in C_a$,*

$$x \ominus a \oplus y = x - a + y.$$

Proof. We first prove:

Claim. *There is a k-long cone $C_a \subseteq G$ containing a, and $\lambda, \mu \in \mathbb{M}(n, \Lambda)$ and $d \in M^n$, such that for all $x, y \in C_a$,*

$$x \ominus a \oplus y = \lambda x + \mu y + d.$$

Proof of the Claim. Let a_1 be a long-generic element of G independent from a. Then $a_2 = a \ominus a_1$ is also a long-generic element of G independent from a. By Lemma 6.2, we can find k-long cones C and C' in G containing a, as well as $\lambda_1, \lambda_2 \in \mathbb{M}(n, \Lambda)$ and $c_1, c_2 \in M^n$, such that $\forall x \in C$, $\forall y \in C'$:

$$x \ominus a_2 = \lambda_1 x + c_1 \text{ and } \ominus a_1 \oplus y = \lambda_2 y + c_2. \tag{9}$$

We let V_{a_1} be the image of C under the map $x \mapsto x \ominus a_2$, and V_{a_2} the image of C' under $y \mapsto \ominus a_1 \oplus y$. Then V_{a_1} and V_{a_2} are open neighborhoods of a_1 and a_2 in G, respectively. Also, since a is long-generic, it must be contained in a k-long cone $C_a \subseteq C \cap C'$, on which, of course, every x and y satisfy equations (9).

Now, by Remark 6.1 and since a_1 and $a_2 = a \ominus a_1$ are independent long-generics of G, there are k-long cones C_{a_1} and C_{a_2} containing a_1 and a_2, respectively, such that for some fixed $\nu, \xi \in \mathbb{M}(n, \Lambda)$ and $\varepsilon \in M^n$, we have: $\forall x \in C_{a_1}$, $\forall y \in C_{a_2}$, $x \oplus y = \nu x + \xi y + \varepsilon$. By continuity of $\oplus$, we could choose C_a, V_{a_1}, V_{a_2} so that $V_{a_1} \subseteq C_{a_1}$ and

$V_{a_2} \subseteq C_{a_2}$, and still every $x, y \in C_a$ satisfy equations (9). Then for all $x, y \in C_a$, we have:

$$\begin{aligned} x \ominus a \oplus y &= x \ominus a \oplus a_1 \ominus a_1 \oplus y \\ &= (x \ominus a_2) \oplus (\ominus a_1 \oplus y) \\ &= \nu(\lambda_1 x + c_1) + \xi(\lambda_2 y + c_2) + \varepsilon \\ &= \nu\lambda_1 x + \xi\lambda_2 y + \nu c_1 + \xi c_2 + \varepsilon \end{aligned}$$

Setting $\lambda = \nu\lambda_1, \mu = \xi\lambda_2$, and $d = \nu c_1 + \xi c_2 + o$ finishes the proof of the claim. □

By the Claim, for all $x, y \in C_a$,

$$y = a \ominus a \oplus y = \lambda a + \mu y + d$$

$$x = x \ominus a \oplus a = \lambda x + \mu a + d$$

$$a = a \ominus a \oplus a = \lambda a + \mu a + d.$$

Hence, $x - a + y = \lambda x + \mu y + d = x \ominus a \oplus y$. □

We conclude with a stronger version of the local theorem that we expect to be useful in a future global analysis for semi-bounded groups. By [Pi2], we know that the t-topology of G coincides with the subspace topology on a large open definable subset W^G. The proof of the following proposition involves all machinery developed so far.

Proposition 6.4. *Assume G is definably compact. There is a left generic k-long cone C contained in G, on which the t-topology agrees with the subspace topology, and for every $a \in C$, there is a relatively open subset V_a of $a + \langle C \rangle$ containing a, such that $\forall x, y \in V_a$,*

$$x \ominus a \oplus y = x - a + y. \tag{10}$$

Proof. By the Refined Structure Theorem, W^G is the union of finitely many long cones $C_1, \ldots, C_l$. Let $\bar{v}_j = (v_{j1}, \ldots, v_{jk_j})$ be the direction of each C_j. By the Generic Lemma, one of the C_j's, say C_1, is a left generic k-long cone.

Claim. *Every long-generic element a of W^G is contained in some k-long cone contained in G with direction some $\bar{v}_j$ on which (10) holds.*

Proof of Claim. Since a is long-generic, Theorem 6.3 implies that a is contained in some k-long cone D on which (10) holds. Since a is in W^G, it is contained in some C_j. By Corollary 3.5, it is not hard to see that some k-long cone with direction $\bar{v}_j$ must be contained in D and contain a. □

Consider now the following property, for an element $a \in C_1$:
(*) there is a relatively open subset V_a of $a + \langle C_1 \rangle$ containing a, such that $\forall x, y \in V_a$, (10) holds.

The set X of elements of C_1 that satisfy (*) is clearly definable. We claim that it is also long-large in C_1.

Clearly, it suffices to prove that every long-generic element of C_1 satisfies (*). Let a be a long-generic element of C_1. If a belongs to a k-long cone of direction $\bar{v}_1$ on which (10) holds, then we are done. Hence, by the Claim, it clearly suffices to show that for every $j \neq 1$, the set A_j of all elements of C_1 that belong to a k-long cone of direction $\bar{v}_j$ but do not satisfy (*), is contained in a definable set of long dimension $< k$. To see this, note that if $a \in A_j$, then by Corollaries 2.14

and 3.5, one of the $v_{j1}, \dots, v_{jk_j}$, say v_{j1}, must be so that for every positive $t \in M$, $v_{j1}t \notin \langle C_1 \rangle$. Let κ be a tall element and $D_j = \{v_{j1}t : t \in (0, \kappa)\}$. The set

$$K_j = (C_1 + D_j) \cap G$$

has long dimension $\leq k$, as a subset of G. Hence, by Lemma 4.3, and since each D_j has long dimension 1, A_j is contained in a definable set of long dimension $\leq \operatorname{lgdim}(K_j) - 1$.

We have proved that X is long-large in C_1. By Lemma 5.14, X is left generic. By the Refined Structure Theorem, the Generic Lemma and the Lemma on Subcones, there is a left generic k-long cone C contained in X with the desired property. □

7. Appendix

If we tried to prove Lemma 2.8 by induction on n, then in the inductive step we would run into a system whose form is more general than that of the original one. Thus, we prove the following, more general statement.

Lemma 7.1. *Let $w_1, \dots, w_n, w_{n+1}, \dots, w_{n+k} \in \Lambda^n$ be M-independent and $\lambda_1, \dots, \lambda_n \in \Lambda^n$. Let $t_1, \dots, t_n \in M$ be non-zero elements and, for every $i = 1, \dots, n$, let $r_i^1, \dots, r_i^k \in M$ be such that:*

$$w_1 t_1 + \sum_{j=1}^{k} w_{n+j} r_1^j = \lambda_1 s_1^1 + \cdots + \lambda_n s_1^n$$

$$\vdots$$

$$w_n t_n + \sum_{j=1}^{k} w_{n+j} r_n^j = \lambda_1 s_n^1 + \cdots + \lambda_n s_n^n$$

for some $s_i^j \in M$. Then there are non-zero $a_1, \dots, a_n \in M$ and $b_i^j \in M$, $i = 1, \dots, n$, $j = 1, \dots, n+k$, such that:

$$\lambda_1 a_1 = w_1 b_1^1 + \cdots + w_{n+k} b_1^{n+k}$$

$$\vdots$$

$$\lambda_n a_n = w_1 b_n^1 + \cdots + w_{n+k} b_n^{n+k}$$

Proof. By induction on n. For $n = 1$, it is trivial. Assume $n > 1$ and that we know the statement for $< n$ equations. Since $w_1, \dots, w_{n+k} \in \Lambda^n$ are M-independent and $t_1 \neq 0$, $w_1 t_1 + \sum_{j=1}^{k} w_{n+j} r_1^j \neq 0$. Hence there is some s_1^j, say s_1^1, which is not zero. By switching the equations, if necessary, we may also assume that $s_i^1 < s_1^1$, for every $i = 2, \dots, n$. Since

$$\lambda_1 s_1^1 = w_1 t_1 + \sum_{j=1}^{k} w_{n+j} r_1^j - (\lambda_2 s_1^2 + \cdots + \lambda_n s_1^n), \tag{11}$$

Lemma 2.7 gives, for every $i = 2, \dots, n$,

$$\lambda_1 s_i^1 = w_1 T_i + \sum_{j=1}^{k} w_{n+j} R_i^j - (\lambda_2 S_i^2 + \cdots + \lambda_n S_i^n)$$

for some $S_i^2, \dots, S_i^n, T_i, R_i^1, \dots R_i^k \in M$. By substituting into the original system of equations, we obtain:

$$w_2 t_2 - w_1 T_1 + \sum_{j=1}^{k} w_{n+j}(r_2^j - R_2^j) = \lambda_2(s_2^2 - S_2^2) + \cdots + \lambda_n(s_2^n - S_2^n)$$

$$\vdots$$

$$w_n t_n - w_1 T_1 + \sum_{j=1}^{k} w_{n+j}(r_n^j - R_n^j) = \lambda_2(s_n^2 - S_n^2) + \cdots + \lambda_n(s_n^n - S_n^n)$$

Now apply the Inductive Hypothesis to find $a_2, \dots, a_n$ such that

(12) each of $\lambda_2 a_2, \dots, \lambda_n a_n$ can be expressed in terms of $w_1, \dots, w_{n+k}$.

By Lemma 2.7, we can replace the elements of M that appear in (11) by arbitrarily small positive ones; that is, there are arbitrarily small $a_1, p_1, q_1^j, u_1^j \in M$ such that

$$\lambda_1 a_1^1 = w_1 p_1 + \sum_{j=1}^{k} w_{n+j} q_1^j - (\lambda_2 u_1^2 + \cdots + \lambda_n u_1^n). \tag{13}$$

In particular, we may choose $0 < u_1^j < a_j$. Hence, by Lemma 2.7 again and (12), we can express each of $\lambda_2 u_1^2, \dots, \lambda_n u_1^n$ in terms of $w_1, \dots, w_{n+k}$. Hence $\lambda_1 a_1^1$ is now also expressed in terms of $w_1, \dots, w_{n+k}$, finishing the proof. □

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Part 2

Definable quotients of locally definable groups

DEFINABLE QUOTIENTS OF LOCALLY DEFINABLE GROUPS

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ABSTRACT. We study locally definable abelian groups $\mathcal{U}$ in various settings and examine conditions under which the quotient of $\mathcal{U}$ by a discrete subgroup might be definable. This turns out to be related to the existence of the type-definable subgroup $\mathcal{U}^{00}$ and to the divisibility of $\mathcal{U}$.

1. INTRODUCTION

This is the first of two papers (originally written as one) around groups definable in o-minimal expansions of ordered groups. The ultimate goal of this project is to reduce the analysis of such groups to semi-linear groups and to groups definable in o-minimal expansions of real closed fields. This reduction is carried out in the second paper ([8]). In the current paper, we prove a crucial lemma in that perspective, Theorem 3.10 below. This theorem is proved by analyzing $\bigvee$-definable abelian groups in various settings and investigating when such groups have definable quotients of the same dimension. The analysis follows closely known work on definably compact groups. We make strong use of their minimal type-definable subgroups of bounded index, and of the solution to so-called Pillay's conjecture in various settings.

In the rest of this introduction we recall the main definitions and state the results of this paper.

Until Section 3, and unless stated otherwise, $\mathcal{M}$ denotes a sufficiently saturated, not necessarily o-minimal, structure.

If $\mathcal{M}$ is κ-saturated, by *bounded* cardinality we mean cardinality smaller than κ. Since "bounded" has a different meaning in the context of an ordered structure we use "small" to refer to subsets of M^n of bounded cardinality. Every small definable set is therefore finite.

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1.1. **$\bigvee$-definable and locally definable sets.** A *$\bigvee$-definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{i \in I} X_i$ of definable subsets of M^n for some fixed n (where $|I|$ is bounded) and for every $i, j \in I$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). Following [6], we say that $\langle \mathcal{U}, \cdot \rangle$ is *locally definable* if $|I|$ is countable. We are mostly interested here in *definably generated* groups, namely $\bigvee$-definable groups which are generated as a group by a definable subset. These groups are locally definable. An important example of such groups is the universal cover of a definable group (see [7]). In [12, Section 7] a more general notion is introduced, of an Ind-definable group, where the X_i's are not assumed to be subsets of the same sort and there are definable maps which connect them to each other.

A map $\phi : \mathcal{U} \to \mathcal{H}$ between $\bigvee$-definable (locally definable) groups is called *$\bigvee$-definable (locally definable)* if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, $graph(\phi) \cap (X \times Y)$ is a definable set. Equivalently, the restriction of ϕ to any definable set is definable.

Remark 1.1. If in the above definition, instead of M^n we allow all X_i's to be subsets of a fixed sort S then the analogous definition of groups and maps works in $\mathcal{M}^{eq}$. This will allow us to discuss locally definable maps from a locally definable group $\mathcal{U}$ onto an interpretable group $\mathcal{V}$.

1.2. **Compatible subgroups.**

Definition 1.2. (See [6]) For a $\bigvee$-definable group $\mathcal{U}$, we say that $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset of $\mathcal{U}$* if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case $\mathcal{V}$ itself is a bounded union of definable sets).

Clearly, the only compatible $\bigvee$-definable subsets of a definable group are the definable ones. Note that if $\phi : \mathcal{U} \to \mathcal{V}$ is a $\bigvee$-definable homomorphism between $\bigvee$-definable groups then $\ker(\phi)$ is a compatible $\bigvee$-definable normal subgroup of $\mathcal{U}$. Compatible subgroups are used in order to obtain $\bigvee$-definable quotients, but for that we need to restrict ourselves to locally definable groups. Together with [6, Theorem 4.2], we have:

Fact 1.3. *If $\mathcal{U}$ is a locally definable group and $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup then $\mathcal{H}$ is a compatible subgroup of $\mathcal{U}$ if and only if there exists a locally definable surjective homomorphism of locally definable groups $\phi : \mathcal{U} \to \mathcal{V}$ whose kernel is $\mathcal{H}$.*

1.3. **Connectedness.** If $\mathcal{M}$ is an o-minimal structure and $\mathcal{U} \subseteq M^n$ is a $\bigvee$-definable group then, by [2, Theorem 4.8], it can be endowed with a manifold-like topology τ, making it into a topological group. Namely, there exists a bounded collection $\{U_i : i \in I\}$ of definable subsets of $\mathcal{U}$, whose union equals $\mathcal{U}$, such that each U_i is in definable bijection with an open subset of M^k ($k = \dim \mathcal{U}$), and the transition maps are continuous. The group operation and group inverse are continuous with respect to this induced topology. Moreover, the U_i's are definable over the same parameters which define $\mathcal{U}$. The topology τ is determined by the ambient topology of M^n in the sense that at every generic point of $\mathcal{U}$ the two topologies coincide. From now on, whenever we refer to a topology on G, it is τ we are considering.

Definition 1.4. (See [1]) In an o-minimal structure, a $\bigvee$-definable group $\mathcal{U}$ is called *connected* if there exists no $\bigvee$-definable compatible subset $\emptyset \subsetneq \mathcal{V} \subsetneq \mathcal{U}$ which is both closed and open with respect to the group topology.

1.4. **Definable quotients.**

Definition 1.5. Given a $\bigvee$-definable group $\mathcal{U}$ and $\Lambda_0 \subseteq \mathcal{U}$ a normal subgroup, we say that $\mathcal{U}/\Lambda_0$ is *definable* if there exists a definable group $\overline{K}$ and a surjective $\bigvee$-definable homomorphism $\mu : \mathcal{U} \to \overline{K}$ whose kernel is Λ_0.

One can define the notion of an *interpretable quotient* by replacing " $\overline{K}$ definable" by "$\overline{K}$ interpretable" in the above definition. Note, however, that in case $\mathcal{M}$ is an o-minimal structure and $\mathcal{U}$ is locally definable, such as in Section 3 below, by [6, Corollary 8.1], the group $\mathcal{U}$ has strong definable choice for definable families of subsets of $\mathcal{U}$. Namely, for every definable family of subsets of $\mathcal{U}$, $\{X_t : t \in T\}$, there is a definable function $f : T \to \bigcup X_t$ such that for every $t \in T$, $f(t) \in X_t$ and if $X_{t_1} = X_{t_2}$ then $f(t_1) = f(t_2)$. In particular, every interpretable quotient of $\mathcal{U}$ would be definably isomorphic to a definable group.

1.5. **Results.** Our results concern the existence of the type-definable group $\mathcal{U}^{00}$, for a $\bigvee$-definable abelian group $\mathcal{U}$. Recall ([12, Section 7]) that for a definable, or $\bigvee$-definable group $\mathcal{U}$, we write $\mathcal{U}^{00}$ for the smallest, if such exists, type-definable subgroup of $\mathcal{U}$ of bounded index. In particular we require that $\mathcal{U}^{00}$ is contained in a definable subset of $\mathcal{U}$. From now on we use the expression "$\mathcal{U}^{00}$ *exists*" to mean that "there exists a smallest type-definable subgroup of $\mathcal{U}$ of bounded index, which we denote by $\mathcal{U}^{00}$". Note that a type definable subgroup $\mathcal{H}$ of $\mathcal{U}$ has bounded index if and only if there are no new cosets of $\mathcal{H}$ in $\mathcal{U}$ in elementary extensions of $\mathcal{M}$.

When $\mathcal{U}$ is a definable group in a NIP structure, then $\mathcal{U}^{00}$ exists (see Shelah's theorem in [18]). When $\mathcal{U}$ is a $\bigvee$-definable group in a NIP structure or even in an o-minimal one, then $\mathcal{U}^{00}$ may not always exist. However, if we assume that *some* type-definable subgroup of bounded index exists, then there is a smallest one (see [12, Proposition 7.4]). Recall that a definable $X \subseteq \mathcal{U}$ is called *left generic* if boundedly many translates of X cover $\mathcal{U}$. In Section 2, we prove the following theorem for $\bigvee$-definable groups:

Theorem 2.6. *Let $\mathcal{U}$ be an abelian $\bigvee$-definable group in a NIP structure. If the definable non-generic sets in $\mathcal{U}$ form an ideal and $\mathcal{U}$ contains at least one definable generic set, then $\mathcal{U}^{00}$ exists.*

We also prove (Corollary 2.12) that when we work in o-minimal expansions of ordered groups, for a $\bigvee$-definable abelian group which contains a definable generic set and is generated by a definably compact set, the non-generic definable subsets do form an ideal (this is a generalization of the same result from [16] for definably compact group, which itself relies heavily on work in [5]).

In Section 3, we use these results to establish the equivalence of the following conditions.

Theorem 3.9. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group in an o-minimal expansion of an ordered group, with $\mathcal{U}$ definably generated. Then there is $k \in \mathbb{N}$ such that the following are equivalent:*
(i) $\mathcal{U}$ contains a definable generic set.
(ii) $\mathcal{U}^{00}$ exists.
(iii) $\mathcal{U}^{00}$ exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, where $\mathbb{T}$ is the circle group and $r \in \mathbb{N}$.
(iv) There is a definable group G, with $\dim G = \dim \mathcal{U}$, and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$.

If $\mathcal{U}$ is generated by a definably compact set, then (ii) is strengthened by the condition that $k + r = \dim \mathcal{U}$.

We conjecture, in fact, that the conditions of Theorem 3.9 are always true.

Conjecture A. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group in an o-minimal structure, which is definably generated. Then*
(i) $\mathcal{U}$ contains a definable generic set.
(ii) $\mathcal{U}$ is divisible.

We do not know if Conjecture A is true, even when $\mathcal{U}$ is a subgroup of a definable group. We do show that it is sufficient to prove (i) under restricted conditions, in order to deduce the full conjecture. In a recent paper (see [9]) we prove that Conjecture A holds for definably generated subgroups of $\langle R^n, +\rangle$, in an o-minimal expansion of a real closed field R.

Finally, we derive the theorem that is used in [8].

Theorem 3.10. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group in an o-minimal expansion of an ordered group, with $\mathcal{U}$ definably generated. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definable group.

If $\mathcal{U}$ is generated by a definably compact set, then $\mathcal{U}/\Lambda'$ is moreover definably compact.

1.6. **Notation.** Given a group $\langle G, \cdot \rangle$ and a set $X \subseteq G$, we denote, for every $n \in \mathbb{N}$,

$$X(n) = \overbrace{XX^{-1} \cdots XX^{-1}}^{n-\text{times}}$$

We assume familiarity with the notion of definable compactness. Whenever we write that a set is definably compact, or definably connected, we assume in particular that it is definable.

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2. $\bigvee$-definable groups and type-definable subgroups of bounded index

In this section, unless stated otherwise, $\mathcal{M}$ denotes a sufficiently saturated, not necessarily o-minimal, structure.

2.1. **Definable quotients of $\bigvee$-definable groups.** We begin with a criterion for definability (and more generally interpretability) of quotients.

Lemma 2.1. *Let $\langle \mathcal{U}, \cdot \rangle$ be a $\bigvee$-definable group and Λ_0 a small normal subgroup of $\mathcal{U}$. Then the following are equivalent:*

1. *(1) The quotient $\mathcal{U}/\Lambda_0$ is interpretable in $\mathcal{M}$.*
2. *(2) There is a definable $X \subseteq \mathcal{U}$ such that (a) $X \cdot \Lambda_0 = \mathcal{U}$ and (b) for every definable $Y \subseteq \mathcal{U}$, $Y \cap \Lambda_0$ is finite.*
3. *(3) There is a definable $X \subseteq \mathcal{U}$ such that (a) $X \cdot \Lambda_0 = \mathcal{U}$ and (b) $X \cap \Lambda_0$ is finite.*

Proof. (1 $\Rightarrow$2). We assume that there is a $\bigvee$-definable surjective $\mu : \mathcal{U} \to \overline{K}$ with kernel Λ_0, and $\overline{K}$ interpretable. By saturation, there is a definable subset $X \subseteq \mathcal{U}$ such that $\mu(X) = \overline{K}$ and hence $X \cdot \Lambda_0 = \mathcal{U}$. Given any definable $Y \subseteq \mathcal{U}$, the restriction of μ to Y is definable and thus the small set $\ker(\mu_{\restriction Y}) = Y \cap \Lambda_0$ is definable and, hence, finite.

(2 $\Rightarrow$3). This is obvious.

(3 $\Rightarrow$1) We claim first that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap \Lambda_0$ is finite. Indeed, since $Y \subseteq X \cdot \Lambda_0$ and Λ_0 is small, by saturation there exists a finite $F \subseteq \Lambda_0$ such that $Y \subseteq X \cdot F$. We assume that $X \cap \Lambda_0$ is finite, and since F is a finite subset of Λ_0 it follows that $(X \cdot F) \cap \Lambda_0$ is finite which clearly implies $Y \cap \Lambda_0$ finite.

Fix a finite $F_1 = XX^{-1} \cap \Lambda_0$ and $F_2 = XXX^{-1} \cap \Lambda_0$.

We now define on X an equivalence relation $x \sim y$ if and only if $xy^{-1} \in \Lambda_0$ if and only if $xy^{-1} \in F_1$. This is a definable relation since F_1 is finite. We can also define a group operation on the equivalence classes: $[x] \cdot [y] = [z]$ if and only if $xyz^{-1} \in \Lambda_0$ if and only if $xyz^{-1} \in F_2$. The interpretable group we get, call it $\overline{K}$, is clearly isomorphic to $\mathcal{U}/\Lambda_0$, and we have a $\bigvee$-definable homomorphism from $\mathcal{U}$ onto $\overline{K}$, whose kernel is Λ_0. □

We will return to definable quotients of $\bigvee$-definable groups in Section 3. We now focus on the existence of $\mathcal{U}^{00}$ for a $\bigvee$-definable group $\mathcal{U}$.

2.2. **Subgroups of bounded index of $\bigvee$-definable groups.** Let $\mathcal{U}$ be a $\bigvee$-definable group in an o-minimal structure. It is not always true that $\mathcal{U}$ has some type-definable subgroup of bounded index. For example, consider a sufficiently saturated ordered divisible abelian group $\langle G, <, +\rangle$ and in it take an infinite increasing sequence of elements $0 < a_1 < a_2 < \cdots$ such that, for every $n \in \mathbb{N}$, we have $na_i < a_{i+1}$. The subgroup $\bigcup_i(-a_i, a_i)$ of G is a $\bigvee$-definable group which does not have any type-definable subgroup of bounded index. However, as is shown in [12] (see Proposition 6.1 and Proposition 7.4), if $\mathcal{U}$ does have some type-definable subgroup of bounded index then it has a smallest one; namely $\mathcal{U}^{00}$ exists.

Our goal here is to show, under various assumptions on $\mathcal{U}$, that the ideal of non-generic definable sets gives rise to type-definable subgroups of bounded index.

As is shown in [16], using Dolich's results in [5], if G is a definably compact, abelian group in an o-minimal expansion of a real closed field then the non-generic definable sets form an ideal. Later, it was pointed out in [10] and [14, Section 8] that the same proof works in expansions of groups. We start by re-proving an analogue of the result for $\bigvee$-definable groups (see Lemma 2.11 below). We first define the corresponding notion of genericity and prove some basic facts about it.

Definition 2.2. Let $\mathcal{U}$ be a $\bigvee$-definable group. A definable $X \subseteq \mathcal{U}$ is called *left-generic* if there is a small subset $A \subseteq \mathcal{U}$ such that $\mathcal{U} = \bigcup_{g\in A} gX$. We similarly define *right-generic*. The set X is called *generic* if it is both left-generic and right-generic.

It is easy to see that a definable $X \subseteq \mathcal{U}$ is generic if and only if for every definable $Y \subseteq \mathcal{U}$, there are finitely many translates of X which cover Y.

Fact 2.3. *(1) If $\mathcal{U}$ is a $\bigvee$-definable group, then every $\bigvee$-definable subgroup of bounded index is a compatible subgroup. In particular, if $X \subseteq \mathcal{U}$ is a definable left-generic set, then the subgroup generated by X is a compatible subgroup.*

(2) Assume that $\mathcal{U}$ is a $\bigvee$-definable group in an o-minimal structure. If $\mathcal{U}$ is connected and $X \subseteq \mathcal{U}$ is a left-generic set, then X generates $\mathcal{U}$.

Proof. (1) Assume that $\mathcal{V}$ is a $\bigvee$-definable subgroup of bounded index. We need to see that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap \mathcal{V}$ is definable. Since $\mathcal{V}$ has bounded index in $\mathcal{U}$ its complement in $\mathcal{U}$ is also a bounded union of definable sets, hence a $\bigvee$-definable set. But then $Y \cap \mathcal{V}$ and $Y \setminus \mathcal{V}$ are both $\bigvee$-definable sets, so by compactness $Y \cap \mathcal{V}$ must be definable.

(2) Assume now that $\mathcal{U}$ is a $\bigvee$-definable connected group in an o-minimal structure and $X \subseteq \mathcal{U}$ is a left-generic set. By (1), the group $\mathcal{V}$ generated by X is compatible, of bounded index. But then $\dim \mathcal{V} = \dim \mathcal{U}$, so by [1, Proposition 1], $\mathcal{V} = \mathcal{U}$. $\square$

Fact 2.4. *Let $\langle \mathcal{U}, +\rangle$ be an abelian, definably generated group. If $X \subseteq \mathcal{U}$ is a definable set then X is generic if and only if there exists a finitely generated (in particular countable) group $\Gamma \leqslant \mathcal{U}$ such that $\mathcal{U} = X + \Gamma$.*

Proof. Clearly, if Γ exists then X is generic. For the converse, assume that $\mathcal{U}$ is generated by the definable set $Y \subseteq \mathcal{U}$, with $0 \in Y$. Because X is generic in $\mathcal{U}$, there is a finite set $F \subseteq \mathcal{U}$ such that the sets $-Y$, Y and $X + X$ are all contained in $X + F$.

Let $Y(n)$ be as in the notation from Section 1.6. If we now let Γ be the group generated by F, then $\mathcal{U} = \bigcup_n Y(n) = X + \Gamma$. □

We next show that under some suitable conditions we can guarantee the existence of $\mathcal{U}^{00}$. We do it first in the general context of NIP theories. We recall a definition [16]:

Definition 2.5. Given a $\bigvee$-definable group $\mathcal{U}$ and a definable set $X \subseteq \mathcal{U}$,

$$Stab_{ng}(X) = \{g \in \mathcal{U} : gX\Delta X \text{ is non-generic in } \mathcal{U}\}.$$

Theorem 2.6. *Let $\mathcal{U}$ be an abelian $\bigvee$-definable group in a NIP structure $\mathcal{M}$. Assume that the non-generic definable subsets of $\mathcal{U}$ form an ideal and that $\mathcal{U}$ contains some definable generic set. Then for any definable generic set X, the set $Stab_{ng}(X)$ is a type-definable group and has bounded index in $\mathcal{U}$. In particular, by* [12, Proposition 7.4], *$\mathcal{U}^{00}$ exists.*

Proof. The fact the definable non-generic sets form an ideal implies that for every definable set X, the set $Stab_{ng}(X)$ is a subgroup. Note however that if X is a non-generic set then $Stab_{ng}(X) = \mathcal{U}$ and therefore will not in general be type-definable (unless $\mathcal{U}$ itself was definable).

We assume now that $X \subseteq \mathcal{U}$ is a definable generic set and show that $Stab_{ng}(X)$ is type-definable. First note that for every $g \in \mathcal{U}$, if $gX\Delta X$ is non-generic, then in particular $gX \cap X \neq \emptyset$ and therefore $g \in XX^{-1}$. It follows that $Stab_{ng}(X)$ is contained in XX^{-1}.

Next, note that a subset of $\mathcal{U}$ is generic if and only if finitely many translates of it cover X (since X itself is generic). Now, for every n, we consider the statement in g: "n many translates of $gX\Delta X$ do not cover X". Here again we note that for $h(gX\Delta X) \cap X$ to be non-empty we must have $h \in XX^{-1} \cup X(gX)^{-1}$. Hence, it is sufficient to write the first-order formula saying that for every $h_1, \ldots, h_n \in XX^{-1} \cup X(gX)^{-1}$, $X \nsubseteq \bigcup_{i=1}^{n} h_i(gX\Delta X)$. The union of all these formulas for every n, together with the formula for XX^{-1} is the type which defines $Stab_{ng}(X)$.

It remains to see that $Stab_{ng}(X)$ has bounded index in $\mathcal{U}$. This is a similar argument to the proof of [12, Corollary 3.4] but in that paper the amenability of definable groups and, as a result, the fact that every generic set has positive measure, played an important role. Since a generic subset of a $\bigvee$-definable group may require infinitely many translates to cover the group, we cannot a-priori conclude that it has positive measure, even if the group is amenable. Assume then towards contradiction that $Stab_{ng}(X)$ had unbounded index and fix a small elementary substructure $\mathcal{M}_0$ over which all data is definable. Then we can find a sequence $g_1, \ldots, g_n, \ldots \in \mathcal{U}$ of indiscernibles over $\mathcal{M}_0$, which are all in different cosets of $Stab_{ng}(X)$. In particular, it means that $g_i X \Delta g_j X$ is generic, for $i \neq j$.

Consider now the sequence $X_i = g_{2i} X \Delta g_{2i+1} X$, $i \in \mathbb{N}$. By NIP, there is a k, such that the sequence $\{X_i : i \in \mathbb{N}\}$ is k-inconsistent.

Consider now the type $tp(g_i/M_0)$ and find some M_0-definable set W containing g_i. Because of indiscernibility, all g_i's are in W. It follows that all the $g_i X$, and therefore also all X_i, are contained in WX. Because each X_i is generic, finitely many translates of X_i cover WX. By indiscernibility, there is some ℓ such that for every i there are ℓ-many translates of X_i which cover WX.

We then have countably many sets $X_i \subseteq WX$, such that on one hand the intersection of every k of them is empty and on the other hand there is some ℓ such that for each i, ℓ-many translates of X_i cover WX. To obtain a contradiction it is sufficient to prove the following lemma (it is here that we need to find an alternative argument to the measure theoretic one):

Lemma 2.7. *Let G be an arbitrary abelian group, $A \subseteq G$ an arbitrary subset. For every k and ℓ there is a fixed number $N = N(k, \ell)$ such that there are at most N subsets of A with the property that each covers A with ℓ-many translates and every k of them have empty intersection.*

Proof. We are going to use the following fact about abelian groups, taken from [13] (see problems 7 and 16 on p. 82):

Fact 2.8. *For every abelian group G, and for every set $A \subseteq G$ and m, it is not possible to find $A_1, \ldots A_{m+1} \subseteq A$ pairwise disjoint such that each A_i covers A by m-many translates.*

Returning to the proof of the lemma, we are going to show that $N = k\ell$ works. Assume for contradiction that there are $k\ell + 1$ subsets $X_1, \ldots, X_{kl+1}$ of A, each covering A by ℓ-many translates, with an empty intersection of every k of them. We work with the group $G' = G \times C_k$, where $C_k = \{0, \ldots, k-1\}$ is the cyclic group. For $i = 1, \ldots, k\ell + 1$, we define $Y_i \subseteq G'$ as follows: For $x \in G$ and $n \in \mathbb{N}$, we have $(x, n) \in Y_i$ if and only if $x \in X_i$ and n is the maximum number such that for some distinct $i_1, \ldots, i_n < i$, we have $x \in X_{i_1} \cap \cdots \cap X_{i_n} \cap X_i$. Notice that even though i might be larger than k, because of our assumption that every k sets among the X_i's intersect trivially, the maximum n we pick is indeed at most $k - 1$. Note also that the projection of each Y_i on the first coordinate is X_i.

We claim that the Y_i's are pairwise disjoint. Indeed, if $x \in X_i \cap X_j$ and $i < j$ then by the definition of the sets, if $(x, n) \in Y_i$ and $(x, n') \in Y_j$ then $n < n'$, so $Y_i \cap Y_j = \emptyset$.

Now, let $A' = A \times C_k$. We claim that each Y_i covers A' by $k\ell$-many translates. Indeed, if $A \subseteq \bigcup_{j=1}^{\ell} g_{ij} \cdot X_i$ then

$$A' \subseteq \bigcup_{p \in C_k} \bigcup_{j=1}^{\ell} (g_{ij}, p) \cdot Y_i.$$

We therefore found $N+1$ pairwise disjoint subsets of A', each covering A' in N translates, contradicting Fact 2.8. ☐

Thus, as pointed out above we reached a contradiction, so $stab_{ng}(X)$ does have bounded index in $\mathcal{U}$. This ends the proof of Theorem 2.6. ☐

Remark 2.9. The last theorem implies that for a $\bigvee$-definable abelian group $\langle \mathcal{U}, + \rangle$ in a NIP structure, if the non-generic definable sets form an ideal, then $\mathcal{U}^{00}$ exists if and only if $\mathcal{U}$ contains a definable generic set (we have just proved the right-to-left direction. The converse is immediate since every definable set containing $\mathcal{U}^{00}$ is generic).

We are now ready to show (Corollary 2.12 below) that when we work in o-minimal expansions of ordered groups, for a $\bigvee$-definable abelian group which contains a definable generic set and is generated by a definably compact set, the assumptions of Theorem 2.6 are satisfied. We begin by proving that we can obtain Dolich's result in this setting.

Fact 2.10. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group and let $\mathcal{M}_0 \preccurlyeq \mathcal{M}$ be a small elementary submodel. If $\mathcal{U}$ is a $\bigvee$-definable group over $\mathcal{M}_0$ and $X_t \subseteq \mathcal{U}$ is a t-definable, definably compact set such that $X_t \cap M_0 = \emptyset$, then there are $t_1, \dots, t_k$, all of the same type as t over M_0 such that $X_{t_1} \cap \dots \cap X_{t_k} = \emptyset$.*

Proof. We need to translate the problem from the group topology to the M^n-topology. As we already noted it is shown in [2] that $\mathcal{U}$ can be covered by a fixed collection of $\mathcal{M}_0$-definable open sets $\bigcup_i V_i$ such that each V_i is definably homeomorphic to an open subset of M^n. By logical compactness, X_t is contained in finitely many V_i's, say $V_1, \cdots, V_m$. Now, by definable compactness, we can replace each of the V_i's by an open set W_i such that $Cl(W_i) \subseteq V_i$ and X_t is still contained in $W_1, \dots, W_m$. Each $X(i) = X_t \cap Cl(W_i)$ is definably compact and we finish the proof as in [10, Lemma 3.10]. ☐

For a $\bigvee$-definable group $\mathcal{U}$, we call a definable $X \subseteq \mathcal{U}$ *relatively definably compact* if the closure of X in $\mathcal{U}$ is definably compact. Clearly, X is relatively definably compact if and only if it is contained in some definably compact subset of $\mathcal{U}$.

Lemma 2.11. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Assume that $\mathcal{U}$ is a $\bigvee$-definable abelian group, and $X, Y \subseteq \mathcal{U}$ are definable, with X relatively definably compact. If X and Y are non-generic, then $X \cup Y$ is still non-generic.*

Proof. This is just a small variation on the work in [16]. Because commutativity plays only a minor role we use multiplicative notation for possible future use.

We may assume that $\mathcal{U}$ contains a definable generic set (otherwise, the conclusion is trivial).

We need to prove that if $X \subseteq \mathcal{U}$ is definable, relatively definably compact and non-generic, and if $Z \supseteq X$ is definable and generic then $Z \setminus X$ is generic.

Fix $\mathcal{M}_0$ over which all sets are definable. Without loss of generality, X is definably compact (since the closure of a non-generic set is non-generic).

We first prove the result for Z of the form $W \cdot W$, when W is generic. Since X is not generic, no finitely many translates of X cover W (because W is generic). It follows from logical compactness that there is $g \in W$ such that $g \notin \bigcup_{h \in M_0} hX$. Changing roles, there is $g \in W$ such that $Xg^{-1} \cap M_0 = \emptyset$. We now apply Fact 2.10 to the definably compact set Xg^{-1}. It follows that there are $g_1, \ldots, g_r$, all realizing the same type as g over $\mathcal{M}_0$, so in particular all are in W, such that $Xg_1^{-1} \cap \cdots \cap Xg_r^{-1} = \emptyset$. This in turn implies that $\bigcup_{i=1}^r (W \setminus Xg_i^{-1}) = W$. For each $i = 1, \ldots, r$ we have

$$W \setminus Xg_i^{-1} = (Wg_i \setminus X)g_i^{-1} \subseteq (WW \setminus X)g_i^{-1}.$$

Therefore, it follows that W is contained in the finite union $\bigcup_{i=1}^r (WW \setminus X)g_i^{-1}$ and since W is generic it follows that $WW \setminus X$ is generic, as needed (it is here that commutativity is used, since left generic sets and right generic sets are the same).

We now consider an arbitrary definable generic set $Z \subseteq \mathcal{U}$, with $X \subseteq Z$ non-generic. Because Z is generic, finitely many translates of Z cover $Z \cdot Z$. Namely, $Z \cdot Z \subseteq \bigcup_{i=1}^t h_i Z$. If $X' = \bigcup_{i=1}^t h_i X$ then X' is still non-generic (and relatively definably compact), so by the case we have just proved, $ZZ \setminus X'$ is generic. However this set difference is contained in

$$\bigcup_{i=1}^t h_i Z \setminus \bigcup_{i=1}^t h_i X \subseteq \bigcup_{i=1}^t h_i (Z \setminus X),$$

hence this right-most union is generic. It follows that $Z \setminus X$ is generic. □

Corollary 2.12. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Assume that $\mathcal{U}$ is a $\bigvee$-definable abelian group which contains a definable generic set and is generated by a definably compact set. Then the definable non-generic subsets of $\mathcal{U}$ form an ideal.*

Proof. Every definable subset of $\mathcal{U}$ must be relatively definably compact, because it is contained in some definably compact set. Then apply Lemma 2.11. □

3. Divisibility, genericity and definable quotients

In this section, $\mathcal{M}$ is a sufficiently saturated o-minimal expansion of an ordered group.

Proposition 3.1. *If $\mathcal{U}$ is an infinite $\bigvee$-definable group of positive dimension, then it has unbounded exponent. In particular, for every n, the subgroup of n torsion points, $\mathcal{U}[n]$, is small.*

Proof. By the Trichotomy Theorem ([15]), there exists a neighborhood of the identity which is in definable bijection with an open subset of R^n for some real closed field R, or of V^n for some ordered vector space V (we use here the definability of a group operation near the identity of $\mathcal{U}$).

In the linear case, the group operation of $\mathcal{U}$ is locally isomorphic near $e_{\mathcal{U}}$ to $+$ near $0 \in M^n$ (see [10, Proposition 4.1 and Corollary 4.4] for a similar argument). Clearly then the map $x \mapsto kx$ is non-constant.

Assume then that we are in the field case. Namely, we assume that some definable neighborhood W of e is definably homeomorphic to an open subset of R^n, with e identified with $0 \in \mathbb{R}^n$, and that a real closed field whose universe is a subset of W is definable in $\mathcal{M}$. The following argument was suggested by S. Starchenko. If $M(x,y) = xy$ is the group product of elements near e, then it is R-differentiable and its differential at (e,e) is $x+y$. It follows that the differential of the map $x \mapsto x^n$ is nx. Therefore, for every n, the map $x \mapsto x^n$ is not the constant map.

As for the last clause, note first that $\mathcal{U}[n]$ is a compatible $\bigvee$-definable subgroup of $\mathcal{U}$ because its restriction to every definable set is obviously definable (by the formula $nx = 0$). Because $\mathcal{U}[n]$ has exponent at most n, it follows from what we have just proved that its dimension must be zero, so its intersection with every definable set is finite. □

Remark 3.2. Although we did not write down the details, we believe that the above result is actually true without any assumptions on the ambient o-minimal $\mathcal{M}$. This can be seen by expressing a neighborhood of $e_{\mathcal{U}}$ as a direct product of neighborhoods, in cartesian powers of orthogonal real closed fields and ordered vector spaces.

Assume that $\mathcal{U} = \bigcup_{i\in I} X_i$ and that $\mathcal{U}^{00}$ exists. Given the projection $\pi : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$, we define the *logic topology* on $\mathcal{U}/\mathcal{U}^{00}$ by: $F \subseteq \mathcal{U}/\mathcal{U}^{00}$ is closed if and only if for every $i \in I$, $\pi^{-1}(F) \cap X_i$ is type-definable. We first prove a general lemma.

Lemma 3.3. *Let $\mathcal{U}$ be a locally definable group for which $\mathcal{U}^{00}$ exists and let $\pi : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$ be the projection map. If $K_0 \subseteq \mathcal{U}/\mathcal{U}^{00}$ is a compact set, then $\pi^{-1}(K_0)$ is contained in a definable subset of $\mathcal{U}$.*

Proof. We write $\mathcal{U} = \bigcup_{n\in\mathbb{N}} X_n$, and we assume that the union is increasing. If the result fails then there is a sequence $k_n \to \infty$ and $x_n \in X_{k_n} \setminus X_{k_n-1}$ such that $\pi(x_n) \in K_0$. Since K_0 is compact we may assume that the sequence $\pi(x_n)$ converges to some $a \in K_0$. The set $\pi^{-1}(a)$ is a coset of $\mathcal{U}^{00}$ and therefore contained in some definable set $Z \subseteq \mathcal{U}$. Since a can be realized as the intersection of countably many open sets, there is, by logical compactness, some open neighborhood $V \ni a$ in $\mathcal{U}/\mathcal{U}^{00}$ such that $\pi^{-1}(V) \subseteq Z$. But then, the whole tail of the sequence $\{\pi(x_n)\}$ belongs to V and therefore the tail of $\{x_n\}$ is contained in Z, contradicting our assumption on the sequence. □

Claim 3.4. *Let $\mathcal{U}$ be an abelian locally definable group. Then there exists a definable torsion-free subgroup $H \subseteq \mathcal{U}$ such that every definable subset of $\mathcal{U}/H$ is relatively definably compact. If, in addition, $\mathcal{U}$ is definably generated, then $\mathcal{U}/H$ can be generated by a definably compact set.*

Proof. As can easily be verified, for a definably generated $\bigvee$-definable group $\mathcal{V}$, the following are equivalent: (a) every definable subset of $\mathcal{V}$ is relatively definably compact, (b) every definable path in $\mathcal{V}$ has limit points in $\mathcal{V}$. A $\bigvee$-definable group with property (b) was called in [6] "definably compact". In Theorem 5.2 of the same reference, it was shown that if $\mathcal{V}$ is a $\bigvee$-definable group which is not definably compact, then $\mathcal{V}$ contains a 1-dimensional torsion-free definable subgroup H_1. Now, if $\mathcal{U}$ is abelian, then by Fact 1.3, $\mathcal{U}/H_1$ is definably isomorphic to a locally definable definable group. Using induction on $\dim(\mathcal{U})$, we see that $\mathcal{U}$ contains a definable torsion-free subgroup H such that $\mathcal{U}/H$ is definably compact in the above sense.

If in addition, $\mathcal{U}$ is definably generated then $\mathcal{U}/H$ is also definably generated by some set X. By replacing X with $Cl(X)$ we conclude that $\mathcal{U}/H$ is generated by a definably compact set. □

Proposition 3.5. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group, which is definably generated. If $\mathcal{U}^{00}$ exists, then*

(1) The group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k \times K$, for some compact group K. (Later we will see that $K \simeq \mathbb{T}^r$ where $\mathbb{T}$ is the circle group and $r \in \mathbb{N}$).

(2) $\mathcal{U}$ and $\mathcal{U}^{00}$ are divisible.

(3) $\mathcal{U}^{00}$ is torsion-free.

Proof. (1) Let us denote the group $\mathcal{U}/\mathcal{U}^{00}$ by L. By [4, Lemma 2.6] (applied to $\mathcal{U}$ instead of G there), the image of every definable, definably connected subset of $\mathcal{U}$ under π is a connected subset of L. As in the proof of Theorem 2.9 in [4], the group L is locally connected, and since $\mathcal{U}$ is connected, the group L must actually be connected.

Since $\mathcal{U}$ is generated by a definable set, say $X \subseteq \mathcal{U}$, its image $\pi(\mathcal{U}) = L$ is generated by $\pi(X)$ which is a compact set ($\pi(X)$ is a quotient of X by a type-definable equivalence relation with bounded quotient, see [17]). Hence, the group L is so-called compactly generated. By [11, Theorem 7.57], the group L is then isomorphic, as a topological group, to a direct product $\mathbb{R}^k \times K$, for some compact abelian group K. This proves (2).

In what follows, we use $+$ for the group operation of $\mathcal{U}$ and write $\mathcal{U}$ as an increasing countable union $\bigcup_{k=1}^{\infty} X(k)$ (with $X(k)$ as in the notation from Section 1.6).

(2) Let us see that $\mathcal{U}$ is divisible. Given $n \in \mathbb{N}$, consider the map $z \mapsto nz : \mathcal{U} \to \mathcal{U}$. For a subset Z of $\mathcal{U}$, let nZ denote the image of Z under this

map. The kernel of this map is $\mathcal{U}[n]$. By Proposition 3.1, $\mathcal{U}[n]$ must have dimension 0, and therefore by connectedness $\dim(n\mathcal{U}) = \dim(\mathcal{U})$.

Since $\mathcal{U}$ is connected, by [1, Proposition 1] it is sufficient to show that for every n, the group $n\mathcal{U}$ is a compatible subgroup of $\mathcal{U}$, namely that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap n\mathcal{U}$ is definable.

We claim that $Y \cap n\mathcal{U}$ is contained in $nX(j)$ for some j. Assume towards a contradiction that this fails. Then for every j there exists $x_j \in \mathcal{U}$ such that $nx_j \in Y \setminus nX(j)$. Hence, $x_j \notin X(j)$ and therefore there is a sequence $k_j \to \infty$ such that $x_j \in X(k_j) \setminus X(k_j - 1)$ and $nx_j \in Y$. Consider the projection $\pi(Y)$ and $\pi(x_j)$ in L. Because Y is definable the set $\pi(Y)$ is compact.

By Lemma 3.3, because the sequence $\{x_j\}$ is not contained in any definable subset of $\mathcal{U}$, its image $\{\pi(x_j)\}$ is not contained in any compact subset of L. At the same time, $n\pi(x_j)$ is contained in the compact set $\pi(Y)$. However, since L is isomorphic to $\mathbb{R}^k \times K$, for a compact group K, the map $x \mapsto nx$ is a proper map on L and hence this is impossible. We therefore showed that

$$Y \cap n\mathcal{U} \subseteq nX(j) \subseteq n\mathcal{U},$$

and so $Y \cap n\mathcal{U} = Y \cap nX(j)$ which is a definable set. We can conclude that the group $n\mathcal{U}$ is a compatible subgroup of $\mathcal{U}$, of the same dimension and therefore $n\mathcal{U} = \mathcal{U}$. It follows that $\mathcal{U}$ is divisible.

Let us see that $\mathcal{U}^{00}$ is also divisible. Indeed, consider the map $x \mapsto nx$ from $\mathcal{U}$ onto $\mathcal{U}$. It sends $\mathcal{U}^{00}$ onto the group $n\mathcal{U}^{00}$ and therefore $[\mathcal{U} : \mathcal{U}^{00}] \leq [\mathcal{U} : n\mathcal{U}^{00}]$. Since $\mathcal{U}^{00}$ is the smallest type-definable subgroup of bounded index we must have $n\mathcal{U}^{00} = \mathcal{U}^{00}$, so $\mathcal{U}^{00}$ is divisible.

(3) This is a repetition of an argument from [16]. Because $\mathcal{U}^{00}$ exists there is a definable generic set $X \subseteq \mathcal{U}$ which we now fix. By Theorem 2.6, the group $Stab_{ng}(X)$ contains $\mathcal{U}^{00}$, so it is sufficient to prove that for every n, there is a definable $Y \subseteq \mathcal{U}$ such that $Stab_{ng}(Y) \cap \mathcal{U}[n] = \{0\}$. We do that as follows. Because $\mathcal{U}$ is divisible, the $\bigvee$-definable map $h \mapsto nh$ is surjective. By compactness, there exists a definable $Y_1 \subseteq \mathcal{U}$ which maps onto X. However, since $\mathcal{U}[n]$ is compatible and has dimension zero, every element of X has only finitely many pre-images in Y_1. By definable choice, we can find a definable $Y \subseteq Y_1$ such that the map $h \mapsto nh$ induces a bijection from Y onto X. The set Y is generic in $\mathcal{U}$ as well (since its image is generic and the kernel of the map has dimension zero) and for every $g \in \mathcal{U}[n]$ we have $(g + Y) \cap Y = \emptyset$. Hence, the only element of $\mathcal{U}[n]$ which belongs to $Stab_{ng}(Y)$ is 0. It follows that $\mathcal{U}^{00}$ is torsion-free. □

As a corollary, we can formulate the following criterion for recognizing $\mathcal{U}^{00}$, generalizing results from [4] and [12]:

Proposition 3.6. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Assume that $H \leqslant \mathcal{U}$ is type-definable of bounded index. Then $H = \mathcal{U}^{00}$ if and only if H is torsion-free.*

In particular, if $\mathcal{U}$ is torsion-free then $\mathcal{U}^{00}$, if it exists, is the only type-definable subgroup of bounded index.

Proof. Since H is type-definable of bounded index, by [12, Proposition 7.4] $\mathcal{U}^{00}$ exists.

If $H = \mathcal{U}^{00}$, then by Proposition 3.5 it is torsion-free.

For the converse, assume that $H \leqslant \mathcal{U}$ is torsion-free. We let $L = \mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology. Because $\mathcal{U}^{00} \leqslant H$, the map $\pi : \mathcal{U} \to L$ sends the type-definable group H onto a compact subgroup of L. If $\pi(H)$ is non-trivial (namely, $H \neq \mathcal{U}^{00}$) then $\pi(H)$ has torsion. However, $\ker(\pi) = \mathcal{U}^{00}$ is divisible (see Proposition 3.5) and therefore H has torsion. Contradiction. □

Lemma 3.7. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group, which is definably generated. Then the following are equivalent.*

(1) $\mathcal{U}$ contains a definable generic set.
(2) $\mathcal{U}^{00}$ exists.
(3) $\mathcal{U}^{00}$ exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for some $k \in \mathbb{N}$ and a compact group K.
(4) There exists a definable group G and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$ with $\ker(\phi) \simeq \mathbb{Z}^{k'}$, for some $k' \in \mathbb{N}$.
(5) There exists a definable group G and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$.

Assume now that the above hold. If k is as in (3) and $\phi : \mathcal{U} \to G$ and k' are as in (4), then $k = k'$.

Proof. (1) ⇒ (2). Note first that by Claim 3.4, the group $\mathcal{U}$ has a definable torsion-free subgroup H with $\mathcal{U}/H$ definably generated by a definably compact set. Because $\mathcal{U}$ contains a definable generic set so does $\mathcal{U}/H$. By Corollary 2.11, the definable non-generic sets in $\mathcal{U}/H$ form an ideal, so by Theorem 2.6, $(\mathcal{U}/H)^{00}$ exists. Its pre-image in $\mathcal{U}$ is a type definable subgroup of bounded index which is also torsion-free (since H and $(\mathcal{U}/H)^{00}$ are both torsion-free). By Proposition 3.6 this pre-image equals $\mathcal{U}^{00}$.

(2) ⇒ (3). By Proposition 3.5.

(3) ⇒ (4). Let $L = \mathbb{R}^k \times K$ and $\pi_{\mathcal{U}} : \mathcal{U} \to L$ be the projection map (whose kernel is $\mathcal{U}^{00}$).

We now fix generators $z_1, \dots, z_k \in \mathbb{R}^k$ for $\mathbb{Z}^k$, and find $u_1, \dots, u_k \in \mathcal{U}$ with $\pi_{\mathcal{U}}(u_i) = (z_i, 0)$. If we let $\Gamma \leqslant \mathcal{U}$ be the subgroup generated by $u_1, \dots, u_k$ then $\pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. Note that since $z_1, \dots, z_k$ are $\mathbb{Z}$-independent, the restriction of $\pi_{\mathcal{U}}$ to Γ is injective, namely $\Gamma \cap \mathcal{U}^{00} = \{0\}$.

By Lemma 3.3, there is a definable $X \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(K) \subseteq X$. It follows from [4, Lemma 1.7] that the set $\pi_{\mathcal{U}}(X)$ contains not only K but also an open neighborhood of K. But then, there is an m such that $m\pi_{\mathcal{U}}(X) + \mathbb{Z}^k = L$. This implies that $\pi_{\mathcal{U}}(mX + \Gamma) = L$ and hence $mX + \mathcal{U}^{00} + \Gamma \subseteq mX + X + \Gamma = \mathcal{U}$. We let $Y = mX + X$ and then $Y + \Gamma = \mathcal{U}$.

We claim that $Y \cap \Gamma$ is finite. Indeed, if $Y \cap \Gamma$ were infinite then, since $\pi_{\mathcal{U}}$ is injective on Γ, the set $\pi_{\mathcal{U}}(Y) \cap \mathbb{Z}^k$ is infinite, contradicting the compactness of

$\pi_{\mathcal{U}}(Y)$. We can now apply Lemma 2.1 and conclude that there is a definable group G and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$ whose kernel is Γ.

(4) $\Rightarrow$ (5) is clear.

(5) $\Rightarrow$ (1). By logical compactness, there is a definable $X \subseteq \mathcal{U}$ such that $\phi(X) = G$. But then $X + \ker(\phi) = \mathcal{U}$, and since $\ker(\phi) = \mathbb{Z}^{k'}$ is small, X is generic in $\mathcal{U}$.

Assume now that the conditions hold, k is as in (3), and $\phi : \mathcal{U} \to G$ and k' are as in (4). We will prove that $k = k'$. Consider the map $\pi_U : U \to \mathbb{R}^k \times K$ and let Γ be the image of $\ker(\phi)$ under π_U.

We first claim that $k \leq k'$. Let $X \subseteq \mathcal{U}$ be so that $\phi(X) = G$. Then $X + \ker(\phi) = \mathcal{U}$. Thus, $\pi_{\mathcal{U}}(X) + \Gamma = \mathbb{R}^k \times K$. Let Y and Γ' be the projections of $\pi_{\mathcal{U}}(X)$ and Γ, respectively, into $\mathbb{R}^k$. We have $Y + \Gamma' = \mathbb{R}^k$. The set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \ldots, \lambda_{k'}$ be the generators of $\ker(\phi)$ and let $v_1, \ldots, v_{k'} \in \mathbb{R}^k$ be their images in Γ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \ldots, v_{k'}$ then $Y + H = \mathbb{R}^k$, and therefore, since Y is compact, we must have $H = \mathbb{R}^k$. This implies that $k \leq k'$.

Now let us prove that $k' \leq k$. Note first that $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\phi \restriction X$ is definable, we must have $\ker(\phi) \cap \mathcal{U}^{00} \subseteq \ker(\phi) \cap X$ finite. However, by Proposition 3.5, the group $\mathcal{U}^{00}$ is torsion-free, hence $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$.

It follows that $\Gamma = \pi_{\mathcal{U}}(\ker \phi)$ is of rank k'. It is also discrete. Indeed, using X as above we can find another definable set X' whose image $\pi_{\mathcal{U}}(X')$ contains an open neighborhood of 0 and no other elements of Γ.

Now, since K is compact, no element of Γ can be in K and therefore the projection of Γ onto $\Gamma' \subseteq \mathbb{R}^k$ is an isomorphism. Furthermore, Γ' is also discrete, which implies that $k' \leq k$. $\square$

At the end of this section, we conjecture that the above conditions always hold.

The result below is proved in [3, Theorem 8.2] for $\mathcal{U}$ the universal covering of an arbitrary definably compact group G in o-minimal expansions of real closed fields.

Proposition 3.8. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group, which is definably generated. Let G be a definable group and $\phi : \mathcal{U} \to G$ a surjective $\bigvee$-definable homomorphism with $\ker(\phi) \simeq \mathbb{Z}^k$.*

Then $\mathcal{U}^{00}$ exists, $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$ and $\phi(\mathcal{U}^{00}) = G^{00}$. Furthermore there is a topological covering map $\phi' : \mathcal{U}/\mathcal{U}^{00} \to G/G^{00}$, with respect to the *logic topologies, such that the following diagram commutes.*

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\ \phi\ } & G \\
\Big\downarrow{\scriptstyle \pi_{\mathcal{U}}} & & \Big\downarrow{\scriptstyle \pi_G} \\
\mathcal{U}/\mathcal{U}^{00} & \xrightarrow{\ \phi'\ } & G/G^{00}
\end{array} \tag{1}
$$

The group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k \times \mathbb{T}^r$, for $\mathbb{T}$ the circle group and $r \in \mathbb{N}$. If $\mathcal{U}$ is generated by a definably compact set, then $k + r = \dim(\mathcal{U})$. If, moreover, $\mathcal{U}$ is torsion-free, then $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^{\dim \mathcal{U}}$.

Proof. By Lemma 3.7, $\mathcal{U}^{00}$ exists. Let $\Gamma = \ker(\phi)$. We first claim that $\Gamma \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\phi \restriction X$ is definable, we must have $\Gamma \cap \mathcal{U}^{00} \subseteq \Gamma \cap X$ finite. However, by Proposition 3.5, the group $\mathcal{U}^{00}$ is torsion-free, hence $\Gamma \cap \mathcal{U}^{00} = \{0\}$.

We claim that $\phi(\mathcal{U}^{00}) = G^{00}$. First note that since $\mathcal{U}^{00}$ has bounded index in $\mathcal{U}$ and ϕ is surjective, the group $\phi(\mathcal{U}^{00})$ has bounded index in G. Because $\Gamma \cap \mathcal{U}^{00} = \{0\}$ the restriction of ϕ to $\mathcal{U}^{00}$ is injective and hence $\phi(\mathcal{U}^{00})$ is torsion-free. By [4], we must have $\phi(\mathcal{U}^{00}) = G^{00}$.

By [17], we have

$$G/G^{00} \simeq \mathbb{T}^l,$$

for some $l \in \mathbb{N}$. We now consider $\pi_G : G \to G/G^{00}$ and define $\phi' : \mathcal{U}/\mathcal{U}^{00} \to G/G^{00}$ as follows: For $u \in \mathcal{U}$, let $\phi'(\pi_{\mathcal{U}}(u)) = \pi_G(\phi(u))$. Since $\phi(\mathcal{U}^{00}) = G^{00}$ this map is a well-defined homomorphism which makes the above diagram commute. It is left to see that ϕ' is a covering map.

It follows from what we established thus far that $\ker(\phi') = \pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. Let us see that this is a discrete subgroup of $\mathcal{U}/\mathcal{U}^{00}$. Indeed, as we already saw, for every compact neighborhood $W \subseteq \mathcal{U}/\mathcal{U}^{00}$ of 0, there is a definable set $Z \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(W) \subseteq Z$. But we already saw that $Z \cap \Gamma$ is finite and hence $W \cap \ker(\phi')$ must be finite. It follows that $\ker(\phi')$ is discrete.

By Lemma 3.7, $\mathcal{U}/\mathcal{U}^{00}$, equipped with the Logic topology, is locally compact. Since $\phi' : \mathcal{U}/\mathcal{U}^{00} \to G/G^{00}$ is a surjective homomorphism with discrete kernel it is sufficient to check that it is continuous as a map between topological groups. If $W \subseteq G/G^{00}$ is open then $V = \pi_G^{-1}(W)$ is a $\bigvee$-definable subset of G and hence $\phi^{-1}(V)$ is a $\bigvee$-definable subset of $\mathcal{U}$ (because $\ker \phi$ is a small group). By commutation, this last set equals $\pi_{\mathcal{U}}^{-1}(\phi'^{-1}(W))$ and therefore $\phi'^{-1}(W)$ is open in $\mathcal{U}/\mathcal{U}^{00}$.

By Lemma 3.7, $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for a compact group K. We now have a covering map $\phi' : \mathbb{R}^k \times K \to G/G^{00} = \mathbb{T}^{k+r}$, with $\ker(\phi') = \mathbb{Z}^k \subseteq \mathbb{R}^k$. It follows that $K \simeq \mathbb{T}^r$.

If $\mathcal{U}$ is generated by a definably compact set, G will be definably compact. In this case, by the work in [10], [12] and [14],

$$G/G^{00} \simeq \mathbb{T}^{\dim(G)}$$

and, hence, $k + r = \dim(G) = \dim(\mathcal{U})$.

If, moreover, $\mathcal{U}$ is torsion-free, we have $r = 0$. □

We summarize the above results in the following theorem.

Theorem 3.9. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Then there is $k \in \mathbb{N}$ such that the following are equivalent:*
(i) $\mathcal{U}$ contains a definable generic set.
(ii) $\mathcal{U}^{00}$ exists.
(iii) $\mathcal{U}^{00}$ exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, for some $r \in \mathbb{N}$.
(iv) There is a definable group G, with $\dim G = \dim \mathcal{U}$, and a $\bigvee$-definable surjective homomorphism $\phi : \mathcal{U} \to G$.

If in addition $\mathcal{U}$ is generated by a definably compact set, then (ii) is strengthened by the condition that $k + r = \dim \mathcal{U}$.

Proof. By Lemma 3.7 and Proposition 3.8. □

Theorem 3.10. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definable group.

If $\mathcal{U}$ generated by a definably compact set, then $\mathcal{U}/\Lambda'$ is moreover definably compact.

Proof. Since $X+\Lambda = \mathcal{U}$, X is generic. By Theorem 3.9, $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, for some $k, r \in \mathbb{N}$. We now consider $\Delta = \pi_{\mathcal{U}}(\Lambda) \subseteq \mathbb{R}^k \times \mathbb{T}^r$ and let $\Delta' \subseteq \mathbb{R}^k$ be the projection of Δ into $\mathbb{R}^k$. Since $X+\Lambda = \mathcal{U}$, we have $\pi_{\mathcal{U}}(X)+\Delta = \mathbb{R}^k \times \mathbb{T}^r$. Hence, if Y is the projection of $\pi_{\mathcal{U}}(X)$ into $\mathbb{R}^k$ then we have $Y + \Delta' = \mathbb{R}^k$. The set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \ldots, \lambda_m$ be generators of Λ and let $v_1, \ldots, v_m \in \mathbb{R}^k$ be their images in Δ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \ldots, v_m$ then $Y + H = \mathbb{R}^k$, and therefore, since Y is compact, we must have $H = \mathbb{R}^k$. This implies that among $v_1, \ldots, v_m$ there are elements $v_{i_1}, \ldots, v_{i_k}$ which are $\mathbb{R}$-independent. It follows that $\lambda_{i_1}, \ldots, \lambda_{i_k} \in \Delta$ are $\mathbb{Z}$-independent. If we let Λ' be the group generated by $\lambda_{i_1}, \ldots, \lambda_{i_k}$ then we immediately see that the restriction of $\pi_{\mathcal{U}}$ to Λ' is injective. We claim that $\mathcal{U}/\Lambda'$ is definable.

First, let us see that for every definable $Z \subseteq \mathcal{U}$, the set $Z \cap \Lambda'$ is finite. Indeed, $\pi_{\mathcal{U}}(Z)$ is a compact subset of $\mathbb{R}^k \times \mathbb{T}^r$ and hence $\pi_{\mathcal{U}}(Z) \cap (\mathbb{Z}v_{i_1} + \cdots + \mathbb{Z}v_{i_k})$ is finite. Because $\pi_{\mathcal{U}}|\Lambda'$ is injective it follows that $Z \cap \Lambda'$ is also finite.

We can now take a compact set $K \subseteq \mathbb{R}^k \times \mathbb{T}^r$ such that $K + \mathbb{Z}^k = \mathbb{R}^k \times \mathbb{T}^r$. It follows that $\pi_{\mathcal{U}}^{-1}(K) + \Lambda' = \mathcal{U}$. By Lemma 3.3, there is a definable set $Z \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(K) \subseteq Z$. We now have $Z + \Lambda' = \mathcal{U}$ and $Z \cap \Lambda'$ finite. By Lemma 2.1, $\mathcal{U}/\Lambda'$ is definable.

For the last clause, let $f : \mathcal{U} \to \mathcal{U}/\Lambda'$ be the quotient map, and X' a definable subset of $\mathcal{U}$ such that $f(X') = \mathcal{U}/\Lambda'$. Since $\mathcal{U}$ is generated by a definably compact set, the closure of X' in $\mathcal{U}$ must be a subset of a definably compact set and, hence, itself definably compact. But then it is easy to verify that $\mathcal{U}/\Lambda' = f(X')$ is definably compact. □

We end this section with a conjecture.

Conjecture A. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Then*
(i) $\mathcal{U}$ contains a definable generic set.
(ii) $\mathcal{U}$ is divisible.

Although we cannot prove the above conjecture, we can reduce it to proving (i) under additional assumptions.

Conjecture B. *Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group, generated by a definably compact set. Then $\mathcal{U}$ contains a definable generic set.*

Claim 3.11. *Conjecture B implies Conjecture A.*

Proof. We assume that Conjecture B is true.

Let $\mathcal{U}$ be a connected abelian $\bigvee$-definable group which is definably generated. Let $\mathcal{V}$ be the universal cover of $\mathcal{U}$ (see [7]). Because $\mathcal{U}$ is the homomorphic image of $\mathcal{V}$ under a $\bigvee$-definable homomorphism whose kernel is a set of dimension 0, it is sufficient to prove that $\mathcal{V}$ contains a generic set and that $\mathcal{V}$ is divisible.

The group $\mathcal{V}$ is connected, torsion-free and generated by a definable set $X \subseteq \mathcal{V}$. We work by induction on $\dim(\mathcal{V})$.

Let Y be the closure of X with respect to the group topology of $\mathcal{V}$.

Case 1 The set Y is definably compact.

Since $\mathcal{V}$ is generated by Y, then by our standing assumption we may conclude that $\mathcal{V}$ contains a definable generic set. By Theorem 3.9 and Proposition 3.5, $\mathcal{V}$ is divisible.

Case 2 The set Y is not definably compact.

In this case, we can apply [6, Theorem 5.2] and obtain a definable 1-dimensional, definably connected, divisible, torsion-free subgroup of $\mathcal{V}$, call it H. Clearly, H is a compatible subgroup of $\mathcal{V}$, hence the group $\mathcal{V}/H$ is $\bigvee$-definable, connected ([6, Corollary 4.8]), torsion-free and definably generated (by the image of X under the projection map). We have $\dim(\mathcal{V}/H) < \dim \mathcal{V}$, so by induction, the conjecture holds for $\mathcal{V}/H$, hence it is divisible and contains a definable generic set Z. Because H is divisible as well, it follows that $\mathcal{V}$ is divisible. It is easy to see that the pre-image of Z in $\mathcal{V}$ is a definable generic subset of $\mathcal{V}$. □

Finally, although we know that $\mathcal{U}$ needs to be definably generated in order to guarantee (i) (by Fact 2.3(2)), we do not know if the same is true for (ii).

Conjecture C. *Let* $\mathcal{U}$ *be a connected abelian* $\bigvee$*-definable group. Then* $\mathcal{U}$ *is divisible.*

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Part 3

Definable groups as homomorphic images of semilinear and field-definable groups

DEFINABLE GROUPS AS HOMOMORPHIC IMAGES OF SEMI-LINEAR AND FIELD-DEFINABLE GROUPS

PANTELIS E. ELEFTHERIOU AND YA'ACOV PETERZIL

ABSTRACT. We analyze definably compact groups in o-minimal expansions of ordered groups as a combination of semi-linear groups and groups definable in o-minimal expansions of real closed fields. The analysis involves structure theorems about their locally definable covers. As a corollary, we prove the Compact Domination Conjecture in o-minimal expansions of ordered groups.

1. INTRODUCTION

This is the second of two papers (originally written as one) analyzing groups definable in o-minimal expansions of ordered groups. The ultimate goal of this project is to reduce the analysis of such groups to semi-linear groups and to groups definable in o-minimal expansions of real closed fields. Such a reduction was proposed in Conjecture 2 from [19] and a first step towards it was carried out in [10].

In the first paper ([12]) we established conditions under which locally definable groups have definable quotients of the same dimension. In this paper, we carry out the aforementioned reduction for definably compact groups by first stating a structure theorem for the universal cover $\widehat{G}$ of a definable group G (Theorem 1.1). We describe $\widehat{G}$ as an extension of a locally definable group $\mathcal{U}$ in an o-minimal expansion of a real closed field by a locally definable semi-linear group $\widehat{H}$. We then apply [12, Theorem 3.10] and derive a stronger structure theorem (Theorem 1.3), replacing the above $\mathcal{U}$ by *a definable group*. We expect that the second theorem will be useful when reducing questions for definable groups to groups in the semi-linear and field settings. We illustrate this effect by applying our second theorem to conclude the Compact Domination Conjecture in o-minimal expansions of ordered groups (Theorem 1.4 below).

Let us provide the details.

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1.1. **The setting.** We let $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ be an o-minimal expansion of an ordered group. When $\mathcal{M}$ expands a real closed field (with $+$ not necessarily one of the field operations) there is strong compatibility of definable sets with the field structure. For example, each definable function is piecewise differentiable with respect to the field structure. Other powerful tools, such as the triangulation theorem, are available as well ([3]). At the other end, when $\mathcal{M}$ is a linear structure, such as a reduct of an ordered vector space over an ordered division ring, then every definable set is *semi-linear.*

By the Trichotomy Theorem for o-minimal structures there is a third possibility (see [20]), where there is a definable real closed field R on some interval in M, and yet the underlying domain of R is necessarily a bounded interval and not the whole of M. Such a structure is called *semi-bounded* (and non-linear), and definable sets in this case turn out to be a combination of semi-linear sets and sets definable in o-minimal expansions of fields (see [4], [19], [10]). An important example is the expansion of the ordered vector space $\langle \mathbb{R}; <, +, x \mapsto ax \rangle_{a \in \mathbb{R}}$ by all bounded semialgebraic sets. Most of our work is intended for a semi-bounded structure which is non-linear.

We assume in the rest of this paper, and unless stated otherwise, that $\mathcal{M} = \langle M, <, +, \cdots \rangle$ is a sufficiently saturated o-minimal expansion of an ordered group.

1.2. **Short sets and long dimension.** Following [19], we call an element $a \in M$ *short* if either $a = 0$ or the interval $(0, a)$ supports a definable real closed field; otherwise a is called *tall.* An element of M^n is called *short* if all its coordinates are short. An interval $[a, b]$ is called *short* if $b - a$ is short, and otherwise it is called *long.* A definable set $X \subseteq M^n$ is called *short* if it is in definable bijection with a subset of I^n for some short interval I. The image of a short set under a definable map is short. As is shown in [4], $\mathcal{M}$ is semi-bounded if and only if all unbounded rays $(a, +\infty)$ are long. However, a semi-bounded and sufficiently saturated $\mathcal{M}$ also has bounded intervals which are long.

Following [10] (see also Section 3 below), we say that *the long dimension* of a definable $X \subseteq M^n$, $\operatorname{lgdim}(X)$, is the maximum k such that X contains a definable homeomorphic image of I^k, for some long interval I (the original definition of $\operatorname{lgdim}(X)$ was given in terms of cones, see Section 3 below, but it is not hard to see the equivalence of the two). The results in [10] show that every definable subset of M^n can be decomposed into "long cones" and as a result it follows that a definable $X \subseteq M^n$ is short if and only if $\operatorname{lgdim}(X) = 0$. We call X *strongly long* if $\operatorname{lgdim}(X) = \dim(X)$; this is for example the case with a cartesian product of long intervals. Note that all these notions are invariant under definable bijections.

Roughly speaking, strongly long sets and short sets are "orthogonal" to each other. The idea is that the structure which $\mathcal{M}$ induces on short sets comes from an o-minimal expansion of a real closed field, while the structure induced on strongly long sets is closely related to the semi-linear structure. More precisely, if $p(x)$ is a complete type over A such that every formula in $p(x)$ defines a strongly long set then its semi-linear formulas determine the type. This is a result which will not be used in this paper, but its proof is straightforward. Indeed, the aforementioned decomposition from [10] implies, in particular, that every strongly long definable set X of dimension k is a union of a strongly long k-dimensional semi-linear set and a definable set whose long dimension is smaller than k. Both sets are definable over the same set of parameters as X. It follows that $p(x)$ is determined by the semi-linear formulas.

We will see in examples (Section 6) that the analysis of definable groups forces us to use the language of $\bigvee$-definable groups, so we recall some definitions.

1.3. **$\bigvee$-definable and locally definable sets.** Let $\mathcal{M}$ be a κ-saturated, not necessarily o-minimal, structure. By *bounded* cardinality we mean cardinality smaller than κ. We alert the reader that there is a second use of the word "bounded" throughout this paper. Namely, a subset of M^n is *bounded* if it is contained in some cartesian product of bounded intervals. It will always be clear from the context what we mean.

A *$\bigvee$-definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{i \in I} X_i$ of definable subsets of M^n for some fixed n (where $|I|$ is bounded) and for every $i, j \in I$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). Following [5], we say that $\langle \mathcal{U}, \cdot \rangle$ is *locally definable* if $|I|$ is countable. In this paper, we consider exclusively locally definable groups. We are mostly interested in *definably generated* groups, namely $\bigvee$-definable groups which are generated as a group by a definable subset. These groups are of course locally definable. An important example of such groups is the universal cover of a definable group (see [6]). In [16] a similar notion is introduced, of an Ind-definable group.

A map $\phi : \mathcal{U} \to \mathcal{H}$ between $\bigvee$-definable (locally definable) groups is called *$\bigvee$-definable (locally definable)* if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, $graph(\phi) \cap (X \times Y)$ is a definable set. Equivalently, the restriction of ϕ to any definable set is definable.

In an o-minimal expansion of an ordered group, a $\bigvee$-definable group $\mathcal{U}$ is called *short* if $\mathcal{U}$ is given as a bounded union of definable short sets. If $\mathcal{U} = \bigcup_{i \in I} X_i$ then we let $\text{lgdim}(\mathcal{U}) = max_i(\text{lgdim}(X_i))$. We say that $\mathcal{U}$ is *strongly long* if $\dim(\mathcal{U}) = \text{lgdim}(\mathcal{U})$.

We are now ready to state the main results of this paper. Note that in the special case where $\mathcal{M}$ expands a real closed field, the results below become trivial (since in this case all definable sets are short), and in the case where $\mathcal{M}$ is semi-linear, they reduce to the main theorem from [13] (since in this case every definable short set is finite).

1.4. **The universal cover of a definably compact group.** We first note (see [19, Lemma 7.1]) that every definably compact group in a semi-bounded structure is necessarily bounded; namely, it is contained in some cartesian product of bounded intervals.

Theorem 1.1. *Let G be a definably compact, definably connected group of long dimension k and let $\widehat{F} : \widehat{G} \to G$ be the universal cover of G. Then there exist an open, connected subgroup $\widehat{\mathcal{H}} \subseteq \langle M^k, + \rangle$, generated by a semi-linear set of long dimension k, and a locally definable embedding $i : \widehat{\mathcal{H}} \to \widehat{G}$, with $i(\widehat{\mathcal{H}})$ central in G, such that $\mathcal{U} = \widehat{G}/i(\widehat{\mathcal{H}})$ is generated by a short definable set. Namely, we have the following exact sequence with locally definable maps i, π and $\widehat{F}$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\ i\ } & \widehat{G} & \xrightarrow{\ \pi\ } & \mathcal{U} & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \widehat{F}} & & & & \\ & & & & G & & & & \end{array}$$

If we let $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$, then $\mathcal{H}$ is the largest connected, strongly long, locally definable subgroup of G, namely it contains every other such group.

Question In Section 6 we present various examples that illustrate this theorem. In all our known examples the universal cover $\widehat{G}$ is the direct sum of the groups $\widehat{\mathcal{H}}$ and $\mathcal{U}$ (rather then just an extension of $\mathcal{U}$ by $\widehat{\mathcal{H}}$). Can $\widehat{G}$ always be realized as a direct sum of $\widehat{\mathcal{H}}$ and $\mathcal{U}$?

Remark 1.2. 1. One immediate corollary of the above theorem is that every definably compact group G which is strongly long is definably isomorphic to a semi-linear group, because in this case $\mathcal{H} = G$.

2. Note that when G is abelian, we have $\ker(\widehat{F}) \simeq \mathbb{Z}^{\dim G}$ (indeed, by [6, Corollary 1.5], we have $ker(\widehat{F}) \simeq \mathbb{Z}^l$, where the k-torsion subgroups of G satisfy $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^l$. By [19], we have $l = \dim G$).

3. Note that since $\mathcal{U}$ above is generated by a definable short set, there is a definable real closed field R such that $\mathcal{U}$ is locally definable in an o-minimal expansion of R. Indeed, let $X \subseteq \mathcal{U}$ be a definable set which generates $\mathcal{U}$, and let R be a definable real closed field such that, up to an $\mathcal{M}$-definable definable bijection, X is a subset of R^m. Let $\mathcal{N}$ be the structure which $\mathcal{M}$ induces on R. Without loss of generality, $0 \in X$. We let $X_1 = X$ and consider the equivalence relation on $X \times X$ given by $(x, y) \sim (x', y')$ if $x - y = x' - y'$. Clearly, $X \times X/\sim$ is in definable bijection with $X - X$. By definable choice in $\mathcal{N}$, there exists a definable set Y in $\mathcal{N}$ and a definable bijection between $X \times X/\sim$ and Y. Hence, in $\mathcal{M}$ the sets $X - X$ and Y are in definable bijection. Now consider the definable embedding of X into $X - X$ ($x \mapsto x - 0$), which induces an $\mathcal{N}$-definable injection $f_1 : X_1 \to Y$. We let

$$X_2 = X_1 \sqcup (Y \setminus f_1(X_1)).$$

The set X_2 is definable in $\mathcal{N}$ and is in definable bijection with Y (so also with $X - X$). We also have $X_1 \subseteq X_2$.

We similarly define X_3 in $\mathcal{N}$ to be in definable bijection with $X - X + X$ and such that $X_1 \subseteq X_2 \subseteq X_3$. We continue in the same way and obtain a locally definable set $\bigcup_{n\in\mathbb{N}} X_n$ in $\mathcal{N}$ that is in locally definable bijection with $\mathcal{U}$.

1.5. **Covers by extensions of definable short groups.** In the next result we want to replace the locally definable group $\mathcal{U}$ from Theorem 1.1 by a definable short group $\overline{K}$. Roughly speaking, it says that G is close to being an extension of a short definable group by a semi-linear group, and the distance from being such a group is measured by the kernel of the map $\overline{F'}$ below.

Theorem 1.3. *Let G be a definably compact, definably connected group of long dimension k. Then G has a locally definable cover $\overline{F} : \overline{G} \to G$ with the following properties: there is an open subgroup $\widehat{\mathcal{H}} \subseteq \langle M^k, +\rangle$, generated by a semi-linear set of long dimension k, and a locally definable embedding $i : \widehat{\mathcal{H}} \to \overline{G}$, with $i(\widehat{\mathcal{H}})$ central in $\overline{G}$, such that $\overline{K} = \overline{G}/i(\widehat{\mathcal{H}})$ is a definably compact* **definable** *short group. Namely, we have the following exact sequence with locally definable maps i, π and $\overline{F}$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\ i\ } & \overline{G} & \xrightarrow{\ \pi\ } & \overline{K} & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \overline{F}} & & & & \\ & & & & G & & & & \end{array}$$

If we take $\mathcal{H} \subseteq G$ as in Theorem 1.1, then there is also a locally definable, central extension G' of $\overline{K}$ by $\mathcal{H}$, with a locally definable homomorphism $F' : G' \to G$.

When G is abelian so is $\overline{G}$ and $\ker(\overline{F}) \simeq \mathbb{Z}^k + F$*, for a finite group F.*

It is at the passage from the locally definable group $\mathcal{U}$ in Theorem 1.1 to the definable group $\overline{K}$ in Theorem 1.3 that we use [12, Theorem 3.10].

1.6. **Compact Domination.** The relationship between a definable group G and the compact Lie group G/G^{00} has been the topic of quite a few papers. In [9], [15], [17] the related so-called Compact Domination Conjecture was solved for semi-linear groups and for groups definable in expansions of real closed fields. Using the above analysis we can complete the proof of the conjecture for groups definable in arbitrary o-minimal expansions of ordered groups (see Section 7 for the original formulation of the conjecture).

Theorem 1.4. *Let G be a definably compact, definably connected group. Let $\pi : G \to G/G^{00}$ denote the canonical homomorphism. Then, G is compactly dominated by G/G^{00}. That is, for every definable set $X \subseteq G$, the set*

$$\pi(X) \cap \pi(G \setminus X)$$

has **Haar** *measure* 0.

1.7. **Notation.** Let us finish this section with a couple of notational remarks. Given a group $\langle G, \cdot\rangle$ and a set $X \subseteq G$, we denote, for every $n \in \mathbb{N}$,

$$X(n) = \overbrace{(XX^{-1})\cdots(XX^{-1})}^{n-\text{times}}$$

We assume familiarity with the notion of definable compactness. Whenever we write that a set is definably compact, or definably connected, we assume in particular that it is definable.

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2. Preliminaries I: locally definable groups, extensions of abelian groups, pushout and pullback

As mentioned in the introduction, we work in a sufficiently saturated o-minimal expansion of an ordered group a $\mathcal{M} = \langle M, <, +, \cdots\rangle$. However, the only use of this assumption is to guarantee a strong version of elimination of imaginaries, which allows us to replace every definable quotient by a definable set. Any structure in which this is true will be just as good here, or, if we are willing to work in $\mathcal{M}^{eq}$, then any o-minimal structure will work.

2.1. Locally definable groups, compatible subgroups and definable quotients.

Definition 2.1. (See [5]) For a locally definable group $\mathcal{U}$, we say that $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset of* $\mathcal{U}$ if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case $\mathcal{V}$ itself is a countable union of definable sets).

Clearly, the only compatible locally definable subsets of a definable group are the definable ones. Note that if $\phi : \mathcal{U} \to \mathcal{V}$ is a locally definable homomorphism between locally definable groups then $\ker(\phi)$ is a compatible locally definable normal subgroup of $\mathcal{U}$. Compatible subgroups are used in order to obtain locally definable quotients. Together with [5, Theorem 4.2], we have:

Fact 2.2. *If* $\mathcal{U}$ *is a locally definable group and* $\mathcal{H} \subseteq \mathcal{U}$ *a locally definable normal subgroup, then* $\mathcal{H}$ *is a compatible subgroup of* $\mathcal{U}$ *if and only if there exists a locally definable surjective homomorphism of locally definable groups* $\phi : \mathcal{U} \to \mathcal{V}$ *whose kernel is* $\mathcal{H}$.

If $\mathcal{M}$ is an o-minimal structure and $\mathcal{U} \subseteq M^n$ is a locally definable group then, by [2, Theorem 4.8], it can be endowed with a manifold-like topology τ, making it into a topological group. Namely, there is a countable collection $\{U_i : i \in I\}$ of definable subsets of $\mathcal{U}$, whose union equals $\mathcal{U}$, such that each U_i is in definable bijection with an open subset of M^k ($k = \dim \mathcal{U}$), and the transition maps are continuous. Moreover the U_i's and the transition maps are definable over the same parameters as $\mathcal{U}$. The group operation and group inverse are continuous with respect to this induced topology. The topology τ is determined by the ambient topology of M^n in the sense that at every generic point of $\mathcal{U}$ the two topologies coincide. From now on, whenever we refer to a topology on G, it is τ we are considering.

Definition 2.3. (See [1]) In an o-minimal structure, a locally definable group $\mathcal{U}$ is called *connected* if there is no locally definable compatible subset $\emptyset \subsetneqq \mathcal{V} \subsetneqq \mathcal{U}$ which is both closed and open with respect to the group topology.

Remark 2.4. It is easy to see that, in an o-minimal structure, if a locally definable group $\mathcal{U}$ is generated by a definably connected set which contains the identity, then it is connected.

Definition 2.5. Given a locally definable group $\mathcal{U}$ and $\Lambda_0 \subseteq \mathcal{U}$ a normal subgroup, we say that $\mathcal{U}/\Lambda_0$ is *definable* if there is a definable group $\overline{K}$ and a surjective locally definable homomorphism $\mu : \mathcal{U} \to \overline{K}$ whose kernel is Λ_0.

We now quote Theorem 3.10 from [12] (in a restricted case).

Fact 2.6. *Let $\mathcal{U}$ be a connected, abelian locally definable group, which is generated by a definably compact set. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definably compact definable group.

2.2. **Pushouts and definability.** *In the following three subsections, all groups are assumed to be abelian and all arrows represent group homomorphisms.*

Several steps of the proof require us to construct extensions of abelian groups with certain maps attached to them. All constructions are standard in the classical theory of abelian groups but because we are concerned here with definability issues we review the basic notions (see [14] for the classical treatment). The proofs of these basic results are given in the appendix. Although we chose to present the constructions below in the more common language of pushouts and pullbacks, it is also possible to carry them out in the less canonical (but possibly more constructive) language of sections and cocycles.

Definition 2.7. Given homomorphisms

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \\ C & & \end{array}$$

the triple (D, γ, δ) (or just D) is called *a pushout* (of B and C over A via $\alpha, \beta, \gamma, \delta$) if the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow{\delta} & D \end{array}$$

and for every commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ C & \xrightarrow{\delta'} & D' \end{array} \tag{1}$$

there is a unique $\phi : D \to D'$ such that $\phi\gamma = \gamma'$ and $\phi\delta = \delta'$.

If A, B, C, D and the associated maps are (locally) definable, and if for every (locally) definable D', γ', δ' there is a (locally) definable $\phi : D \to D'$ as required then we say that *the pushout is (locally) definable.*

Proposition 2.8. *Assume that we are given the following diagram*

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \\ C & & \end{array}$$

(i) Let (D, γ, δ) be a pushout. Then

$$\ker(\gamma) = \alpha(\ker(\beta)).$$

Moreover, if β is surjective, then so is γ. If α is injective, then so is δ.
(ii) Suppose that all data are definable. Then there exists a definable pushout (D, γ, δ), which is unique up to definable isomorphism.
(iii) Suppose that all data are locally definable and $\alpha(A)$ is a compatible subgroup of B. Then there exists a locally definable pushout (D, γ, δ), which is unique up to locally definable isomorphism.

Assume now that α is injective. If we let $E = B/\alpha(A)$ and $\pi : B \to E$ the projection map then there is a locally definable surjection $\pi' : D \to E$

such that the diagram below commutes and both sequences are exact. In particular, $\ker(\pi') = \delta(C)$ *is a compatible subgroup of* D*.*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\pi} & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle id_E} & & \\ 0 & \longrightarrow & C & \xrightarrow{\delta} & D & \xrightarrow{\pi'} & E & \longrightarrow & 0 \end{array}$$

We also need the following general fact, for which we could not find a reference (see appendix for proof):

Lemma 2.9. *Assume that we are given the following commutative diagram*

(2)
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ C & \xrightarrow{\delta} & D \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\mu} \\ E & \xrightarrow{\xi} & F \end{array}$$

with D *the pushout of* B *and* C *over* A *(via* $\alpha, \beta, \gamma, \delta$*), and* F *the pushout of* B *and* E *over* A *(via* α, $\eta\beta$, $\mu\gamma$ *and* ξ*). Then* F *is also the pushout of* E *and* D *over* C *(via* η, δ, μ, ξ*).*

2.3. **Pullbacks and definability.**

Definition 2.10. Given homomorphisms

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

the triple (D, γ, δ) (or just D) is called *a pullback* (of B and C over A via $\alpha, \beta, \gamma, \delta$) if the following diagram commutes

$$\begin{array}{ccc} D & \xrightarrow{\gamma} & B \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

and for every commutative diagram

(3)
$$\begin{array}{ccc} D' & \xrightarrow{\gamma'} & B \\ \downarrow{\scriptstyle\delta'} & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

there is a unique $\phi : D' \to D$ such that $\gamma\phi = \gamma'$ and $\delta\phi = \delta'$.

If A, B, C, D and the associated maps are (locally) definable, and if for every (locally) definable D', γ', δ' there is a (locally) definable $\phi : D' \to D$ as required then we say that *the pullback is (locally) definable.*

Proposition 2.11. *Assume that we are given the following diagram*

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

(i) Let (D, γ, δ) be a pullback. Then

$$\gamma(\ker(\delta)) = \ker(\alpha).$$

Moreover, if β is surjective, then so is γ. If α is injective, then so is δ.
(ii) Suppose that all data are definable. Then there exists a definable pullback (D, γ, δ), which is unique up to definable isomorphism.
(iii) Suppose that all data are locally definable. Then there exists a locally definable pullback (D, γ, δ), which is unique up to locally definable isomorphism.

Assume now that β is surjective. Let $G = \ker(\gamma)$ and $H = \ker(\beta)$. Then G, H are locally definable and compatible in D and C, respectively. Moreover, there is a locally definable isomorphism $j : G \to H$ such that the following diagram commutes and both sequences are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{id_G} & D & \xrightarrow{\gamma} & B & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \alpha} & & \\ 0 & \longrightarrow & H & \xrightarrow{id_H} & C & \xrightarrow{\beta} & A & \longrightarrow & 0 \end{array}$$

2.4. **Additional lemmas.**

Lemma 2.12. *Assume that the sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

is exact and that we have a surjective homomorphism $\mu : C \to D$. Let $A' = \ker(\mu\pi) \subseteq B$. Then the following diagram commutes and both sequences are exact. If all data are locally definable then so is A' and the associated maps.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{\pi} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mu} & & \\ 0 & \longrightarrow & A' & \xrightarrow{id} & B & \xrightarrow{\mu\pi} & D & \longrightarrow & 0 \end{array}$$

Proof. This is trivial. □

Lemma 2.13. *Assume that we have surjective homomorphisms $F : B \to G$ and $F' : B \to G'$ with $\ker(F') \subseteq \ker(F)$. Then there is a canonical surjective homomorphism $h : G' \to G$, given by $h(g') = g$ if and only if there exists $b \in B$ with $F'(b) = g'$ and $F(b) = g$. The kernel of h equals $F'(\ker(F))$ and if all data are locally definable then so is h.*

Proof. Algebraically, this is just the fact that if $B_1 \subseteq B_2 \subseteq B$ then there is a canonical homomorphism $h : B/B_1 \to B/B_2$, whose kernel is B_2/B_1.

As for definability, assume that B, G, G', and F, F' are $\bigvee$-definable, and take definable sets $X \subseteq G$ and $X' \subseteq G'$. We want to show that the intersection of *graph*(h) with $X' \times X$ is definable. Since F', F are $\bigvee$-definable and surjective, there exists a definable $Y \subseteq B$ such that $F'(Y) \supset X'$ and $F(Y) \supseteq X$. Now, for every $g' \in X'$ there exists $b \in Y$ such that $F'(b) = g'$, and we have $h(g') = F(b)$. Thus, the intersection of *graph*(h) with $X' \times X$ is definable. □

Remark 2.14. All statements from Proposition 2.8 to Lemma 2.13 hold under the more general assumption that $\mathcal{M}$ is any sufficiently saturated structure (not necessarily o-minimal) which has strong definable choice. This is because the definability issues in the statements are all based on Fact 2.2, which can be proved for such a more general $\mathcal{M}$.

3. Preliminaries II: Semi-bounded sets

3.1. **Long cones and long dimension.** In this section we recall some notions from [10] and prove basic facts that follow from that paper.

A *k-long cone in M^n* is a set of the form

$$C = \left\{ b + \sum_{i=1}^{k} \lambda_i(t_i) : b \in B,\ t_i \in J_i \right\},$$

where B is a short cell, each $J_i = (0, a_i)$ is a long interval (with a_i possibly ∞) and $\lambda_1, \dots, \lambda_k$ are M-independent partial linear maps from $(-a_i, a_i)$ into M^n (by M-independent we mean: for all $t_1, \dots, t_k \in M$, if $\lambda_1(t_1) + \cdots + \lambda_k(t_k) = 0$ then $t_1 = \cdots = t_k = 0$). It is required further that for each $x \in C$ there are unique b and t_i's with $x = b + \sum_{i=1}^{k} \lambda_i(t_i)$ (we refer to this as "long cones are normalized"). So $\dim C = \dim B + k$. A *long cone* is a k-long cone for some k. By the normality condition, if C is a k-long cone of dimension k then B must be a singleton.

The *long dimension* of a definable set $X \subseteq M^n$, denoted $\operatorname{lgdim}(X)$, is the maximum k such that X contains a k-long cone. This notion coincides with what we defined as long dimension in the Introduction. We call X *strongly long* if $\operatorname{lgdim}(X) = \dim(X)$.

Note that if C as above is a bounded cone (namely, all a_i's belong to M) then we can take $B' = \{b + (\lambda_1(a_1/2), \dots, \lambda_k(a_k/2)) : b \in B\}$ and write

$C = B' + \langle C\rangle$ where

$$\langle C\rangle = \left\{ \sum_{i=1}^{k} \lambda_i(t_i) : t_i \in (-a_i/2, a_i/2) \right\}.$$

In this paper, we are interested in bounded cones so we replace B with B' and write $C = B + \langle C\rangle$.

As is shown in [10, Section 5] the notion of short and long intervals gives rise to a pregeometry based on the following closure operation:

Definition 3.1. Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Given $A \subseteq M$ and $a \in M$, we say that a is in the short closure of A, $a \in scl(A)$, if there exists an A-definable short interval containing a (in particular, $dcl(A) \subseteq scl(A)$).

We say that $B \subseteq M$ is scl-independent over A if for every $b \in B$, we have $b \notin scl(B \cup A \setminus \{b\})$. We let $\mathrm{lgdim}(B/A)$ be the cardinality of a maximal scl-independent subset of B over A.

Notice that if $\mathcal{M}$ expands a real closed field then every set has long dimension 0 over $\emptyset$. On the other hand if $\mathcal{M}$ is a reduct of an ordered vector space then $scl(-) = dcl(-)$. Thus, this notion is interesting when $\mathcal{M}$ is non-linear and yet does not expand a real closed field (namely, non-linear and semi-bounded).

As for the usual o-minimal dimension, the notion of long dimension for definable sets is compatible with the scl-pregeometry in the following sense (see [10, Corollary 5.10]):

Fact 3.2. *If X is an A-definable set in a sufficiently saturated o-minimal expansion of an ordered group then*

$$\mathrm{lgdim}(X) = \max\{\mathrm{lgdim}(x/A) : x \in X\}.$$

We say that $a \in X$ is *long-generic over A* if $\mathrm{lgdim}(a/A) = \mathrm{lgdim}(X)$.

By [10, Theorem 3.8], if X is A-definable of long dimension k and a is long generic in X over A then a belongs to an A-definable k-long cone in X.

We are now ready to prove two facts which will be used later on.

Fact 3.3. *Let $F : B \times C \to M^l$ be a definable map, where $B \subseteq M^m$ is a short set and $C \subseteq M^n$ is strongly long (namely $\mathrm{lgdim}(C) = dim(C)$). Then there is an open subset B_1 of B and a strongly long $X \subseteq C$, with $\dim X = \dim C$, such that F is continuous on $B_1 \times X$.*

Proof. We may assume that B, C and F are $\emptyset$-definable. Pick b generic in B and c which is long-generic in C over b. Since B is short we have

$$\mathrm{lgdim}(bc/\emptyset) = \mathrm{lgdim}(c/b) = \mathrm{lgdim}(C) = \mathrm{lgdim}(B \times C).$$

Because $\dim C = \text{lgdim}C$, c is also generic over b and, hence, we have

$$\dim(bc/\emptyset) = \dim B \times C.$$

That is, $\langle b, c\rangle$ is generic in $B \times C$ so there exists a $\emptyset$-definable relatively open set $Y \subseteq B \times C$ containing $\langle b, c\rangle$, on which F is continuous. In particular, there exists a relatively open neighborhood $B_1 \subseteq B$, $b \in B_1$, such that $B_1 \times \{c\} \subseteq Y$. We may assume that B_1 is given as the intersection of a short rectangular neighborhood V_0 and B. By shrinking V_0 if needed, we may assume that the set of parameters A defining V_0 is *scl*-independent from $\langle b, c\rangle$ (and contains short elements). Hence $\text{lgdim}(c/Ab) = \text{lgdim}(c/b)$ so c is still long-generic in C over Ab. By genericity, we can find an Ab-definable set $X \subseteq C$ such that $B_1 \times X \subseteq Y$. Because $c \in X$, the set X must be strongly long of the same (long) dimension as C. □

Fact 3.4. *Let $h : X \to W$ be a definable map, where* $\text{lgdim}X = \dim X > 0$ *and $W \subseteq M^m$ is short. Then there exists a definable set $Y \subseteq X$, with* $\text{lgdim}Y < \text{lgdim}X$ *such that h is locally constant on $X \setminus Y$.*

Proof. Without loss of generality, X, W and h are $\emptyset$-definable. Take x long-generic in X and let $w = h(x)$. Because $w \in W$, we have $\text{lgdim}(w/\emptyset) = 0$ and therefore x is still long-generic in X over w. It follows that there is a w-definable set $X_0 \subseteq X$, such that for every $x' \in X_0$, $h(x') = w$. The set X is strongly long, so x is also generic in X over w. Hence, the set X_0 contains a relative neighborhood of x in X, so h is locally constant at x. This is true for every long-generic element in X so the set of points at which h is not locally constant must have smaller long dimension than that of X. □

3.2. **A preliminary result about definably compact groups.** We assume that $\langle G, +\rangle$ is a definable abelian group. Recall that $X \subseteq G$ is *generic* if finitely many group translates of X cover G. Using terminology from [18], a definable set $X \subseteq G$ is called *G-linear* if for every $g, h \in X$ there is an open neighborhood U of 0 (here and below, we always refer to the group topology of G), such that $(g - X) \cap U = (h - X) \cap U$. Clearly, every open subset of a definable subgroup of G is a G-linear set. More generally, every group translate of such a set is also G-linear. As is shown in [18], if a G-linear subset contains 0 then it contains an infinitesimal subgroup of G. When the group G is $\langle M^n, +\rangle$ a G-linear subset is also called *affine*. We call a definable G-linear subset $X \subseteq G$ *a local subgroup of* G if it is definably connected and $0 \in X$.

The G-linear set $G_0 \subseteq G$ and the H-linear set $H_0 \subseteq H$ are *definably isomorphic* if there exists a definable bijection $\phi : G_0 \to H_0$ such that for every $g, h, k \in G_0$, $g - h + k \in G_0$ if and only if $\phi(g) - \phi(h) + \phi(k) \in H_0$, in which case we have $\phi(g - h + k) = \phi(g) - \phi(h) + \phi(k)$. *An isomorphism of local subgroups* $G_0 \subseteq G$ and $H_0 \subseteq H$, is further required to send 0_G to 0_H. If $\phi : G_0 \to H_0$ is an isomorphism of local subgroups then for all $g, k \in G_0$, if $g + k \in G_0$ then $\phi(g) + \phi(h) \in H_0$ and we have $\phi(g + h) = \phi(g) + \phi(h)$.

Our starting point is Proposition 5.4 from [10], which comes out of the analysis of definable sets in semi-bounded structures. Recall our notation $\mathcal{C} = B + \langle C\rangle$ from Section 3. Below we use $\oplus$ and $\ominus$ for group addition and subtraction in G and use $+$ and $-$ for the group operations in $\mathcal{M}$.

Fact 3.5. [10, Proposition 5.4] *Let $\langle G, \oplus\rangle$ be a definably compact abelian group of long dimension k. Then G contains a definable, generic, bounded k-long cone $\mathcal{C}$ on which the group topology of G agrees with the o-minimal topology. Furthermore, for every $a \in \mathcal{C}$ there exists an open neighborhood $V \subseteq G$ of a such that for all $x, y \in V \cap a + \langle C\rangle$,*

$$x \ominus a \oplus y = x - a + y. \tag{4}$$

Our goal is to prove:

Proposition 3.6. *Let $\langle G, \oplus\rangle$ be a definably compact, definably connected abelian group. Then there exists a definably connected, k-dimensional local subgroup $H \subseteq G$ and a definable short set $B \subseteq G$, $\dim(B) = \dim(G) - k$, satisfying:*

1. *(1) $\langle H, \oplus\rangle$ is definably isomorphic, as a local group, to $\langle H', +\rangle$, where $H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k) \subseteq M^k$, with each $e_i > 0$ tall in M. In particular, $\dim H = lgdim H = k$.*
2. *(2) The set $B \oplus H = \{b \oplus h : b \in B\ h \in H\}$ is generic in G.*

Proof. We fix a definably connected short set B and a k-long cone $\mathcal{C} = B + \langle C\rangle$ as in Fact 3.5.

For $b \in B$, let $\mathcal{C}_b$ be the fiber $b + \langle C\rangle$. Note that for every $x \in \mathcal{C}_b$, and a sufficiently small neighborhood V of x, we have $V \cap \mathcal{C}_b = V \cap x + \langle C\rangle$. Note also that each $\mathcal{C}_b$ is an affine subset of $\langle M^n, +\rangle$. Thus, condition (4) implies that each $\mathcal{C}_b$, locally near every $a \in \mathcal{C}_b$, is a G-linear subset of G, and furthermore the identity map is locally an isomorphism of $\langle \mathcal{C}_b, +\rangle$ and $\langle \mathcal{C}_b, \oplus\rangle$. Because the affine topology and the group topology agree on $\mathcal{C}$ (and because $\mathcal{C}$ is definably connected in M^n), each fiber $\mathcal{C}_b$ is definably connected with respect to the group topology. By [18, Lemma 2.4], each $\mathcal{C}_b$ is therefore a G-linear (not only locally) subset of G and the identity map is an isomorphism of the affine set $\langle \mathcal{C}_b, +\rangle$ and the G-linear set $\langle \mathcal{C}_b, \oplus\rangle$.

Let us summarize what we have so far: On one hand, the set $\mathcal{C} = B + \langle C\rangle$ is a generic set in G, which can be written as a disjoint union of affine sets $\bigcup_{b\in B} \mathcal{C}_b$. Furthermore, for each $a, b \in B$ the map

$$f_{a,b}(x) = x - a + b$$

is an isomorphism of the affine sets $\mathcal{C}_a$ and $\mathcal{C}_b$. On the other hand, each $\mathcal{C}_b$ is also a G-linear set, and the same maps $f_{a,b} : \mathcal{C}_a \to \mathcal{C}_b$ are isomorphisms of G-linear sets (because the identity is an isomorphism of $\langle \mathcal{C}_a, +\rangle$ and $\langle \mathcal{C}_a, \oplus\rangle$).

Our next goal is to show that, for many a, b in B, each map $f_{a,b}(x)$ is not only a translation in the sense of the group $\langle M^n, +\rangle$ but also a translation in $\langle G, \oplus\rangle$.

We define on B the following equivalence relation: $a \sim b$ if there exists $g \in G$ such that we have $f_{a,b}(x) = x \oplus g$ for all $x \in C_a$. Note that for every $a, b, c \in B$, we have $f_{b,d} \circ f_{a,b} = f_{a,d}$, so it is easy to check that $\sim$ is an equivalence relation.

Claim 3.7. *There are only finitely many $\sim$-equivalence classes in B.*

Proof. Assume towards contradiction that there are infinitely many classes. By definable choice, we can find an infinite definable set of representatives for $B/\sim$. We then replace B by a definably connected component of this set, calling it B again. So, we may assume that any two $a, b \in B$ are in distinct $\sim$-classes and that B is still infinite and definably connected. We fix some $a_0 \in B$ and consider the map $F : B \times C_{a_0} \to G$, given by $F(b, x) = f_{a_0,b}(x)$.

Since C_{a_0} is strongly long, we can find an open subset $B_1 \subseteq B$ and a strongly long set $X \subseteq C_{a_0}$, $\dim X = \dim C_{a_0}$, such that F is continuous on $B_1 \times X$ with respect to the group topology (Fact 3.3). Without loss of generality we can assume that X is definably compact (we first take a bounded X, then shrink it slightly, and take its topological closure).

Let us fix a G-open chart $V \subseteq G$ containing 0_G, and a homeomorphism with an open affine set $\phi : V \to V' \subseteq M^\ell$ ($\ell = dimG$). Without loss of generality $\phi(0_G) = 0 \in M^\ell$. By identifying V and V', we may assume that $V' \subseteq G$ is an open set with respect to both the affine and the G-topology.

By the definable compactness of X, for every neighborhood $W \subseteq M^\ell$ of 0, there is a neighborhood $B_2 \subseteq B_1$ of a_0, such that for all $b', b'' \in B_2$ and $x \in X$, we have $F(x, b') \ominus F(x, b'') \in W$. Indeed, if not then there are definable curves $x(t) \in X$, $b_1(t), b_2(t) \in B_1$, with $b_1(t)$, $b_2(t)$ tending to b and such that for all t,

$$F(x(t), b_1(t)) \ominus F(x(t), b_2(t)) \notin W.$$

Definable compactness of X implies that $x(t) \to x_0 \in X$, so by continuity we have $F(x_0, b) \ominus F(x_0, b) \notin W$, contradiction.

We now fix $W \subseteq M^\ell$ a short neighborhood of 0, and choose B_2 accordingly. If we take distinct b', b'' in B_2 then we obtain a map $h : X \to W$, defined by $h(x) = F(x, b') \ominus F(x, b'')$. Because X is strongly long, and W is short, the map h must be locally constant outside a subset of X of long dimension smaller than k (Fact 3.4). So, we have an open neighborhood $V'' \subseteq C_{a_0}$ and an element $g \in G$, such that for all $x \in V''$, $f_{a_0,b'}(x) \ominus f_{a_0,b''}(x) = g$.

We claim that for all $x \in C_{a_0}$, we have $f_{a_0,b'}(x) \ominus f_{a_0,b''}(x) = g$.

First take $x \in V''$ and choose any $y, z \in C_{a_0}$ which are sufficiently close to each other. Since C_{a_0} is a G-linear set, $x \ominus y \oplus z$ is still in C_{a_0} and still in V''. So we have

$$f_{a_0,b'}(x \ominus y \oplus z) = f_{a_0,b'}(x) \ominus f_{a_0,b'}(y) \oplus f_{a_0,b'}(z)$$

and

$$f_{a_0,b''}(x \ominus y \oplus z) = f_{a_0,b''}(x) \ominus f_{a_0,b''}(y) \oplus f_{a_0,b''}(z).$$

By subtracting the two equations (in G), we obtain

$$g = g \oplus (f_{a_0,b'}(z) \ominus f_{a,b''}(z)) \ominus (f_{a,b'}(y) \ominus f_{a,b''}(y)),$$

so

$$f_{a_0,b'}(z) \ominus f_{a,b''}(z) = f_{a,b'}(y) \ominus f_{a,b''}(y)$$

for all $y, z \in C_{a_0}$ which are sufficiently close to each other. This implies that the function $f_{a_0,b'} \ominus f_{a_0,b''}$ is locally constant on C_{a_0} so by definable connectedness, it must be constant on C_{a_0}. We therefore showed that $f_{a_0,b'} \ominus f_{a_0,b''} = g$, so in fact $b' \sim b''$ contradicting our assumption. Thus $\sim$ has only finitely many classes in B. □

We now return to the relation $\sim$ with its finitely many classes $B_1, \ldots, B_m$, and consider the partition of C into $\bigcup_{b \in B_i} C_b$, $i = 1, \ldots, m$. Note that for each $i = 1, \ldots, m$ and every $b', b'' \in B_i$, there exists $g \in G$ such that $x \mapsto x \oplus g$ is an isomorphism of the G-linear sets $C_{b'}$ and $C_{b''}$.

Since C was generic in G, one of these sets is also generic in G (here we use the definable compactness of G). So we assume from now on that for every $b_1, b_2 \in B$ there exists an element $g \in G$ such that $C_{b_1} = C_{b_2} \oplus g$.

Fix $b_0 \in B$ and for every $b \in B$ choose an element $g(b)$ in G such that $C_b = C_{b_0} \oplus g(b)$. If we let $B' = \{g(b) \oplus b_0 : b \in B\}$ and $H = C_{b_0} \ominus b_0$, then $C = B' \oplus H$.

Let's see that H is as required. Indeed, the map $x \mapsto x \oplus b_0$ is an isomorphism of the local subgroups $\langle H, \oplus \rangle$ and $\langle C_{b_0}, \oplus \rangle$. As we already pointed out, the identity map is an isomorphism of $\langle C_{b_0}, \oplus \rangle$ and $\langle C_{b_0}, + \rangle$. Finally, $y \mapsto y - b_0$ is an isomorphism of the affine sets $\langle C_{b_0}, + \rangle$ and $\langle \langle C \rangle, + \rangle$. The composition of these maps is an isomorphism of the local groups $\langle H, \oplus \rangle$ and

$$H' = \left\langle \left(-\frac{a_1}{2}, \frac{a_1}{2}\right) \times \cdots \times \left(-\frac{a_k}{2}, \frac{a_k}{2}\right), + \right\rangle$$

(it sends 0_G to 0). This ends the proof of Proposition 3.6. □

4. The universal cover of G

4.1. **Proof of Theorem 1.1.** We first prove the abelian case. We proceed with the same notation as in the previous seciton. Namely, $\langle G, \oplus \rangle$ is a definably connected, definably compact abelian group, and $H \subseteq G$ is the definable strongly long set from Proposition 3.6.

Let $f' : \langle H', + \rangle \to \langle H, \oplus \rangle$ be the acclaimed isomorphism of local groups. We let $\mathcal{H} = \langle H \rangle$ be the subgroup of G generated by H. Since H is a local abelian subgroup of G of dimension k, $\mathcal{H}$ is a locally definable abelian subgroup of G of dimension k (see [18, Lemma 2.18]). One can show that the universal cover of $\mathcal{H}$ is a locally definable subgroup $\widehat{\mathcal{H}}$ of $\langle M^k, + \rangle$. Indeed, let $\widehat{\mathcal{H}} = \langle H' \rangle$ be the subgroup of $\langle M^k, + \rangle$ generated by H'. Then we can extend f' to a map $f : \widehat{\mathcal{H}} \to \mathcal{H}$ with, for every $x_1, \ldots, x_l \in H'$,

$$f(x_1 + \cdots + x_l) = f'(x_1) \oplus f'(x_2) \oplus \cdots \oplus f'(x_l)$$

is a $\bigvee$-definable covering map for $\mathcal{H}$. (The fact that f is well-defined is provided by the same argument as for [13, Lemma 4.27]). Since $\widehat{\mathcal{H}}$ is divisible and torsion-free, it is the universal cover of $\mathcal{H}$.

We let $\mathcal{H}_0'$ be the subset of M^k that consists of all short elements (by this we mean all elements of M^k all of whose coordinates are short). By [19, Lemma 3.4], $\langle \mathcal{H}_0', +\rangle$ is a subgroup of $\langle M^k, +\rangle$ and moreover, it is a subset of H'. It follows that $\mathcal{H}_0 = f(\mathcal{H}_0')$ is a subgroup of $\mathcal{H}$ which is isomorphic to $\mathcal{H}_0'$ (note that by [19], $\mathcal{H}_0$ is a $\bigvee$-definable set, but not, in general, a definable one).

From now on, in order to simplify the notation, we will write $+$ for the group operation of G. In few cases we will also use $+$ for the usual operation on M^k, and this will be clear from the context.

We define $\mathcal{B} = \bigcup_{n\in\mathbb{N}} B(n)$, where B is the definable short set from Proposition 3.6, and the notation $B(n)$ is given in Section 1.7. Since each $B(n)$ is a short definable set, $\mathcal{B}$ is a short locally definable subgroup of G.

Claim 4.1. $\mathcal{H} + \mathcal{B} = G$.

Proof. By Proposition 3.6, the set $H + B$ is a generic subset of G and is contained in $\mathcal{H} + \mathcal{B}$ (we use here the fact that $B \subseteq \mathcal{B}$ since $0 \in B$). Since G is definably connected we have $\mathcal{H} + \mathcal{B} = G$. □

The following claim is crucial to the rest of the analysis.

Claim 4.2. *The group $\mathcal{H}_0 \cap \mathcal{B}$ is compatible in $\mathcal{B}$, so in particular locally definable.*

Proof. Let $X \subseteq \mathcal{B}$ be a definable set. The set $\mathcal{B}$ is a bounded union of short definable sets, so X is contained in one of these and must also be short. We prove that, in general, the intersection of any definable short $X \subseteq G$ with $\mathcal{H}_0$ is definable.

Since $\mathcal{H}_0 \subseteq H$ we may assume that X is a subset of H. Let us consider $X' = (f')^{-1}(X) \subseteq M^k$. Because f' is injective the set X' is a finite union of definably connected short subsets of M^k. It is easy to see that if one of these short sets contains a short element then every element of it is short. Thus, if one of these components intersects $\mathcal{H}_0'$ non-trivially then it must be entirely contained in $\mathcal{H}_0'$ (since $\mathcal{H}_0'$ is the collection of all short elements). Hence, $X' \cap \mathcal{H}_0'$ is a finite union of components of X' and therefore definable. Its image under f' is the definable set $X \cap \mathcal{H}_0$. □

Note: It is not true in general that $\mathcal{H} \cap \mathcal{B}$ is a compatible subgroup of $\mathcal{B}$ (see Example 6.1 below).

The decomposition of $\widehat{G}$ is done through a series of steps.

Step 1 By Claim 4.2 and Fact 2.2, the quotient $\mathcal{K} = \mathcal{B}/(\mathcal{H}_0 \cap \mathcal{B})$ is locally definable and hence we obtain the following short exact sequence of locally definable groups:

$$0 \longrightarrow \mathcal{H}_0 \cap \mathcal{B} \xrightarrow{i_0} \mathcal{B} \xrightarrow{\pi_{\mathcal{B}}} \mathcal{K} \longrightarrow 0 \tag{5}$$

Claim 4.3. $\dim H + \dim \mathcal{K} = \dim G$.

Proof. Because $\mathcal{H} + \mathcal{B} = G$, we have

$$\dim \mathcal{H} + \dim \mathcal{B} - \dim(\mathcal{H} \cap \mathcal{B}) = \dim G.$$

Indeed, this is true for definable groups, and can be proved similarly here by considering a sufficiently small neighborhood of 0 in the locally definable group $\mathcal{H} \cap \mathcal{B}$.

But $\mathcal{H}_0$ is open in $\mathcal{H}$ and therefore $\dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim(\mathcal{H} \cap \mathcal{B})$, so we also have $\dim \mathcal{H} + \dim \mathcal{B} - \dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim G$. Because $\mathcal{K} = \mathcal{B}/(\mathcal{H}_0 \cap \mathcal{B})$, we have $\dim \mathcal{B} - \dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim \mathcal{K}$. We can now conclude $\dim \mathcal{H} + \dim \mathcal{K} = \dim G$. □

Step 2. Since $\mathcal{H}_0 \cap \mathcal{B}$ embeds into $\mathcal{H}$ and $\mathcal{H}_0 \cap \mathcal{B}$ is a compatible subgroup of $\mathcal{B}$, we can apply Lemma 2.8 and obtain a locally definable group D (the pushout of $\mathcal{H}$ and $\mathcal{B}$ over $\mathcal{H}_0 \cap \mathcal{B}$) with the following diagram commuting

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_0 \cap \mathcal{B} & \xrightarrow{i_0} & \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{K} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle id_{\mathcal{K}}} & & \\ 0 & \longrightarrow & \mathcal{H} & \xrightarrow{j} & D & \xrightarrow{\pi_D} & \mathcal{K} & \longrightarrow & 0 \end{array} \tag{6}$$

The maps γ and j are injective. Note that since $\mathcal{H}$ and $\mathcal{B}$ are subgroups of G, we also have a commutative diagram (with all maps being inclusions)

$$\begin{array}{ccc} \mathcal{H}_0 \cap \mathcal{B} & \longrightarrow & \mathcal{B} \\ \downarrow & & \downarrow \\ \mathcal{H} & \longrightarrow & G \end{array} \tag{7}$$

It follows from the definition of pushouts that there exists a locally definable map $\phi : D \to G$ such that $\phi\gamma : \mathcal{B} \to G$ and $\phi j : \mathcal{H} \to G$ are the inclusion maps. The restriction of ϕ to $j(\mathcal{H})$ is therefore injective and furthermore, the set $\phi(D)$ contains $\mathcal{H} + \mathcal{B}$ and hence, by Claim 4.1, ϕ is surjective on G.

Step 3 Consider now the universal cover $f : \widehat{\mathcal{H}} \to \mathcal{H}$ where $\widehat{\mathcal{H}}$ is identified with an open subgroup of $\langle M^k, +\rangle$ as before. As we saw, the group $\widehat{\mathcal{H}}$ has a subgroup $\mathcal{H}_0'$ which is isomorphic via f to $\mathcal{H}_0$. Hence, there is a locally definable embedding $\beta : \mathcal{H}_0 \cap \mathcal{B} \to \widehat{\mathcal{H}}$ such that $f\beta = id_{\mathcal{H}_0 \cap \mathcal{B}}$. Our goal is to use this embedding in order to interpolate an exact sequence between the two sequences in (6) (see (10) below).

We let $\widehat{D}$ be the pushout of $\widehat{\mathcal{H}}$ and $\mathcal{B}$ over $\mathcal{H}_0 \cap \mathcal{B}$. Namely, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_0\cap\mathcal{B} & \xrightarrow{i_0} & \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{K} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma''} & & \downarrow{\scriptstyle id_{\mathcal{K}}} & & \\ 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\widehat{\delta}} & \widehat{D} & \xrightarrow{\pi_{\widehat{D}}} & \mathcal{K} & \longrightarrow & 0 \end{array} \tag{8}$$

Step 4 Next, we consider the diagram

$$\begin{array}{ccc} \mathcal{H}_0\cap\mathcal{B} & \xrightarrow{i_0} & \mathcal{B} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \widehat{\mathcal{H}} & \xrightarrow{jf} & D \end{array} \tag{9}$$

Since $f\beta = id$, it follows from (6) that the above diagram commutes. Since $\widehat{D}$ was a pushout, there exists a locally definable $\gamma' : \widehat{D} \to D$ such that $\gamma'\gamma'' = \gamma$ and $\gamma'\widehat{\delta} = jf$.

Putting the above together with (6) and (8), we obtain

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_0\cap\mathcal{B} & \xrightarrow{i_0} & \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{K} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma''} & & \downarrow{\scriptstyle id_{\mathcal{K}}} & & \\ 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\widehat{\delta}} & \widehat{D} & \xrightarrow{\pi_{\widehat{D}}} & \mathcal{K} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle\gamma'} & & \downarrow{\scriptstyle id_{\mathcal{K}}} & & \\ 0 & \longrightarrow & \mathcal{H} & \xrightarrow{j} & D & \xrightarrow{\pi_D} & \mathcal{K} & \longrightarrow & 0 \end{array} \tag{10}$$

Note that in order to conclude that the above diagram commutes, we still need to verify that the bottom right square commutes, namely, $(id_{\mathcal{K}})\pi_{\widehat{D}} = (\pi_D)\gamma'$.

We now apply Lemma 2.9 and conclude that the group D is the pushout of $\mathcal{H}$ and $\widehat{D}$ over $\widehat{\mathcal{H}}$. As a corollary we conclude, by Lemma 2.8 (and the fact that f is surjective),

$$(i)\ \pi_{\widehat{D}} = (\pi_D)\gamma' \ \ (ii)\ \ker(\gamma') = \widehat{\delta}(\ker f) \ \ (iii)\ \gamma' \text{ is surjective.} \tag{11}$$

In particular, (10) commutes.

If we now return to the surjective $\phi : D \to G$ and compose it with γ', we obtain a surjection $\phi\gamma' : \widehat{D} \to G$.

Let us summarize what we have so far:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\widehat{\delta}} & \widehat{D} & \xrightarrow{\pi_{\widehat{D}}} & \mathcal{K} & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \phi\gamma'} & & & & \\
 & & & & G & & & &
\end{array} \tag{12}
$$

Step 5 Let $\mu : \mathcal{U} \to \mathcal{K}$ be the universal cover of $\mathcal{K}$, (see [6, Theorem 3.11] for its existence and its local definability) and apply the pullback construction from Proposition 2.11 to $\mathcal{U}$, $\mathcal{K}$ and $\widehat{D}$.

We obtain a $\bigvee$-definable group $\widehat{G}$ (the pullback of $\mathcal{U}$ and $\widehat{D}$ over $\mathcal{K}$), with associated $\bigvee$-definable maps such that the following sequences are exact and commute (since the kernels of $\pi_{\widehat{G}}$ and $\pi_{\widehat{G}}$ are isomorphic we identify them both with $\widehat{\mathcal{H}}$ and assume that the map between them is the identity). By Proposition 2.11, we also have

$$\pi_{\widehat{G}}(\ker(\eta)) = \ker(\mu). \tag{13}$$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i} & \widehat{G} & \xrightarrow{\pi_{\widehat{G}}} & \mathcal{U} & \longrightarrow & 0 \\
 & & {\scriptstyle id}\Big\downarrow & & \Big\downarrow{\scriptstyle \eta} & & \Big\downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\widehat{\delta}} & \widehat{D} & \xrightarrow{\pi_{\widehat{D}}} & \mathcal{K} & \longrightarrow & 0
\end{array} \tag{14}
$$

Because μ is surjective, so is η, so we obtain a surjective homomorphism $\widehat{F} := \phi\gamma'\eta : \widehat{G} \to G$. It can be inferred from what we have so far that $\mathcal{H} = \widehat{F}(i(\mathcal{H}))$.

Note that $\dim \widehat{G} = \dim \mathcal{U} + \dim \widehat{\mathcal{H}}$ and, since $\mathcal{U}$ is the universal cover of $\mathcal{K}$, $\dim \mathcal{U} = \dim \mathcal{K}$. By Claim 4.3, we have $\dim \widehat{G} = \dim G$. Note also that $\mathcal{U}$ and $\widehat{\mathcal{H}}$ are divisible (as connected covers of divisible groups) and torsion-free and therefore so is $\widehat{G}$. It follows that $\widehat{F} : \widehat{G} \to G$ is isomorphic to the universal cover of G.

We therefore obtain

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i} & \widehat{G} & \xrightarrow{\pi_{\widehat{G}}} & \mathcal{U} & \longrightarrow & 0 \\
 & & & & \Big\downarrow{\scriptstyle \widehat{F}} & & & & \\
 & & & & G & & & &
\end{array} \tag{15}
$$

This ends the proof of the first part Theorem 1.1 for an abelian definably connected, definably compact G.

Assume now that G is an arbitrary definably compact, definably connected group. By [17, Corollary 6.4], the group G is the almost direct product of the definably connected groups $Z(G)^0$ and $[G, G]$, and $[G, G]$ is a semisimple group. The group G is then the homomorphic image of the direct sum $A \oplus S$ with A abelian, S semi-simple, both definably compact, and the kernel of this homomorphism is finite. We may therefore assume that $G = A \oplus S$. By [17, Theorem 4.4 (ii)], the group S is definably isomorphic to a semialgebraic group over a definable real closed field so it must be short. It follows that $\operatorname{lgdim}(G) = \operatorname{lgdim}(A)$. By the abelian case, we obtain the following for the universal cover $\widehat{A}$ of A.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \longrightarrow & \widehat{A} & \longrightarrow & \mathcal{U} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \widehat{F}} & & & & \\ & & & & A & & & & \end{array}$$

Next, we consider $p : \widehat{S} \to S$ the universal cover of S (note that $\widehat{S}$ is also a compact group). By taking the direct product we obtain:

(16)

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i} & \widehat{G} = \widehat{A} \oplus \widehat{S} & \xrightarrow{\pi} & \mathcal{U} \oplus \widehat{S} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle \hat{F}\cdot p} & & & & \\ & & & & G = A \oplus S & & & & \end{array}$$

In order to finish the proof of Theorem 1.1 we need to see:

Lemma 4.4. *The group $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$ contains every connected, $\bigvee$-definable strongly long subgroup of G.*

Proof. We first prove the analogous result for the universal cover $\widehat{G}$ of G, namely we prove that $i(\widehat{\mathcal{H}})$ contains every connected, locally definable, strongly long subgroup of $\widehat{G}$. For simplicity, we assume that $\widehat{\mathcal{H}} \subseteq \widehat{G}$.

Assume that $\mathcal{V} \subseteq \widehat{G}$ is a connected, $\bigvee$-definable subgroup with $\dim(\mathcal{V}) = \operatorname{lgdim}(\mathcal{V}) = \ell$. Because $\operatorname{lgdim}(\widehat{G}) = k$ we must have $\ell \leq k$. We will show that the group $\mathcal{V} \cap \widehat{\mathcal{H}}$ has bounded index in $\mathcal{V}$, so by connectedness the two must be equal.

Consider $\mathcal{U}$ from Theorem 1.1. Because $\mathcal{U}$ is short, there exists at least one $u \in \mathcal{U}$ such that $\operatorname{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$ (see [10, Lemma 4.2]). Since $\mathcal{V}$ is a group we can use translation in $\mathcal{V}$ to show that *for every* $u \in \pi(\mathcal{V})$, we must have $\operatorname{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$. In particular, $\operatorname{lgdim}(\widehat{\mathcal{H}} \cap \mathcal{V}) = \operatorname{lgdim}(\pi^{-1}(0) \cap \mathcal{V}) = \ell$.

Write $\mathcal{V} = \bigcup_i V_i$ a bounded union of definable sets which we may assume to be all strongly long of dimension ℓ. For every V_i, consider the definable projection $\pi(V_i) \subseteq \mathcal{U}$. By Lemma 9.1 (proved in the appendix), the set F_i of all $u \in \pi(V_i)$ such that $\operatorname{lgdim}(\pi^{-1}(u) \cap V_i) = \ell$ is definable, so because $\dim(V_i) = \ell$, this set must be finite.

Let $F = \bigcup_i F_i \subseteq \pi(\mathcal{V})$. We claim that $F = \pi(\mathcal{V})$. Indeed, if $u \in \pi(\mathcal{V}) \setminus F$ then by the definition of the F_i's, $\operatorname{lgdim}(\pi^{-1}(u) \cap V_i) < \ell$ for all i, which implies that $\operatorname{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) < \ell$. This is impossible by our above observation, so we must have $F = \pi(\mathcal{V})$.

Because F is a bounded union of finite sets it follows that the index of $\mathcal{V}\cap\widehat{\mathcal{H}}$ in $\mathcal{V}$ is bounded. Since $\mathcal{V}$ is connected it follows that $\mathcal{V}\cap\widehat{\mathcal{H}}=\mathcal{V}$, so $\mathcal{V}\subseteq\widehat{\mathcal{H}}$.

Assume now that $\mathcal{V}\subsetneqq G$ is a connected, locally definable, strongly long subgroup of G and let $\widehat{\mathcal{V}}\subseteq\widehat{G}$ be the pre-image of $\mathcal{V}$ under $\widehat{F}$. The group $\widehat{\mathcal{V}}$ is strongly long and locally definable, and the connected component of the identity (see [1, Proposition 1]), call it $\widehat{\mathcal{V}}^0$, is still strongly long (since it has the same dimension and long dimension as $\widehat{\mathcal{V}}$). By what we just saw, $\widehat{\mathcal{V}}^0$ is contained in $\widehat{\mathcal{H}}$ and hence $\widehat{F}(\mathcal{V}^0)$ is a $\bigvee$-definable subgroup of $\mathcal{H}\cap\mathcal{V}$, which has bounded index in $\mathcal{V}$. Because $\mathcal{V}$ is connected it follows $\widehat{F}(\mathcal{V}^0)=\mathcal{V}\subseteq\mathcal{H}$. □

This ends the proof of Theorem 1.1.

5. Replacing the locally definable group $\mathcal{U}$ with a definable group

We now proceed to prove Theorem 1.3. We first assume again that G is abelian. The goal is to replace the locally definable group $\mathcal{U}$ in (15) with *a definable* short group. We refer to the notation of (14) and (15).

Step 1 Let $\Lambda=\ker(\widehat{F})$ and let $\Lambda_1=\pi_{\widehat{G}}(\Lambda)\subseteq\mathcal{U}$.

Claim 5.1. *The universal cover $\mathcal{U}$ of $\mathcal{K}$ from (14), together with Λ_1, satisfy the assumptions of Fact 2.6. Namely, $\mathcal{U}$ is connected, generated by a definably compact set and there is a definable set $X\subseteq\mathcal{U}$ such that $X+\Lambda_1=\mathcal{U}$. Moreover, Λ_1 is finitely generated.*

Proof. The group $\widehat{G}$ is the universal cover of G. We first find a definable, definably connected, definably compact $X\subseteq\widehat{G}$ which contains the identity, such that $\widehat{F}(X)=G$. We start with a definable $X\subseteq\widehat{G}$ such that $\widehat{F}(X)=G$ and then replace it with $Cl(X)$. We claim that $Cl(X)$ is definably compact. Indeed, if not then by [5, Lemma 5.1 and Theorem 5.2], $\widehat{G}$ has a definable, 1-dimensional subgroup G_0 which is not definably compact. Because μ is locally definable, its restriction to G_0 is definable so $ker(\widehat{F})\cap G_0$ is finite and therefore trivial. Hence $\widehat{F}(G_0)$ is a definable subgroup of G that is not definably compact, contradicting the fact that G is definably compact. Thus, we can find a definably compact X' with $X'+\ker(\widehat{F})=\widehat{G}$. By [12, Fact 2.3(2)], X' generates $\widehat{G}$.

By [6, Claim 3.8], $\widehat{G}$ is path connected so we can easily replace X' by $X_1\supseteq X'$ which is definably compact and path connected (connect any two definably connected components of X' by a definable path). To simplify we call this new set X again.

Also, by [6, Theorem 1.4 and Corollary 1.5], $\ker(\widehat{F})$ is isomorphic to the fundamental group of G, $\pi_1^{def}(G)$, which is finitely generated. It follows that Λ_1 is finitely generated, $\mathcal{U}=\pi_{\widehat{G}}(X)+\Lambda_1$, and $\pi_{\widehat{G}}(X)$ is definably compact and definably connected. Since X generates $\widehat{G}$, the set $\pi_{\widehat{G}}(X)$ generates $\mathcal{U}$. By Remark 2.4, $\mathcal{U}$ is connected. □

We can now apply Fact 2.6 and conclude that there is a definably compact group $\overline{K}$ and a $\bigvee$-definable surjection $\widehat{\mu} : \mathcal{U} \to \overline{K}$ with $\ker(\widehat{\mu}) = \Lambda_0 \subseteq \Lambda_1$.

Our goal is to prove: *There are locally definable extensions $\overline{G}$ and G' of $\overline{K}$, by the group $\widehat{\mathcal{H}}$ and $\mathcal{H}$, respectively, and surjective homomorphisms from $\overline{G}$ and G' onto G.*

First, by Lemma 2.12, we have a locally definable group $\widehat{\mathcal{H}}' = \ker(\widehat{\mu}\pi_{\widehat{G}}) = \pi_{\widehat{G}}^{-1}(\Lambda_0) \subseteq \widehat{G}$ such that (we write i for the identity on $\widehat{\mathcal{H}}$ on the top left) the diagram commutes and the following sequences are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i} & \widehat{G} & \xrightarrow{\pi_{\widehat{G}}} & \mathcal{U} & \longrightarrow & 0 \\ && \downarrow{\scriptstyle i} && \downarrow{\scriptstyle id} && \downarrow{\scriptstyle \widehat{\mu}} && \\ 0 & \longrightarrow & \widehat{\mathcal{H}}' & \xrightarrow{id} & \widehat{G} & \xrightarrow{\widehat{\mu}\pi_{\widehat{G}}} & \overline{K} & \longrightarrow & 0 \end{array} \tag{17}$$

Because $ker(\widehat{\mu}) \subseteq \pi_{\widehat{G}}(\Lambda)$, the group $\widehat{\mathcal{H}}'$ is contained in the group $i(\widehat{\mathcal{H}})+\Lambda$. Since $i(\widehat{\mathcal{H}})$ is a divisible subgroup of $\widehat{\mathcal{H}}'$, there exists a subgroup $\Lambda' \subseteq \Lambda$ such that $\widehat{\mathcal{H}}'$ equals the direct sum of $i(\widehat{\mathcal{H}})$ and Λ'. Because $\ker(\pi_{\widehat{G}}) = i(\widehat{\mathcal{H}})$, the group Λ' is isomorphic, via $\pi_{\widehat{G}}$, to Λ_0, so Λ' is finitely generated. We now have a group homomorphism $p : \widehat{\mathcal{H}}' \to \widehat{\mathcal{H}}$, given via the identification of $\widehat{\mathcal{H}}'$ with $i(\widehat{\mathcal{H}}) \oplus \Lambda'$. Namely, $p(i(h) + \lambda) = h$.

We claim that p is a locally definable map. Indeed, $\widehat{\mathcal{H}}'$ is the union of sets of the form $i(H_i) + F_i$, where H_i is definable and F_i is a finite subset of Λ'. Because the sum of $\widehat{\mathcal{H}}$ and Λ' is direct, each element g of $i(H_i) + F_i$ has a unique representation as $g = i(h) + f$, for $h \in H_i$ and $f \in F_i$. Therefore the restriction of p to $i(H_i)+F_i$ is definable. It follows that p is locally definable.

Step 2. We apply Proposition 2.8 to the diagram

$$\begin{array}{ccc} \widehat{\mathcal{H}}' & \xrightarrow{id} & \widehat{G} \\ \downarrow{\scriptstyle p} && \\ \widehat{\mathcal{H}} && \end{array}$$

and obtain a locally definable pushout $\overline{G}$, such that the following diagram commutes and the sequences are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}}' & \xrightarrow{id} & \widehat{G} & \xrightarrow{\widehat{\mu}\pi_{\widehat{G}}} & \overline{K} & \longrightarrow & 0 \\ && \downarrow{\scriptstyle p} && \downarrow{\scriptstyle \widehat{\alpha}} && \downarrow{\scriptstyle id} && \\ 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i_1} & \overline{G} & \xrightarrow{\pi_{\overline{G}}} & \overline{K} & \longrightarrow & 0 \end{array}$$

Because p is surjective the map $\widehat{\alpha} : \widehat{G} \to \overline{G}$ is also surjective. Moreover, by Lemma 2.8, the kernel of $\widehat{\alpha}$ equals $\ker p = \Lambda'$ so is contained in $\Lambda = \ker(\widehat{F})$.

Step 3. We now have surjective maps $\widehat{F} : \widehat{G} \to G$ and $\widehat{\alpha} : \widehat{G} \to \overline{G}$, both $\bigvee$-definable with $\ker(\widehat{\alpha}) \subseteq \ker(\widehat{F})$. By Lemma 2.13 we have a $\bigvee$-definable surjective $\overline{F} : \overline{G} \to G$, with $\ker(\overline{F}) = \widehat{\alpha}(\ker(\widehat{F}))$. We therefore obtained the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i_1} & \overline{G} & \xrightarrow{\pi_{\overline{G}}} & \overline{K} & \longrightarrow & 0 \\ & & & & \Big\downarrow{\scriptstyle \overline{F}} & & & & \\ & & & & G & & & & \end{array} \tag{18}$$

Finally, let us calculate $\ker(\overline{F})$: Recall that Λ' is isomorphic to Λ_0 the kernel of the universal covering map $\widehat{\mu} : \mathcal{U} \to \overline{K}$. Because $\overline{K}$ is a short definably compact group, it follows from [8] that $\ker(\widehat{\mu}) = \pi_1^{def}(\overline{K}) = \mathbb{Z}^d$, where $\pi_1^{def}(\overline{K})$ is the o-minimal fundamental group of $\overline{K}$ and

$$d = \dim(\overline{K}) = \dim(\mathcal{U}) = \dim(G) - k,$$

for $k = \operatorname{lgdim}(G)$. The map $\widehat{F} : \widehat{G} \to G$ is the universal covering map of G and therefore, as shown in [6, Theorem 1.4, Corollary 1.5], $\ker(\widehat{F}) = \pi_1^{def}(G) = \mathbb{Z}^\ell$, for some ℓ. Furthermore, for every $m \in \mathbb{N}$, the group of m-torsion points $G[m]$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^\ell$. By [19, Theorem 7.6], $G[m] = (\mathbb{Z}/m\mathbb{Z})^{\dim(G)}$, hence we can conclude

$$\Lambda = \ker(\widehat{F}) = \pi_1^{def}(G) = \mathbb{Z}^{\dim G}.$$

We now have $\ker(\overline{F}) = \widehat{\alpha}(\Lambda) \simeq \Lambda/\Lambda'$, with $\Lambda \simeq \mathbb{Z}^{\dim(G)}$ and $\Lambda' \simeq \mathbb{Z}^{\dim(G)-k}$. Hence, $\ker(\overline{F})$ is isomorphic to the direct sum of $\mathbb{Z}^k$ and a finite group, as required.

Question Can $\overline{K}$ be chosen so that $\ker(\overline{F}) \simeq \mathbb{Z}^k$?

Next, consider $\mathcal{H} \subseteq G$ as in Theorem 1.1. We want to see that we can obtain a similar diagram to (18), with $\mathcal{H}$ instead of $\widehat{\mathcal{H}}$. For simplicity, assume that i_1 is the identity. First notice that by the last clause of Theorem 1.1, we must have $\overline{F}(\widehat{\mathcal{H}}) \subseteq \mathcal{H}$. However, using exactly the same proof as in Lemma 4.4, we can show that that $\overline{F}(\widehat{\mathcal{H}})$ is also the largest connected strongly long, locally definable, subgroup of G, hence it equals $\mathcal{H}$. We therefore have

$$\begin{array}{ccc} \widehat{\mathcal{H}} & \xrightarrow{i_1} & \overline{G} \\ {\scriptstyle \overline{F}\restriction\widehat{\mathcal{H}}}\Big\downarrow & & \\ \mathcal{H} & & \end{array}$$

We can now obtain G', the pushout of $\overline{G}$ and $\mathcal{H}$ over $\widehat{\mathcal{H}}$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i_1} & \overline{G} & \xrightarrow{\pi_{\overline{G}}} & \overline{K} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \overline{F}\restriction\widehat{\mathcal{H}}} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & \mathcal{H} & \xrightarrow{i'} & G' & \xrightarrow{\pi_{G'}} & \overline{K} & \longrightarrow & 0 \end{array}$$

Clearly, $\ker(\overline{F} \restriction \widehat{\mathcal{H}}) \subseteq \ker(\overline{F})$, so by Proposition 2.8, $\ker(\alpha') = i(ker\overline{F} \restriction H) \subseteq \ker \overline{F}$. By Lemma 2.13, we have a homomorphism from G' onto G as we want. We therefore have:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H} & \xrightarrow{i'} & G' & \xrightarrow{\pi_{\overline{G}}} & \overline{K} & \longrightarrow & 0 \\ & & & & \downarrow{\scriptstyle h'} & & & & \\ & & & & G & & & & \end{array} \tag{19}$$

This ends the switch from (18) to (19), and with that the proof of Theorem 1.3 in the case that G is abelian. In order to conclude the same result for arbitrary definably compact, definably connected G, we repeat the same arguments as in the last part of the proof of Theorem 1.1. □

5.1. **Special cases.** As was pointed out earlier, we use Fact 2.6 to guarantee that there is a definable group $\overline{K}$ and a $\bigvee$-definable surjection $\widehat{\mu} : \mathcal{U} \to \overline{K}$ with $\Lambda_0 := \ker(\widehat{\mu})$ a subgroup of $\pi_{\widehat{G}}(\ker \widehat{F})$ (see notation of Theorem 1.1). In certain simple cases we can see directly why such Λ_0 exists, without referring to Fact 2.6:

Assume G is abelian. Let $\mathcal{K}$ and $\mathcal{H}$ be as in Section 4.1. Namely, $\mathcal{K}$ is the group obtained as the quotient of the locally definable subgroup $\mathcal{B}$ of G by the compatible subgroup $\mathcal{H}_0 \cap \mathcal{B}$, and $\mathcal{H}$ is the largest locally definable, connected strongly long subgroup of G.

(1) **Assume that $\mathcal{K}$ is definable**.

In this case we take $\Lambda_0 = \ker(\mu)$, where $\mu : \mathcal{U} \to \mathcal{K}$. Obviously, $\mathcal{U}/\Lambda_0$ is definable, so we need only to see that $\Lambda_0 \subseteq \pi_{\widehat{G}}(\ker \widehat{F})$. Let $u \in \ker(\mu)$. By (13), $u = \pi_{\widehat{G}}(v)$, for some $v \in \ker(\eta)$. But then $\widehat{F}(v) = \phi\gamma'\eta(v) = 0$,so $U \in \pi_{\widehat{G}}(\ker \widehat{F})$.

(2) **Assume that $\mathcal{H}$ is definable**.

We denote by $\overline{K}$ the definable group $G/\mathcal{H}$. From Theorem 1.1 and its proof we obtain the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i} & \widehat{G} & \xrightarrow{\pi_{\widehat{G}}} & \mathcal{U} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \widehat{F}} & & & & \\ 0 & \longrightarrow & \mathcal{H} & \xrightarrow{id} & G & \xrightarrow{\pi_G} & \overline{K} & \longrightarrow & 0 \end{array}$$

But now there is a unique map $\mu : \mathcal{U} \to \overline{K}$ which makes the above diagram commute, and it is easy to verify by construction that $\ker(\mu) \subseteq \pi_{\widehat{G}}(\ker(\widehat{F}))$. We now take $\Lambda_0 = \ker(\mu)$.

6. Examples

In this section we provide examples that motivate the statements of Theorem 1.1 and 1.3. More specifically, we give examples of definably compact groups which cannot themselves be written as extensions of short (locally) definable groups by strongly long (locally) definable subgroups. This is what forces us to move our analysis to the level of universal covers.

In the following examples, we fix $\mathcal{M} = \langle M, +, <, 0, R\rangle$ to be an expansion of an ordered divisible abelian group by a real closed field R, whose domain is a bounded interval $(0, a) \subseteq M$. In particular, $\mathcal{M}$ is semi-bounded, o-minimal, and $(0, a)$ is short. Let also $b \in M$ be any tall positive element. In the first two examples, we define semi-linear groups which have the same domain $[0, a) \times [0, b)$ but different operations.

Example 6.1. Pick any $0 < v_1 < a$ such that a and v_1 are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, +\rangle$ generated by the vectors $\langle a, 0\rangle$ and $\langle v_1, b\rangle$, and let $G = \langle [0, a) \times [0, b), \star, 0\rangle$ be the group with

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

By [13, Claim 2.7(ii)], G is definable.

Let us see what the various groups of Theorems 1.1 and 1.3 are in this case.

We let $\widehat{G}$ be the subgroup of M^2 generated by $[0, a] \times [0, b]$. The group $\widehat{G}$ is torsion-free and it is easy to see that there is a locally definable covering map $\widehat{F} : \widehat{G} \to G$. Hence, $\widehat{G}$ is the universal cover of G. The group $\widehat{\mathcal{H}} = \{0\} \times \bigcup_n (-nb, nb)$, is a locally definable compatible subgroup of $\widehat{G}$ and the quotient $\widehat{G}/\widehat{\mathcal{H}}$ is isomorphic to the short group $\bigcup_n (-na, na)$.

We have $\text{lgdim}(\widehat{\mathcal{H}}) = \dim(\widehat{\mathcal{H}}) = 1$, so $\widehat{\mathcal{H}}$ is strongly long. As in the proof of Proposition 4.4, the group $\widehat{\mathcal{H}}$ is the largest strongly long, connected, locally definable subgroup of $\widehat{\mathcal{H}}$.

Now, we let $\mathcal{H} = \widehat{F}(\widehat{\mathcal{H}})$. This is the subgroup of G generated by the set $H = \{0\} \times [0, b)$ and we can describe it explicitly. Let $S \subseteq [0, a)$ be the set containing all elements of the form $n(a - v_1) \mod a$. By the choice of v_1, the set S has to be infinite. By the definition of the operation $\star$, it is easy to see that

$$\mathcal{H} = \bigcup_{s \in S} \{s\} \times [0, b),$$

which is not definable (so in particular not compatible in G). This shows the need in Theorem 1.1 to work with the universal cover of G rather than with G itself. Note that $\widehat{F}$ restricted to $\widehat{H}$ is an isomorphism onto $\mathcal{H}$.

In fact, G does not contain any infinite strongly long definable subgroup. Indeed, if it did, then its connected component should be contained in $\mathcal{H}$ and therefore the pre-image of this component under $\widehat{F} \restriction \widehat{\mathcal{H}}$ would be a proper definable subgroup of $\widehat{\mathcal{H}}$ and, thus, of $\langle M, + \rangle$, a contradiction.

Now consider the subgroup $K = \langle [0, a) \times \{0\}, \star, 0 \rangle$ of G and let $\widehat{K}$ be its universal cover. We can write

$$G = \mathcal{H} \star K.$$

Of course $\mathcal{H} \cap K$ is infinite, so this is not a direct sum. However, the universal cover $\widehat{G}$ of G is a direct sum

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K},$$

whereas, if we let

$$\overline{G} = \widehat{\mathcal{H}} \oplus K,$$

then we can define a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}(0, b)$.

We finally observe in this example that $\mathcal{H} \cap K = S$ is not a compatible subgroup of K, which indicates the need for passing to $\mathcal{H}_0$ in the proof of Theorem 1.1 (see Claim 4.2).

Example 6.2. Pick any $0 < u_2 < b$ such that u_2 and b are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, + \rangle$ which is generated by the two vectors $\langle a, u_2 \rangle$ and $\langle 0, b \rangle$, and let again $G = \langle [0, a) \times [0, b), \star, 0 \rangle$ be the group with

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

Here we observe that $H = \{0\} \times [0, b)$ itself is the largest strongly long locally definable subgroup of G and, hence, G is itself an extension of a short definable group by H. However, H does not have a definable complement in G; namely, G cannot be written as a direct sum of H with some definable subgroup of it. The proof of this goes back to [22]. See also [21].

The universal cover $\widehat{\mathcal{H}}$ of H is again the subgroup of M^2 generated by H. Let $\mathcal{K}$ be the subgroup of G generated by $K = [0, a) \times \{0\}$, and $\widehat{K}$ its universal cover. Then we can write

$$G = H \star \mathcal{K},$$

where again $H \cap \mathcal{K}$ is not finite, so this is not a direct sum. The universal cover $\widehat{G}$ of G is again a direct sum

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K}.$$

If we let $\overline{K} = \langle [0,a) \times \{0\}, \star_K, 0\rangle$ be the group with operation $\star_K = +$ mod a, then we can define a *suitable* extension $\overline{G}$ of $\overline{K}$ by $\widehat{\mathcal{H}}$

$$0 \longrightarrow \widehat{\mathcal{H}} \longrightarrow \overline{G} \longrightarrow \overline{K} \longrightarrow 0$$

and a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}(0,b)$.

We finally give an example for Theorems 1.1 and 1.3 of a definable group G which contains no infinite proper definable subgroup.

Example 6.3. Pick any $0 < v_1 < a$ such that a and v_1 are $\mathbb{Z}$-independent, and any $0 < u_2 < b$ such that u_2 and b are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, +\rangle$ which is generated by the vectors $\langle a, u_2\rangle$ and $\langle v_1, b\rangle$. We define the group G with domain

$$\big([0,a) \times [0, b-u_2)\big) \cup \big([v_1, a) \times [b-u_2, b)\big),$$

and group operation again

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

It is not too hard to verify that the above is indeed a definable group - this will appear in a subsequent paper ([11]).

In this case, G does not contain any infinite proper definable subgroup. This again originates in [22]. We let $\mathcal{H}$ the subgroup of G generated by $H = \{0\} \times [0, b-u_2)$, and $\widehat{\mathcal{H}}$ its universal cover. We also let $\mathcal{K}$ be the subgroup of G generated by $K = [0,a) \times \{0\}$, and $\widehat{K}$ its universal cover. Then we have:

$$G = \mathcal{H} \star \mathcal{K},$$

with $\mathcal{H} \cap \mathcal{K}$ infinite, and

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K}.$$

Finally, if we let $\overline{K} = \langle [0,a) \times \{0\}, \star_K, 0\rangle$ be the group with operation $\star_K = +$ mod a, then we can define a suitable extension $\overline{G}$ of $\overline{K}$ by $\widehat{\mathcal{H}}$

$$0 \longrightarrow \widehat{\mathcal{H}} \longrightarrow \overline{G} \longrightarrow \overline{K} \longrightarrow 0$$

and a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}(v_1, b)$.

7. Compact Domination

Let us first recall ([16, Section 7]) that for a definable, or $\bigvee$-definable group $\mathcal{U}$, we write $\mathcal{U}^{00}$ for the smallest, if such exists, type-definable subgroup of $\mathcal{U}$ of bounded index (in particular we require that $\mathcal{U}^{00}$ is contained in a definable subset of $\mathcal{U}$). Note that a type-definable subgroup $\mathcal{H}$ of $\mathcal{U}$ has bounded index if and only if there are no new cosets of $\mathcal{H}$ in $\mathcal{U}$ in elementary extensions of $\mathcal{M}$. A definable $X \subseteq \mathcal{U}$ is called *generic* if boundedly many translates of X cover $\mathcal{U}$. In [12, Theorems 2.9 and 3.9] we established conditions so that $\mathcal{U}^{00}$ and generic sets exist.

Let G be a definably connected, definably compact, abelian definable group and $\pi : G \to G/G^{00}$ the natural projection. We equip the compact

Lie group G/G^{00} with the Haar measure, denoted by $m(Z)$, and prove: for every definable $X \subseteq G$, the set of $h \in G/G^{00}$ for which $\pi^{-1}(h) \cap X \neq \emptyset$ and $\pi^{-1}(h) \cap (G \setminus X) \neq \emptyset$ has measure zero. As is pointed out in [16], it is sufficient to prove that

(20) for every definable $X \subseteq G$, if $\dim X < \dim G$, then $m(\pi X) = 0$.

We say then that G (and π) *satisfy Compact Domination*. When G is locally definable and G^{00} exists then G/G^{00} is locally compact (see [16, Lemma 7.5]) and so admits Haar measure as well. We still say that G satisfies compact domination if (20) holds.

We split the argument into two cases:

I. G is abelian.

Consider the universal covering map $\phi : \widehat{G} \to G$ and the commutative diagram in [12, Proposition 3.8]

(21)
$$\begin{array}{ccc} \widehat{G} & \xrightarrow{\ \phi\ } & G \\ \downarrow{\scriptstyle \pi_{\widehat{G}}} & & \downarrow{\scriptstyle \pi_G} \\ \widehat{G}/\widehat{G}^{00} & \xrightarrow{\ \phi'\ } & G/G^{00} \end{array} .$$

Using the fact that $\ker \phi$ has dimension zero and $ker\phi'$ is countable, it is not hard to see that G satisfies Compact Domination if and only if $\widehat{G}$ does. Our goal is then to prove (20) for the universal cover $\widehat{G}$.

Recall by Theorem 1.1 the sequence:

$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{\ i\ } \widehat{G} \xrightarrow{\ f\ } \mathcal{U} \longrightarrow 0$$

with $\widehat{\mathcal{H}}$ an open subgroup of $\langle M^k, + \rangle$, $\mathrm{lgdim}(\widehat{\mathcal{H}}) = k = \mathrm{lgdim}(\widehat{G})$ and $\mathcal{U}$ a short $\bigvee$-definable group of dimension n. Note that $\widehat{G}$ contains a definable generic set (any definable set which projects onto G), and hence so does $\mathcal{U}$. By [12, Theorem 3.9], $\mathcal{U}$ has a definable, definably compact quotient K, and the homomorphism from $\mathcal{U}$ onto K has kernel of dimension zero. By [15], the group K, with its map onto K/K^{00} satisfies Compact Domination, and therefore $\pi_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$ also satisfies Compact Domination.

We now consider $\widehat{\mathcal{H}}$ and first claim:

(22) $\widehat{\mathcal{H}}^{00}$ exists and contains the set of all short elements in M^k.

Indeed, recall from Section 3.2 that $\widehat{\mathcal{H}}$ is generated by a subset $H' \subseteq M^k$,

$$H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k),$$

with each $e_i > 0$ tall in M. We define, for each $n \in \mathbb{N}$, $H_i = \frac{1}{n} H'$, and claim that

$$\widehat{\mathcal{H}}^{00} = \bigcap_n H_n.$$

Indeed, $\bigcap_n H_n$ is a torsion-free subgroup of $\widehat{\mathcal{H}}$. Moreover, each H_n is generic in $\widehat{\mathcal{H}}$ because we have $\widehat{\mathcal{H}} = H_n + \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$. It follows that $\bigcap_n H_n$ has bounded index in $\widehat{\mathcal{H}}$, and thus [12, Proposition 3.6] gives $\widehat{\mathcal{H}}^{00} = \bigcap_n H_n$. Finally, since each e_i is tall, it is easy to verify that each short tuple in M^k must be contained in $\bigcap_n H_n$.

We now claim that $\widehat{G}^{00} \cap i(\widehat{\mathcal{H}}) = i(\widehat{\mathcal{H}}^{00})$. This follows from the fact that $\widehat{G}^{00} \cap i(\widehat{\mathcal{H}})$ has bounded index in $i(\widehat{\mathcal{H}})$ and it is torsion-free ([12, Proposition 3.6]). Next, we claim that $f(\widehat{G}^{00}) = \mathcal{U}^{00}$. Since $f(\widehat{G}^{00})$ has bounded index it must contain $\mathcal{U}^{00}$. Because $\widehat{G}^{00}$ is torsion-free and $\ker(f) = i(\widehat{\mathcal{H}}^{00}) = i(\widehat{\mathcal{H}}) \cap \widehat{G}^{00}$ is divisible ([12, Proposition 3.5]), it follows that $f(\widehat{G}^{00})$ is torsion-free so must equal $\mathcal{U}^{00}$. We therefore have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\ i\ } & \widehat{G} & \xrightarrow{\ f\ } & \mathcal{U} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \pi_{\widehat{\mathcal{H}}}} & & \downarrow{\scriptstyle \pi_{\widehat{G}}} & & \downarrow{\scriptstyle \pi_{\mathcal{U}}} & & \\ 0 & \longrightarrow & \widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00} & \xrightarrow{\ \widehat{i}\ } & \widehat{G}/\widehat{G}^{00} & \xrightarrow{\ \widehat{f}\ } & \mathcal{U}/\mathcal{U}^{00} & \longrightarrow & 0 \end{array} \tag{23}$$

As in the proof of [12, Proposition 3.8], the map $\widehat{f}$ is continuous.

Assume now that $X \subseteq \widehat{G}$ is a definable set of dimension smaller than $\dim \widehat{G}$. We want to show that $\pi_{\widehat{G}}(X)$ has measure 0. We are going to use several variations of Fubini's theorem so let us see that the setting is correct. By [12], the group $\widehat{G}/\widehat{G}^{00}$ is isomorphic to $\mathbb{R}^k \times \mathbb{R}^n$ and the bottom sequence in the above diagram is just

$$0 \longrightarrow \mathbb{R}^k \xrightarrow{\ \widehat{i}\ } \widehat{G}/\widehat{G}^{00} \xrightarrow{\ \widehat{f}\ } \mathbb{R}^n \longrightarrow 0 \tag{24}$$

The above sequence necessarily splits as a Lie group, so by Fubini, a set $Y \subseteq \widehat{G}/\widehat{G}^{00}$ has measure zero if and only if the set

$$\{u \in \mathbb{R}^n : m_{\mathbb{R}^k}(\widehat{f}^{-1}(u) \cap Y) > 0\}$$

has measure zero in $\mathbb{R}^n$. (By $m_{\mathbb{R}^k}(\widehat{f}^{-1}(u) \cap Y)$ we mean the measure after identifying $\mathbb{R}^k \times \{u\}$ with $\mathbb{R}^k$)

We are now ready to start the proof.

Case 1 $\dim f(X) < \dim \mathcal{U}$.

Here we use Compact Domination in expansions of real closed fields (see [15]), so by an earlier observation, $\mathcal{U}$ also satisfies it. Hence, we have $m(\pi_{\mathcal{U}}(f(X))) = 0$, and therefore, by the commutation of the above diagram and Fubini we must have $m(\pi_{\widehat{G}}(X)) = 0$.

Most of the work goes towards the proof of the second case. For simplicity, let us assume that $\widehat{\mathcal{H}} \subseteq \widehat{G}$.

Case 2 $\dim f(X) = \dim \mathcal{U}$.

We first establish two preliminary results.

Claim We may assume that $\operatorname{lgdim}(X) < k = \operatorname{lgdim}(\widehat{G})$.

Indeed, by Lemma 9.1, we can decompose $f(X)$ into two definable sets $Y_1 \cup Y_2$ such that for every $u \in Y_1$, we have $\operatorname{lgdim}(f^{-1}(u) \cap X) < k$ and for every $u \in Y_2$, $\operatorname{lgdim}(f^{-1}(u) \cap X) = k = \dim(f^{-1}(u))$. Because $\dim X < \dim \widehat{G}$ and $\dim f(X) = \dim \mathcal{U}$, the dimension of Y_2 must be smaller than $\dim \mathcal{U}$. By Case (1), we can ignore Y_2 and assume now that for every $u \in f(X)$, $\operatorname{lgdim}(f^{-1}(u) \cap X) < k$. Since $\mathcal{U}$ is short, it follows from [10] that $\operatorname{lgdim}(X) < k$.

In the rest of the argument we prove the more general statement:

Lemma 7.1. *If $X \subseteq \widehat{G}$ is definable and lgdim$(X) < k$ then the measure of $\pi_{\widehat{G}}(X)$ is zero.*

Proof. We first prove a result for the group $\widehat{\mathcal{H}}$. By [12, Proposition 3.8], the group $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k$.

Claim 7.2. *If $Y \subseteq \widehat{\mathcal{H}}$ is definable and* $\operatorname{lgdim}(Y) < k$ *then* $m(\pi_{\widehat{\mathcal{H}}}(Y)) = 0$.

Proof. Recall that $\widehat{\mathcal{H}}$ is a subgroup of $\langle M^k, + \rangle$ and that the set of all short elements of M^k is contained in $\widehat{\mathcal{H}}^{00}$. Hence, if B is any definably connected short set, then $\pi_{\widehat{\mathcal{H}}}(B) = \{b\}$ is a singleton.

The set Y is a finite union of m-long cones, with $m < k$, hence we may assume that Y is such a cone $C = B + \langle C \rangle$, where $\langle C \rangle = \left\{ \sum_{i=1}^{k} \lambda_i(t_i) : t_i \in I_i \right\}$, for long $I_i = (-a_i, a_i)$ and partial linear maps $\lambda_i : I_i \to M^k$. We have

$$\pi_{\widehat{\mathcal{H}}}(C) = b + \sum_{i=1}^{m} \pi_{\widehat{\mathcal{H}}}(\lambda_i(t_i)).$$

Because $\pi_{\widehat{\mathcal{H}}}$ is a homomorphism from $\langle \widehat{\mathcal{H}}, + \rangle$ onto $\langle \mathbb{R}^k, + \rangle$, it follows that for each $i = 1, \ldots, m$, $t_i \mapsto \pi_{\widehat{\mathcal{H}}}(\lambda_i(t_i))$ is a partial homomorphism from I_i into $\langle \mathbb{R}^k, + \rangle$. Hence, the image of the $\widehat{G}$-linear set $\{\lambda_i(t) : t \in I_i\}$ is a closed affine subset of $\mathbb{R}^k$ of dimension m. Since $m < k$ we have $m(\pi_{\widehat{\mathcal{H}}}(Y)) = m(\pi_{\widehat{\mathcal{H}}}(C)) = 0$. □

Claim 7.3. *There exists a definable set $U_0 \subseteq \mathcal{U}$ with $\mathcal{U}^{00} \subseteq U_0$, and a definable section $s : U_0 \to \widehat{G}$ (i.e. $fs(u) = u$ for every $u \in U_0$), such that* *(i) the function s is continuous with respect to the topologies induced by $\mathcal{U}$ and $\widehat{G}$ and (ii) $s(\mathcal{U}^{00}) \subseteq \widehat{G}^{00}$.*

Proof. Let $U_1 \subseteq \mathcal{U}$ be a definable generic set. By definable choice. there exists a definable partial section $s : U_1 \to \widehat{G}$, namely, $sf(u) = u$ for all $u \in U_1$. The map s is piecewise continuous (with respect to the τ-topologies of $\mathcal{U}$ and $\hat{G}$) and therefore U_1 has a definable, definably connected $U_0 \subseteq U_1$, still generic in $\widehat{G}$ such that s is continuous on U_0. Using Compact Domination for $\mathcal{U}$, it follows from [16, Claim 3, p.590] that the set U_0 contains a coset of $\mathcal{U}^{00}$ so we may assume after translation in U that U_0 contains $\mathcal{U}^{00}$ and $s : U_0 \to \widehat{G}$ is continuous. We may also assume that $s(0) = 0$.

It is left to see that $s(\mathcal{U}^{00})$ is contained in $\widehat{G}^{00}$. Consider the map $\sigma(x, y) = s(x - y) - (s(x) - s(y))$, a definable and continuous map from $U_0 \times U_0$ into $\widehat{\mathcal{H}}$. Because the group topology on $\widehat{\mathcal{H}}$ is the subspace topology of M^k and because $U_0 \times U_0$ is a short definably connected set its image under σ is a short, definably connected subset of $\widehat{\mathcal{H}}$ containing 0. As we pointed out earlier, it must therefore be contained in $\widehat{\mathcal{H}}^{00}$.

We consider the set $\widehat{G}_1 = s(\mathcal{U}^{00}) + \widehat{\mathcal{H}}^{00}$ and claim that $\widehat{G}_1 = \widehat{G}^{00}$. To see first that $\widehat{G}_1$ is a subgroup, we note that

$$(s(u_1) + h_1) - (s(u_2) + h_2) = s(u_1 - u_2) + (h_1 + h_2 - \sigma(u_1, u_2)).$$

When $u_1, u_2 \in \mathcal{U}^{00}$ we have $\sigma(u_1, u_2) \in \widehat{\mathcal{H}}^{00}$ and hence this sum is also in $\widehat{G}_1$. Because s is definable and both $\mathcal{U}^{00}$ and $\widehat{\mathcal{H}}^{00}$ are type-definable the group $\widehat{G}_1$ is also type-definable. Because $\mathcal{U}^{00}$ has bounded index in $\mathcal{U}$ and $\widehat{\mathcal{H}}^{00}$ has bounded index in $\widehat{\mathcal{H}}$ it follows that $\widehat{G}_1$ has bounded index in $\widehat{G}$. Since $\widehat{G}_1$ is torsion-fee it follows from [12, Proposition 3.6] that $\widehat{G}_1 = \widehat{G}^{00}$. In particular, $s(\mathcal{U}^{00})$ is contained in $\widehat{G}^{00}$. □

Our goal is to show that $m(\pi_{\widehat{G}}(X)) = 0$. By Fubini, it is sufficient to show that for every $u \in \mathcal{U}/\mathcal{U}^{00}$, the fiber $\pi_{\widehat{G}}(X) \cap \widehat{f}^{-1}(u)$ has zero measure in the sense of $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$. Namely, it is the translate in $\widehat{G}/\widehat{G}^{00}$ of a zero measure subset of $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$.

Claim Fix $u \in \mathcal{U}/\mathcal{U}^{00}$. Then there exists a a definable set $Y \subseteq \widehat{\mathcal{H}}$ with $\mathrm{lgdim}(Y) < k$, and an element $g \in \widehat{G}$ such that the fiber $\pi_{\widehat{G}}(X) \cap \widehat{f}^{-1}(u)$ is contained in the set

$$\pi_{\widehat{H}}(Y) + \pi_{\widehat{G}}(g).$$

Proof. Fix $\bar{u} \in \mathcal{U}$ such that $\pi_{\mathcal{U}}(\bar{u}) = u$. By translation in $\widehat{G}$ and in $\mathcal{U}$ we may assume that the domain of the partial section s which was defined above, call it still U_0, contains $\bar{u} + \mathcal{U}^{00}$. If we let $g = s(\bar{u})$ then $s(\bar{u} + \mathcal{U}^{00}) \subseteq g + \widehat{G}^{00}$.

Consider the definable map $x \mapsto x - s(f(x))$ from $X \cap f^{-1}(U_0)$ into $\widehat{\mathcal{H}}$ and let Y be its image. Because $\mathrm{lgdim}(X) < k$, we must also have $\mathrm{lgdim}(Y) < k$. We claim that this is the desired Y. Indeed, we assume that $\widehat{f}(\pi_{\widehat{G}}(x)) = u$ for some $x \in X$ and show that $\pi_{\widehat{G}}(x) \in \pi_{\widehat{H}}(Y) + \pi_{\widehat{G}}(g)$.

By the commuting diagram above, $f(x) \in \bar{u} + \mathcal{U}^{00} \subseteq U_0$ and therefore $x - s(f(x)) \in Y$. Since $s(\bar{u} + \mathcal{U}^{00})$ is contained in $s(\bar{u}) + \widehat{G}^{00}$, we also have

$s(f(x)) \in g + \widehat{G}^{00}$. We now have

$$\pi_{\widehat{G}}(x) = \pi_{\widehat{G}}(x - sf(x)) + \pi_{\widehat{G}}(sf(x)) \in \pi_{\widehat{H}}(Y) + \pi_{\widehat{G}}(g).$$

□

We can now complete the proof that $\pi_{\widehat{G}}(X)$ has measure 0. For every $u \in \mathcal{U}/\mathcal{U}^{00}$ we find a definable $Y \subseteq \widehat{\mathcal{H}}$ as above. By Claim 7.2, the set $\pi_{\widehat{\mathcal{H}}}(Y)$ has measure 0 in $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$, hence the fiber $\pi_{\widehat{G}}(X) \cap \widehat{f}^{-1}(u)$ is a translate of a measure zero subset of $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$. By Fubini the measure of $\pi_{\widehat{G}}(X)$ is zero.

This ends the proof of Lemma 7.1 and with it that of Compact Domination for abelian G.

II. The general case (G not necessarily abelian).

Assume now that G is an arbitrary definably compact group. By [17], G is the almost direct product of a definably connected abelian group G_0 and a definable semi-simple group S. It is enough to prove the result for a finite cover of G hence we may assume that $G = G_0 \times S$. By [17, Theorem 4.4 (ii)], the group S is definably isomorphic to a semialgebraic group over a definable real closed field so it must be short, and therefore $\text{lgdim} G = \text{lgdim} G_0 = k$. To simplify the diagram, we use $\overline{G_0} = G_0/G_0^{00}$, $\overline{S} = S/S^{00}$, so we have $G/G^{00} = \overline{G_0} \times \overline{S}$.

We have

$$\begin{array}{ccccccccc}
0 & \longrightarrow & G_0 & \xrightarrow{i} & G_0 \times S & \xrightarrow{f} & S & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \pi_{G_0}} & & \downarrow{\scriptstyle \pi_G} & & \downarrow{\scriptstyle \pi_S} & & \\
0 & \longrightarrow & \overline{G_0} & \xrightarrow{\widehat{i}} & \overline{G_0} \times \overline{S} & \xrightarrow{\widehat{f}} & \overline{S} & \longrightarrow & 0
\end{array}$$

Assume now that $X \subseteq G$ is a definable set and $\dim(X) < \dim(G)$. If $\dim(f(X)) < \dim(S)$ then by Compact Domination in expansions of fields, the Haar measure of $\pi_S(f(X))$ in $\overline{S}$ is 0 and therefore $m(\pi_G(X))$ in G/G^{00} is 0.

If $\dim(X) = \dim(S)$ then, as in the abelian case, we may assume, after partition, that for every $s \in S$, $\text{lgdim}(f^{-1}(s) \cap X) < k$. Because S is short, it follows that $\text{lgdim}(X) < k$ and therefore the projection of X into G_0, call it X', has long dimension smaller than k. But now, by Lemma 7.1, the Haar measure in $\overline{G_0}$ of $\pi_{G_0}(X')$ equals to 0. By Fubini, the Haar measure of $\pi_G(X)$ must also be zero.

This ends the proof of Compact Domination for definably compact groups in o-minimal expansions of ordered groups. □

8. Appendix A - pullback and pushout

8.1. **Pushout.**

Proof of Proposition 2.8. We start with

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \\ C & & \end{array}$$

and prove the existence of the pushout D. We first review the standard construction of D (without verifying the algebraic facts). We consider the direct product $B \times C$ and take $D = (B \times C)/H$ where H is the subgroup $H = \{(\alpha(a), -\beta(a)) : a \in A\}$. If we denote by $[b, c]$ the coset of (b, c) mod H then the maps γ, δ are defined by $\gamma(b) = [b, 0]$ and $\delta(c) = [0, c]$. Assume now that we also have

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma'} \\ C & \xrightarrow{\delta'} & D' \end{array}$$

We define $\phi : D \to D'$ by $\phi([b, c]) = \gamma'(b) + \delta'(c)$. Clearly, if all data are definable then so are $B \times C$ and H, and therefore, using definable choice, D and the associated maps are definable.

If α is injective then δ is also injective, and if β is surjective then so is γ (see observation (b) on p. 53 in [14])

Suppose that A, B, C and α, β are $\bigvee$-definable and that $\alpha(A)$ is compatible subgroup of B. Clearly $B \times C$ is $\bigvee$-definable and it is easy to see that H is a $\bigvee$-definable subgroup. We want to show that H is a compatible subgroup of $B \times C$. For that we write $A = \bigcup A_i$, $B = \bigcup B_j$ and $C = \bigcup C_k$. It follows that $B \times C = \bigcup_{j,k} B_j \times C_k$. To show compatibility of H it is enough to show that for every j, k, the intersection $(B_j \times C_k) \cap H$ is definable. Because $\alpha(A)$ is compatible in B, the set $B_j \cap \alpha(A)$ is definable. Hence, there is some i_0 such that $\alpha(A_{i_0}) \supseteq B_j \cap \alpha(A)$. Moreover, because α is injective $\alpha^{-1}(B_j) \subset A_{i_0}$. It follows that the intersection $H \cap (B_j \times C_k)$ equals

$$\{(\alpha(a), -\beta(a)) \in B_j \times C_k : a \in A\} = \{(\alpha(a), -\beta(a)) \in B_j \times C_k : a \in A_{i_0}\}.$$

The set on the right is clearly definable, hence H is a compatible subgroup of $B \times C$, so $D = (B \times C)/H$ is $\bigvee$-definable (see Fact 2.2). It is now easy to check that $\gamma : B \to D$ and $\delta : C \to D$ are $\bigvee$-definable.

If $E = B/\alpha(A)$ then, by the compatibility of $\alpha(A)$, we see that E is $\bigvee$-definable. If $\pi : B \to E$ is the projection then we define $\pi' : D \to E$ by $\pi'([b, c]) = \pi(b)$. It is routine to verify that π' is a well-defined surjective homomorphism whose kernel is $\delta(C)$. It follows, using Fact 2.2, that $\delta(C)$ is a compatible subgroup of D. Finally, it is routine to verify commutation of all maps. $\square$

Proof of Lemma 2.9. We have

$$\begin{array}{ccccc} B & \xrightarrow{\gamma} & D & \xrightarrow{\mu} & F \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\xi} \\ A & \xrightarrow{\beta} & C & \xrightarrow{\eta} & E \end{array} \tag{25}$$

with D the pushout of B and C over A and F the pushout of B and E over A and we want to see that F is also the pushout of D and E over C.

It is sufficient to show that for every given commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\mu'} & F' \\ \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\xi'} \\ C & \xrightarrow{\eta} & E \end{array} \tag{26}$$

there is a map $\phi' : F \to F'$ such that $\phi'\mu = \mu'$ and $\phi'\xi = \xi'$ (according to the definition we also need to prove uniqueness but this follows).

By commutativity we have $\mu'\delta = \xi'\eta$ and hence $\mu'\delta\beta = \xi'\eta\beta$. Since $\delta\beta = \gamma\alpha$ we also have $(\mu'\gamma)\alpha = (\xi')\eta\beta$. We now use the fact that F is the pushout of B and E over A and conclude that there is $\phi' : F \to F'$ such that

$$(i)\phi'\xi = \xi' \quad \text{and} \quad (ii)\phi'\mu\gamma = \mu'\gamma \tag{27}$$

(i) gives half of what we need to show so it is left to see that $\phi'\mu = \mu'$.

Consider the commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\mu'\gamma} & F' \\ {\scriptstyle\alpha}\uparrow & & \uparrow{\scriptstyle\xi'\eta} \\ A & \xrightarrow{\beta} & C \end{array} \tag{28}$$

Because D is the pushout of B and C over A, there is a unique map $\psi : D \to F'$ with the property

$$(i)\psi\delta = \xi'\eta \quad \text{and} \quad (ii)\psi\gamma = \mu'\gamma.$$

If we can show that both maps μ' and $\phi'\mu$ from D into F' satisfy these properties of ψ then by uniqueness we will get their equality. For $\psi = \mu'$, (i) is part of the assumptions, and (ii) is obvious. For $\psi = \phi'\mu$, we obtain (ii) directly from (27)(ii). To see (i), start from (27)(i), $\phi'\xi = \xi'$, and conclude $\phi'\xi\eta = \xi'\eta$. By commutation, $\xi\eta = \mu\delta$ so we obtain $\phi'\mu\delta = \xi'\eta$, as needed. We therefore conclude that $\mu' = \phi'\mu$ and hence F is the pushout of E and D over C. $\square$

8.2. **Pullback.**

Proof of Proposition 2.11. Consider the diagram

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

We again review the algebraic construction of a pullback (which is simpler because we take no quotients). We let

$$D = \{(b, c) \in B \times C : \alpha(b) = \beta(c)\},$$

and the maps are just $\gamma(b, c) = b$ and $\delta(b, c) = c$. Given

$$\begin{array}{ccc} D' & \xrightarrow{\gamma'} & B \\ {\scriptstyle \delta'}\downarrow & & \downarrow{\scriptstyle \alpha} \\ C & \xrightarrow[\beta]{} & A \end{array}$$

we define $\phi(d') = (\gamma'(d'), \delta'(d')) \in D$.

Clearly, if all data are definable then so is D and the associated maps. Similarly, if all data are $\bigvee$-definable then so are D and the associated maps.

If $G = \ker(\gamma)$ then

$$G = \{(b, c) \in D : b = 0\} = \{(0, c) \in B \times C : \beta(c) = 0\},$$

and then clearly $j(0, c) = c$ is an isomorphism of G and $H = \ker(\beta)$. If all given data are $\bigvee$-definable then so are G, H and the associated maps. Furthermore, since G and H are kernels of $\bigvee$-definable maps they are clearly compatible in D, C, respectively.

If β is surjective then so is γ and the sequences in the diagram are exact (and the diagram is commutative). □

9. Appendix B - Short and long set

We assume that $\mathcal{M}$ is an o-minimal semi-bounded expansion of an ordered group

Lemma 9.1. *Let $S \subseteq M^r$ be a definable short set and let $A \subseteq S \times M^n$ be a definable set. For $s \in S$, we let $A_s = \{x \in M^n : (s, x) \in A\}$. Then, for every $\ell \geq 0$, the set $\ell(A) = \{s \in S : \operatorname{lgdim}(A_s) = \ell\}$ is definable.*

Proof. By [10], the set A can be written as a union of long cones $\bigcup C_i$. Since $\operatorname{lgdim}(X_1 \cup \cdots \cup X_m) = max_i(\operatorname{lgdim}(X_i))$, we may assume that A itself is a long cone $A = B + \sum_{i=1}^{k} \lambda_i(t_i)$, where $B \subseteq M^{r+n}$ is a short cell, $\lambda_1, \ldots, \lambda_k$ are M-independent partial linear maps $\lambda_i : I_i \to M^{r+n}$ and $I_i = (0, a_i)$ are long intervals. We write $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^{r+n})$, for $i = 1, \ldots, k$, so each λ_i^j is a partial endomorphism from I_i into M.

We claim that for every $s \in S$, $\operatorname{lgdim}(A_s) = k$. This clearly implies what we need.

For $b = (b_1, \ldots, b_{r+n}) \in B$, $i = 1, \ldots, k$ and $t_i \in I_i$, we have $b_i + \lambda_i(t_i) : I_i \to A$. Therefore, we have $(b_1, \ldots, b_r) + (\lambda_i^1(t_i), \ldots, \lambda_i^r(t_i)) \in S$. Each λ_i^j is either injective or constantly 0 and hence, because S is short and each I_i is long, for each $j = 1, \ldots, r$ and $i = 1, \ldots, k$, we have $\lambda_i^j \equiv 0$. It follows that for every $b \in B$, we have $(b_1, \ldots, b_r) \in S$.

For $i = 1 \ldots, k$, we let

$$\hat{\lambda}_i = (\lambda_i^{r+1}, \ldots, \lambda_i^{r+n}) : I_i \to M^n.$$

Because $\lambda_1, \ldots, \lambda_k$ were M-independent, it is still true that $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$ are M-independent. We now have, for every $s \in S$,

$$A_s = \left\{ b + \sum_{i=1}^{k} \hat{\lambda}_i(t_i) : b \in B_s, t \in I_i \right\}$$

and therefore the set A_s is a k-long cone, so $\operatorname{lgdim}(A_s) = k$. □

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Part 4

Lattices in locally definable subgroups of $\langle R^n, + \rangle$

LATTICES IN LOCALLY DEFINABLE SUBGROUPS OF $\langle R^n, +\rangle$

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Abstract. Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R. We define the notion of a lattice in a locally definable group and then prove that every connected, definably generated subgroup of $\langle R^n, +\rangle$ contains a definable generic set and therefore admits a lattice.

The goal of this note is to re-formulate some problems which appeared in [4], introduce the notion of a lattice in a locally definable group (a notion which also appeared in that paper, but not under this name) and establish connections between various related concepts. Finally, we return to the main conjecture from [4]:

Every locally definable connected, abelian group, which is generated by a definable set contains a definable generic set.

We prove the conjecture for subgroups of $\langle R^n, +\rangle$, in the context of an o-minimal expansion $\mathcal{M}$ of a real closed field R.

1. Locally definable groups and lattices

We first recall some definitions: Let $\mathcal{M}$ be an arbitrary κ-saturated o-minimal structure (for κ sufficiently large). By a *locally definable* group we mean a group $\langle \mathcal{U}, \cdot\rangle$, whose universe $\mathcal{U} = \bigcup_{n\in\mathbb{N}} X_n$, is a countable union of definable subsets of M^k, for some fixed k, and the group operation is definable when restricted to each $X_m \times X_n$ (equivalently, to each definable subset of $\mathcal{U}\times\mathcal{U}$). We say that a function $f : \mathcal{U} \to M^n$ is *locally definable* if its restriction to each X_i (equivalently, to each definable subset of $\mathcal{U}$) is definable. We let $\dim\mathcal{U}$ be the maximum of $\dim X_n$, $n \in \mathbb{N}$. While some notions treated here make sense under the more general "$\bigvee$-definable group" (no restriction on the number of X_i's), we mostly work in the context of a group which is generated, as a group, by a definable subset and hence it is locally definable. Note that another related concept, that of an *ind-definable group* (see [6]) is identical to our definition when one further assumes that the group is a subset of a fixed M^k.

As was shown in [7], every locally definable group admits a group topology. This topology agrees with the M^k-topology in neighborhoods of generic points, namely, points $g \in \mathcal{U}$ such that $\dim(g/A) = \dim(\mathcal{U})$ (we assume here that all the X_i's above are defined over A). We therefore obtain a definable family of neighborhoods $\{U_t : t \in T\}$ of the identity element, such that $\{gU_t : t \in T, g \in \mathcal{U}\}$ is a basis for the group topology on $\mathcal{U}$. In [2] it was further shown that the topology can be realized by countably many

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definable open charts, each definably homeomorphic to an open subset of M^n, where $n = \dim(\mathcal{U})$.

A subset $X \subseteq \mathcal{U}$ is called *compatible* (see [3]) if for every $Y \subseteq \mathcal{U}$ which is definable, the set $X \cap Y$ is also definable. It easily follows that X itself is also locally definable (namely, given as a countable union of definable subsets of $\mathcal{U}$). As was shown in [3], if $\mathcal{U}$ is locally definable and $\mathcal{H}$ is a normal compatible subgroup of $\mathcal{U}$ then there is a locally definable group $\mathcal{K}$ and a locally definable surjective homomorphism $f : \mathcal{U} \to \mathcal{K}$ whose kernel is $\mathcal{H}$. The converse is true as well, namely if such a homomorphism exists then $\mathcal{H}$ is necessarily compatible.

A locally definable group is called *connected* (see [1]) if it has no compatible subset which is both closed and open, with respect to the group topology. As is shown in [2, Remark 4.3], a locally definable group $\mathcal{U}$ is connected if and only if it is path connected, namely for any two points $x, y \in \mathcal{U}$ there exists a *definable* continuous $\sigma : [0,1] \to \mathcal{U}$ such that $\sigma(0) = x$ and $\sigma(1) = y$.

A typical example of a locally definable group is obtained by taking a definable subset of a definable group (say, of $\langle R^n, +\rangle$) and letting $\mathcal{U}$ be the subgroup generated by X. When the generating set is definably connected and contains the identity one obtains a connected locally definable group. We call a locally definable group $\mathcal{U}$ *definably generated* if it is generated, as a group, by some definable subset.

Definition 1.1. *For $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup, we say that* the quotient $\mathcal{U}/\mathcal{H}$ is definable *if there exists a definable group G and a locally definable surjective homomorphism from $\mathcal{U}$ onto G, whose kernel is $\mathcal{H}$.*

Definition 1.2. *A locally definable normal subgroup $\Lambda \subseteq \mathcal{U}$ is called* a lattice in $\mathcal{U}$ *if* $\dim(\Lambda) = 0$ *and $\mathcal{U}/\Lambda$ is definable.*

Notice that any countable group can be realized as a locally definable group, and therefore it is also a lattice in itself.

If $\mathcal{U}$ is the subgroup of R^n generated by the unit n-cube $[-1,1]^n$ then $\mathbb{Z}^n$ is a lattice in $\mathcal{U}$. The quotient is definably isomorphic to the group H^n, where $H = [0,1)$, with addition modulo 1.

In [4, Lemma 2.1] we prove the following equivalence:

Lemma 1.3. *Let $\mathcal{U}$ be a locally definable group in an o-minimal expansion of an ordered group and Λ a locally definable normal subgroup of dimension 0. The following are equivalent.*

(1) Λ is a lattice in $\mathcal{U}$.
(2) Λ is compatible, and there exists a definable set $X \subseteq G$ such that $\Lambda \cdot X = \mathcal{U}$.

It is easy to see that every lattice in a locally definable group is countable (the intersection with every definable set is finite). We prove a stronger statement:

Lemma 1.4. *If Λ is a lattice in a locally definable connected group $\mathcal{U}$ then Λ is finitely generated as a group.*

Proof. Let $\phi : \mathcal{U} \to G$ be a locally definable surjective homomorphism onto a definable group G, with $ker\phi = \Lambda$. By compactness there exists a definable set $X \subseteq \mathcal{U}$ such that $\phi(X) = G$. Because we have definable choice for subsets of $\mathcal{U}$ ([3, Corollary 8.1]) we can find a definable section $s : G \to X$ (i.e. $\phi \circ s = id$), and so we replace X by the image of this section, and call it X again. We may assume that $e \in X$.

Consider the topological closure (with respect to the group topology), $Cl(X) \subseteq \mathcal{U}$.

Claim There exists a finite set $F \subseteq \Lambda$ such that for every $g \in \Lambda$, if the intersection $gCl(X) \cap Cl(X) \neq \emptyset$ then $g \in F$.

Proof of Claim. Let $X' \subseteq \mathcal{U}$ be any definable open set containing $Cl(X)$. By saturation, there is a finite $F \subseteq \Lambda$, which we may assume is minimal, such that $X' \subseteq F \cdot X$. Because $gX \cap hX = \emptyset$ for every $g \neq h \in \Lambda$, if $gX \cap X' \neq \emptyset$ then necessarily $g \in F$. Now, if $gCl(X) \cap Cl(X) \neq \emptyset$ then necessarily $gX \cap X' \neq \emptyset$ so $g \in F$. □

We now claim that F generates Λ, namely every element of Λ is a finite word in F and F^{-1}.

Take $\lambda \in \Lambda$. Since $\mathcal{U}$ is path connected, there exists a definable path $\gamma : [0,1] \to \mathcal{U}$, with $\gamma(0) = e$ and $\gamma(1) = \lambda$. Let $\Gamma \subseteq \mathcal{U}$ be the image of γ. Because Γ is definable it can be covered by finitely many Λ-translates of X. By taking a minimal number of translates, we obtain $\lambda_1, \ldots, \lambda_k \in \Lambda$ (possibly with repetitions), such that $e \in \lambda_1 X$, $\lambda \in \lambda_k X$ and for $i = 1, \ldots, k-1$, we have $Cl(\lambda_i X) \cap Cl(\lambda_{i+1} X) \neq \emptyset$.

By the Claim, it follows that $\lambda_{i+1}^{-1}\lambda_i \in F$, for $i = 1, \ldots, k-1$. But since $e \in X$, we must have $\lambda_1 = e$ and $\lambda_k = \lambda$, so $\lambda_1, \ldots, \lambda_k$ are all in the group generated by F, and in particular, λ belongs to that group. □

We say that $\mathcal{U}$ *admits a lattice* if there is a lattice in $\mathcal{U}$. Note that not every locally definable group admits a lattice. For example, if $r \in R$ is larger than all elements of $\mathbb{N}$ then the subgroup of $\langle R, +\rangle$ given by $\bigcup[-r^n, r^n]$ does not admit any lattice.

As we point out in [4], there are many consequences, for a given group $\mathcal{U}$, to the fact that it admits a lattice. Hence, our main question is:

Question 1 Which locally definable groups in $\mathcal{M}$ admit a lattice?

We start with some basic observations.

Definition 1.5. *A definable subset X of a locally definable group $\mathcal{U}$ is called* left generic in $\mathcal{U}$ *if there exists a bounded set $\Delta \subseteq \mathcal{U}$ (namely, $|\Delta| < \kappa$) such that $\mathcal{U} = \Delta \cdot X$. Equivalently, for every definable $Y \subseteq \mathcal{U}$ there is a finite set $F \subseteq \mathcal{U}$ such that $Y \subseteq F \cdot X$.*

Lemma 1.3 immediately gives:

Lemma 1.6. *If a locally definable group $\mathcal{U}$ admits a lattice then $\mathcal{U}$ contains a definable left generic set.*

Lemma 1.7. *Let $\mathcal{U}$ be a connected locally definable group which contains a left generic definable set X (e.g. if $\mathcal{U}$ admits a lattice). Then $\mathcal{U}$ is definably generated.*

Proof. Let $X \subseteq \mathcal{U}$ be a definable, left generic set, namely there is a bounded set $\Delta \subseteq \mathcal{U}$ such that $\Delta \cdot X = \mathcal{U}$. The group generated by X, call it $\mathcal{H}$, is therefore locally definable, of bounded index in $\mathcal{U}$ (since $\langle\Delta\rangle \cdot \mathcal{H} = \mathcal{U}$, where $\langle\Delta\rangle$ is the group generated by Δ). But then, if $Y \subseteq \mathcal{U}$ is a definable set then $Y \cap \mathcal{H}$ and $Y \cap (\mathcal{U} \setminus \mathcal{H})$ are both bounded unions of definable sets. By saturation, this forces $Y \cap \mathcal{H}$ to be definable, hence $\mathcal{H}$ is compatible. It is easy to see that $\mathcal{H}$ is both closed and open so by connectedness of $\mathcal{U}$ must equal $\mathcal{U}$. □

It is now natural to ask:

Question 2 Does every connected, definably generated group admit a lattice?

2. Lattices in abelian groups

We still work in a sufficiently saturated structure $\mathcal{M}$.

Recall that for a locally definable group $\mathcal{U}$, we say that $\mathcal{U}^{00}$ *exists*, if there is a smallest type-definable normal subgroup of $\mathcal{U}$ of bounded index (note that a type-definable subgroup of $\mathcal{U}$ is necessarily contained in a definable subset of $\mathcal{U}$). We denote that subgroup by $\mathcal{U}^{00}$.

One of the main results in [4] is the following: (the equivalence of the bottom three clauses is given in [4, Theorem 3.9]; the addition of Clause (1) is obtained using Lemma 1.6):

Theorem 2.1. *Let $\mathcal{U}$ be a connected, abelian definably generated group. Then there is k so that the following are equivalent:*

(1) $\mathcal{U}$ admits a lattice.
(2) $\mathcal{U}$ admits a lattice, isomorphic to $\mathbb{Z}^k$.
(3) $\mathcal{U}$ contains a definable generic set.
(4) $\mathcal{U}^{00}$ exists, and $\mathcal{U}/\mathcal{U}^{00}$ is isomorphic to $\mathbb{R}^k \times K$, for some compact Lie group K.

In particular, we see that a connected, abelian, locally definable $\mathcal{U}$ admits a lattice if and only if it contains a definable generic set. Note that by (4), the above k is determined by $\mathcal{U}/\mathcal{U}^{00}$ and thus unique.

In [4] we made the conjecture that the conclusions of the above theorem are always true:

Conjecture A. *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then $\mathcal{U}$ contains a definable generic set (so in particular admits a lattice).*

The number k in Theorem 2.1 can be viewed as a measure of how "non-definable" the group $\mathcal{U}$ is. Namely, if $k = 0$ then $\mathcal{U}$ is outright definable, while if $k = \dim\mathcal{U} > 0$, then $\mathcal{U}$ will not contain any infinite definable subgroup. We prove the latter statement in Corollary 2.6 below.

In fact, we can define an invariant for every locally definable group $\mathcal{U}$ (not necessarily satisfying Conjecture A) which gives some indication as to how "non-definable" $\mathcal{U}$ is.

Definition 2.2. *The* $\bigvee$-dimension *of* $\mathcal{U}$*, denoted by* $\mathrm{vdim}(\mathcal{U})$*, is the maximum* k *such that* $\mathcal{U}$ *contains a compatible subgroup isomorphic to* $\mathbb{Z}^k$*, if such* k *exists, and* ∞*, otherwise.*

We prove in Theorem 2.8 below that Conjecture A is equivalent to the following.

Conjecture B. *Let* $\mathcal{U}$ *be a connected, abelian, definably generated group. Then,*

(1) $\mathrm{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$*. In particular,* $\mathrm{vdim}(\mathcal{U})$ *is finite.*
(2) If $\mathcal{U}$ *is not definable, then* $\mathrm{vdim}(\mathcal{U}) > 0$*.*

In Section 3 we will prove Conjecture A for definably generated subgroups of $\langle R^n, +\rangle$, where R is a real closed field and $\mathcal{M}$ is an o-minimal expansion of R.

Unless otherwise stated, $\mathcal{U}$ *denotes a connected, abelian, definably generated group.*

We first prove:

Lemma 2.3. *Assume that* $\mathcal{U}$ *contains a definable group* H*. Then* $\mathcal{U}$ *admits a lattice* Γ *isomorphic to* $\mathbb{Z}^k$ *if and only if* $\mathcal{U}/H$ *(which is also definably generated) contains a lattice* Δ *isomorphic to* $\mathbb{Z}^k$*.*

Proof. Let $\psi : \mathcal{U} \to \mathcal{U}/H$ be a locally definable surjective homomorphism.

Assume that $\mathcal{U}$ contains a lattice $\Gamma \simeq \mathbb{Z}^k$. Because H is definable the intersection $\Gamma \cap H$ is finite so must equal $\{0\}$. Let $\Delta = \psi(\Gamma) \simeq \mathbb{Z}^k$. To see that Δ is compatible in $\mathcal{U}/H$, take a definable $Y \subseteq \mathcal{U}/H$ and find a definable $X \subseteq \mathcal{U}$ such that $\psi(X) = Y$. Our goal is to show that $Y \cap \Delta$ is finite. But $Y \cap \Delta = \psi((X + H) \cap \Gamma)$ and since Γ is compatible its intersection with $X + H$ is finite. Thus $Y \cap \Delta$ is finite and so Δ is compatible in $\mathcal{U}/H$.

Let $\phi : \mathcal{U} \to G$ be a locally definable surjective homomorphism onto a definable group, with $\Gamma = ker\phi$. Notice that $\phi(H)$ is a definable subgroup of G. To see that Δ is a lattice in $\mathcal{U}/H$, we note that

$$(\mathcal{U}/H)/\Delta \simeq \mathcal{U}/(H + \Gamma) \simeq G/\phi(H),$$

and therefore $(\mathcal{U}/H)/\Delta$ is definable.

Assume now that $\mathcal{U}/H$ admits a lattice $\Delta \simeq \mathbb{Z}^k$. We can find $u_1, \ldots, u_k \in \mathcal{U}$ with $\phi(u_1), \ldots, \phi(u_k)$ generators of Δ. Let $\Gamma \subseteq \mathcal{U}$ be the group generated by the u_i's.

We first show that Γ is compatible. Because Δ is torsion free, ϕ is injective on Γ. Therefore, if $X \subseteq \mathcal{U}$ is definable the intersection $X \cap \Gamma$ must be finite, or else $\phi(X) \cap \Delta$ is infinite, contradicting the compatibility of Δ. To see that Γ is a lattice it is sufficient, by Lemma 1.3, to see that $\mathcal{U}$ contains a definable set X with $X + \Gamma = \mathcal{U}$. We first find a definable $Y \subseteq \mathcal{U}/H$ such that $Y + \Delta = \mathcal{U}/H$, then a definable $X' \subseteq \mathcal{U}$ with $\psi(X') = Y$, and finally take $X = X' + H$. It is easy to verify that $X + \Gamma = \mathcal{U}$. □

Lemma 2.4. *Assume that $\mathcal{U}$ contains a definable generic set. Then $\mathcal{U}$ is definable if and only if* $\text{vdim}(\mathcal{U}) = 0$.

Proof. One direction is obvious for if $\mathcal{U}$ is definable then it cannot contain any infinite 0-dimensional compatible subgroup. For the converse, assume that $\mathcal{U}$ is not definable.

By Theorem 2.1, the group $\mathcal{U}^{00}$ exists and for some $k \in \mathbb{N}$, we have $\mathcal{U}/\mathcal{U}^{00} \simeq R^k \times K$, for a compact Lie group K. We claim that $k > 0$. Indeed, if $k = 0$ then $\mathcal{U}/\mathcal{U}^{00} = K$ is compact. But then, by [4, Lemma 3.3], the preimage of K would be contained in a definable subset of $\mathcal{U}$, and thus $\mathcal{U}$ would be definable, a contradiction.

If we now apply Theorem 2.1 (4) $\Rightarrow$ (2), we see that $\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$ so $\text{vdim}(\mathcal{U}) \geq k > 0$. □

Proposition 2.5. *Assume that $\mathcal{U}$ admits a lattice.*

(i) If Λ is a 0-dimensional, compatible subgroup of $\mathcal{U}$, then $\Lambda \simeq \mathbb{Z}^l + F$, with $l \leq \text{vdim}(\mathcal{U})$ and F a finite subgroup of $\mathcal{U}$.

(ii) $\text{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$.

(iii) If Λ is a lattice in $\mathcal{U}$, then $\Lambda \simeq \mathbb{Z}^l + F$, with $l = \text{vdim}(\mathcal{U})$ and F a finite subgroup of $\mathcal{U}$.

(iv) If $\mathcal{U}$ is torsion-free and generated by a definably compact set then every lattice in $\mathcal{U}$ is isomorphic to $\mathbb{Z}^l$, with $l = \dim(\mathcal{U}) = \text{vdim}(\mathcal{U})$.

Proof. By [4, Claim 3.4], there exists a definable torsion-free subgroup $H \subseteq \mathcal{U}$ such that the group $\mathcal{U}/H$ is generated by a definably compact set.

By [4, Theorem 3.9], there exists a unique k such that $\mathcal{U}/H$ admits a lattice isomorphic to $\mathbb{Z}^k$ and moreover, because $\mathcal{U}/H$ is generated by a definably compact set, we have $k \leq \dim(\mathcal{U}/H)$ and hence $k \leq \dim(\mathcal{U})$. Also, by Lemma 2.3, the group $\mathcal{U}$ also admits a lattice isomorphic to $\mathbb{Z}^k$, so $k \leq \text{vdim}(\mathcal{U})$. Our proof below implies that $k = \text{vdim}(\mathcal{U})$.

Again, by [4, Theorem 3.9], the groups $\mathcal{U}/\mathcal{U}^{00}$ is isomorphic to $\mathbb{R}^k \times K$, where K is a compact Lie group. The rest of the argument is extracted from the proof of [4, Lemma 3.7].

(i) Assume that $\Lambda \subseteq \mathcal{U}$ is a 0-dimensional compatible subgroup. Consider $\phi : \mathcal{U} \to \mathcal{U}/\Lambda$. We claim that $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\phi \restriction X$ is definable, the intersection $\ker(\phi) \cap \mathcal{U}^{00} \subseteq \ker(\phi) \cap X$ is finite. However, by [4, Proposition 3.5], the group $\mathcal{U}^{00}$ is torsion-free, so $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$.

Consider the map $\pi_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^k \times K$ and let Γ be the image of $\ker(\phi)$ under π_U. We just showed that Γ is isomorphic to $\Lambda = \ker(\phi)$. We claim that Γ is discrete. Indeed, using X as above we can find another definable set X' whose image $\pi_{\mathcal{U}}(X')$ contains an open neighborhood of 0 and no other elements of Γ, so Γ is discrete.

Now, since K is compact, the projection Γ' of Γ into $\mathbb{R}^k$ has a finite kernel $F \subseteq K$. Furthermore, Γ' is a discrete subgroup of $\langle \mathbb{R}^k, + \rangle$, and hence $\Gamma' \simeq \mathbb{Z}^l$, for some $l \leq k$. Therefore, $\Gamma \simeq \mathbb{Z}^l + F$, so $\Lambda \simeq \mathbb{Z}^l + F$. In particular, if $\Lambda \simeq \mathbb{Z}^l$, then $l \leq k$, which implies $\text{vdim}(\mathcal{U}) \leq k$. Since $\mathcal{U}$ does contain a compatible copy of $\mathbb{Z}^k$ it follows that $k = \text{vdim}(\mathcal{U})$, so $l \leq \text{vdim}(\mathcal{U})$, as required.

(ii) Since $k \leq \dim(\mathcal{U})$ we have $\text{vdim}(\mathcal{U}) \leq \dim(U)$.

(iii) Assume now that $\Lambda \simeq \mathbb{Z}^l + F$ is a lattice in $\mathcal{U}$. Namely, $\mathcal{U}/\Lambda$ is a definable group G. We proceed to show that $l = k$. Let $X \subseteq \mathcal{U}$ be a definable set so that $\phi(X) = G$. Then $X + \ker(\phi) = \mathcal{U}$. Thus, $\pi_{\mathcal{U}}(X) + \Gamma = \mathbb{R}^k \times K$. Let Y, F' and Γ' be the projections of $\pi_{\mathcal{U}}(X)$, F and Γ, respectively, into $\mathbb{R}^k$. We have $Y + \Gamma' = \mathbb{R}^k$. Since X is definable, the set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \ldots, \lambda_l$ be the generators of $\ker(\phi)$ and let $v_1, \ldots, v_l \in \mathbb{R}^k$ be their images in Γ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \ldots, v_l$ then $Y + H + F' = \mathbb{R}^k$, and therefore, since Y is compact and F' finite, we must have $H = \mathbb{R}^k$. This implies that $l = k$.

(iv) By [4, Proposition 3.8], if $\mathcal{U}$ is generated by a definably compact set and is torsion-free then $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^{\dim(\mathcal{U})}$, so by Theorem 2.1 every lattice is isomorphic to $\mathbb{Z}^{\dim U}$. By (iii), $\dim(\mathcal{U}) = \text{vdim}(\mathcal{U})$. □

We can now see better why $\text{vdim}(\mathcal{U})$ gives an indication as to how "non-definable" $\mathcal{U}$ is.

Corollary 2.6. *Assume that $\mathcal{U}$ admits a lattice and H is a definable subgroup of $\mathcal{U}$. Then*

(i) $\text{vdim}(\mathcal{U}) = \text{vdim}(\mathcal{U}/H)$.

(ii) If $\text{vdim}(\mathcal{U}) = \dim(\mathcal{U})$*, then H must be finite.*

(iii) If $\mathcal{U}$ is torsion-free, and H has maximal dimension among all definable subgroups of $\mathcal{U}$, then $\dim H = \dim(\mathcal{U}) - \text{vdim}(\mathcal{U})$.

Proof. (i) By Theorem 2.1 $\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$, and by Proposition 2.5 (iii), $k = \text{vdim}(\mathcal{U})$. By Lemma 2.3, $\mathcal{U}/H$ also admits a lattice isomorphic to $\mathbb{Z}^k$ and so by again by the same proposition, we have $\text{vdim}(\mathcal{U}/H) = k$.

(ii) Assume that H is an infinite definable subgroup of $\mathcal{U}$. Then by (i), we have $\text{vdim}(\mathcal{U}/H) = \text{vdim}(\mathcal{U}) = \dim(\mathcal{U}) > \dim(\mathcal{U}/H)$, which contradicts Proposition 2.5 (ii) for $\mathcal{U}/H$.

(iii) If $\dim H$ has maximal dimension among the definable subgroups of $\mathcal{U}$ then, as we already noted, $\mathcal{U}/H$ is generated by a definably compact set. Because H is torsion-free, as a subgroup of $\mathcal{U}$, it must be definably connected and therefore divisible. It follows that $\mathcal{U}/H$ is torsion-free as well. By Proposition 2.5 (iv), $\text{vdim}(\mathcal{U}/H) = \dim(\mathcal{U}/H)$. But then, by (i) we have

$$\dim H = \dim(\mathcal{U}) - \dim(\mathcal{U}/H) = \dim(\mathcal{U}) - \text{vdim}(\mathcal{U}).$$

□

The torsion-free condition in (iii) above is necessary. For example, the group $\bar{G}$ in [5, Example 6.2] does not contain any non-trivial definable subgroups, yet $\dim(\overline{G}) = 2$ and $\text{vdim}(\overline{G}) = 1$. We describe below a general method to obtain a locally definable group $\mathcal{V}$, generated by a definably compact set, such that $\mathcal{V}$ has no infinite definable subgroups and $\text{vdim}(\mathcal{V}) < \dim(\mathcal{V})$.

Example 2.7. Let G be a k-dimensional definably compact abelian group which has no proper definable subgroups of positive dimension and let $\mathcal{U}$ be the universal covering of G, so $\dim(\mathcal{U}) = k$. Let $\Gamma \simeq \mathbb{Z}^k$ be the kernel of the covering map, so Γ is compatible in $\mathcal{U}$. Write $\Gamma = \Gamma_1 \oplus \Gamma_2$ with $\Gamma_1 \simeq \mathbb{Z}^m$, $\Gamma_2 \simeq \mathbb{Z}^{k-m}$ and $0 < m < k$. Obviously, Γ_1 is still compatible in $\mathcal{U}$ and therefore $\mathcal{V} = \mathcal{U}/\Gamma_1$ is a locally definable group with $\dim(\mathcal{V}) = \dim(\mathcal{U})$. It is not hard to see that the covering map $\mathcal{U} \to G$ factors through $\mathcal{V}$ and hence $\mathcal{V}$ cannot have any proper definable subgroup of positive dimension. We claim that $\text{vdim}(\mathcal{V}) = k - m$.

Let $\phi : \mathcal{U} \to \mathcal{V}$ be a locally definable projection. The image of Γ under ϕ is a group $\Delta \simeq \mathbb{Z}^{k-m}$ which we claim to be compatible in $\mathcal{V}$. We start with $Y \subseteq \mathcal{V}$ definable and claim that $Y \cap \Delta$ is finite.

Let $\pi_2 : \Gamma \to \Gamma_2$ be the projection with respect to the direct sum decomposition. For every $W \subseteq \Gamma$, $\phi(W)$ is in bijection with $\pi_2(W)$, so it is enough to prove that $\pi_2(\phi^{-1}(Y \cap \Delta)) = \pi_2(\phi^{-1}(Y) \cap \Gamma)$ is finite.

If we choose a definable $X \subseteq \mathcal{U}$ such that $\phi(X) = Y$ then $\phi^{-1}(Y) = X + \Gamma_1$. But $(X + \Gamma_1) \cap \Gamma = (X \cap \Gamma) + \Gamma_1$ and because Γ is compatible the set $X \cap \Gamma$ is finite. It follows that

$$\pi_2(\phi^{-1}(Y) \cap \Gamma) = \pi_2((X \cap \Gamma) + \Gamma_1) = \pi_2(X \cap \Gamma)$$

is finite so $Y \cap \Delta$ is finite, showing that Δ is compatible in $\mathcal{V}$. Hence, $\text{vdim}(\mathcal{V}) \geq k - m$.

For the opposite inequality, assume $\mathcal{V}$ contains a compatible subgroup Δ isomorphic to $\mathbb{Z}^r$ and choose $u_1, \ldots, u_r \in \mathcal{U}$ so that $\phi(u_1), \ldots, \phi(u_r)$ are generators of Δ. It is not hard to see that $\Gamma_1 + \mathbb{Z}u_1 + \cdots + \mathbb{Z}u_r$ is a compatible subgroup of $\mathcal{U}$, isomorphic to $\mathbb{Z}^{m+r}$, so necessarily $m + r \leq k$. Hence, $r \leq k - m$, so $\text{vdim}(\mathcal{V}) = k - m$.

Note that $\mathcal{V}$ has non-trivial torsion since any $a \in \mathcal{U}$ for which $na \in \Gamma_1$ will be mapped to an n-torsion element of $\mathcal{V}$.

We end by noting that the two conjectures mentioned above are equivalent.

Theorem 2.8. *Conjecture A is equivalent to Conjecture B. More precisely,*

(i) If $\mathcal{U}$ admits a definable generic set then $\mathcal{U}$ satisfies clauses (1), (2) of Conjecture B.

(ii) Conjecture B implies Conjecture A.

Proof. (i). By Proposition 2.5 and Lemma 2.4.

(ii). Let $\Lambda \simeq \mathbb{Z}^k$ be a compatible subgroup of $\mathcal{U}$ with $k = \text{vdim}(\mathcal{U})$. We will prove that the locally definable group $\mathcal{U}/\Lambda$ is actually definable.

Assume that $\mathcal{U}/\Lambda$ is not definable. By Conjecture B(2) (applied to $\mathcal{U}/\Lambda$), there exists some $a \in \mathcal{U}/\Lambda$ such that $\mathbb{Z}a$ is a compatible subgroup of $\mathcal{U}/\Lambda$, and for every n, $na \neq 0$. Let $b \in \mathcal{U}$ be an element that projects via $\phi : \mathcal{U} \to \mathcal{U}/\Lambda$ to a. Clearly, $\mathbb{Z}b \cap \Lambda = \{0\}$. We claim that $\Lambda + \mathbb{Z}b$ is a compatible subgroup of $\mathcal{U}$, contradicting $k = \text{vdim}(\mathcal{U})$. Let $X \subseteq \mathcal{U}$ be definable. The image of $X \cap (\Lambda + \mathbb{Z}b)$ under ϕ is contained in $\phi(X) \cap \mathbb{Z}a$. Since ϕ is locally definable, $\phi(X)$ is definable. Therefore $\phi(X) \cap \mathbb{Z}a$ is finite, by compatibility of $\mathbb{Z}a$. The preimage of this finite set under π is a union of sets $\Lambda + x$, $x \in B$, for some finite $B \subseteq \mathbb{Z}b$. So $X \cap (\Lambda + \mathbb{Z}b)$ is equal to the finite union of the sets $X \cap (\Lambda + x)$, $x \in B$, each of which is finite, because so is $(X - x) \cap \Lambda$

by compatibility of Λ. Hence $X \cap (\Lambda + \mathbb{Z}b)$ is finite, and thus $\Lambda + \mathbb{Z}b$ is compatible. □

3. Locally definable subgroups of $\langle R^n, +\rangle$

We assume here that $\mathcal{M}$ is an o-minimal expansion of a real closed field R

Our goal is to prove Conjecture A for subgroup of $\langle R^n, +\rangle$ but in fact we prove a stronger result (as was suggested to us by the referee):

Theorem 3.1. *Let $\mathcal{U}$ be a connected definably generated subgroup of $\langle R^n, +\rangle$ of dimension k. Then there are linearly independent one-dimensional R-subspaces $R_1, \ldots, R_k$ and intervals $I_i = (-a_i, a_i) \subseteq R_i$ (with a_i possibly ∞) such that $\mathcal{U}$ is generated by the set $X = I_1 + \cdots + I_k$. The set X is generic in $\mathcal{U}$.*

Proof. Recall that for $X \subseteq R^n$, we write $X(m)$ for the addition of $X - X$ to itself m times. If $0 \in X$ then $X \subseteq X(m)$.

Definition 3.2. *A subset of R^n is called* convex with respect to R (or R-convex) *if for all $x, y \in X$, the line segment connecting x and y is also in X.*

The R-convex hull *of X is the smallest R-convex subset of R^n containing X. It consists of all finite combinations $\sum_{i=1}^{m} t_i x_i$, where the x_i's are in X, all $t_1 \geq 0$ and $\sum t_i = 1$.*

Lemma 3.3. *If $X \subseteq R^n$ is definable then the R-convex hull of X is also definable.*

Proof. More precisely, we claim that the following set equals the R-convex hull of X:

$$X' = \left\{ \sum_{i=1}^{n+1} t_i x_i : t_1 + \cdots + t_{n+1} = 1, t_i \in [0,1], x_i \in X \right\}.$$

Indeed, by Caratheodory's Theorem, every convex combination of any number of points from X can also be realized as a combination of $n+1$ of these points, hence the R-convex hull of X equals X'. (Note that although Caratheodory's theorem is usually proved over the reals the same proof works over any ordered field. Alternatively, the statement over the real numbers implies, by transfer, the same result over any real closed field). □

Lemma 3.4. *Assume that $X \subseteq R^n$ is a definably connected set containing 0. Then there is m such that $X(m)$ (in the sense of the additive group $\langle R, +\rangle$) contains the R-convex hull of X.*

Proof. Given $f : X \to Z$, the fiber power of X is defined as:

$$X \times_f X = \{\langle x, y\rangle \in X \times X : \ f(x) = f(y)\}.$$

Clearly, the diagonal Δ is contained in $X \times_f X$.

Note that for $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in X \times_f X$, there is a continuous definable path in $X \times_f X$, connecting the two points if and only if there are definable continuous curves $\gamma_1, \gamma_2 : [0,1] \to X$ such that $\gamma_i(0) = x_i$, $\gamma_i(1) = y_i$, and for every $t \in [0,1]$ we have $f(\gamma_1(t)) = f(\gamma_2(t))$.

Claim 3.5. *For $X \subseteq R^n$, consider the projection $\pi : R^n \to R$ onto the first coordinate. Assume that $\pi(x_1) = \pi(x_2)$, $\pi(y_1) = \pi(y_2)$ (in particular, $\pi_1(x_1 - x_2) = \pi(y_1 - y_2) = 0$). Assume further that $\langle x_1, x_2 \rangle$ and $\langle y_1, y_2 \rangle$ are in the same connected component of $X \times_\pi X$. Then the elements $x_1 - x_2$ and $y_1 - y_2$ are in the same connected component of the set $(X - X) \cap \{0\} \times R^{n-1}$.*

Proof. Note that the image of $X \times_\pi X$ under the binary map $\langle x, y \rangle \mapsto x - y$ is contained in the set $\{0\} \times R^{n-1}$. Consider the restriction of this map to the connected component of $X \times_\pi X$ which contains $\langle x_1, x_2 \rangle$ and $\langle y_1, y_2 \rangle$. The image is connected and clearly contains $x_2 - x_1$ and $y_2 - y_1$. □

Claim 3.6. *Assume that $x, y \in X$, $\pi(x) = \pi(y)$ and that there is a curve*

$$\gamma = (\gamma_1, \ldots, \gamma_n) : [0, 1] \to X$$

connecting x and y inside X (note that $\gamma_1(0) = \gamma_1(1)$). Let Γ be the image of γ.

(1) If γ_1 is constant on $[0, 1]$ then $\Gamma \times_\pi \Gamma$ is definably connected. In particular, for every $x, y, z \in \Gamma$, $\langle x, y \rangle$ and $\langle z, z \rangle$ are in the same definably connected component of $X \times_\pi X$.

(2) If for some $a \in (0, 1)$, γ_1 is increasing on $(0, a)$ and decreasing on $(a, 1)$ then $y - x$ and 0 are in the same connected component of $(X - X) \cap \{0\} \times R^{n-1}$.

(3) If for some $a_1 < a_2$ in $(0, 1)$, γ_1 is increasing on $(0, a_1)$, constant on (a_1, a_2) and decreasing on $(a_2, 1)$ then $y - x$ and 0 are in the same connected component of $(X - X) \cap \{0\} \times R^{n-1}$.

Proof. (1) By assumption the map π is constant on Γ and therefore $\Gamma \times_\pi \Gamma = \Gamma \times \Gamma$, which is clearly definably connected.

(2) Let $[b_1, b_2]$ be the image of γ under π. By assumptions, $\pi(\gamma_1(0)) = \pi(\gamma_1(1)) = b_1$, $\pi(\gamma_1(a)) = b_2$ and the restrictions of π to the pieces $\gamma([0, a])$ and $\gamma([a, 1])$ are both injective. Let α_1, α_2 be their inverse maps, respectively (so these are maps from $[b_1, b_2]$ into Γ). We have $\alpha_1(b_1) = x$, $\alpha_2(b_1) = y$, $\alpha_1(b_2) = \alpha_2(b_2) = \gamma(a)$. Moreover, for every $t \in [b_1, b_2]$ we have $\pi(\alpha_1(t)) = \pi(\alpha_2(t)) = t$. It follows that $\langle x, y \rangle$ and $\langle \gamma(a), \gamma(a) \rangle$ are in the same component of $X \times_\pi X$, so by Claim 3.5, $y - x$ and 0 are in the same component of $(X - X) \cap \{0\} \times R^{n-1}$.

(3) As in (2), let $[b_1, b_2]$ be the image of γ under π. It is easy to see that $\gamma_1(t) = b_2$ for all $t \in [a_1, a_2]$. Similarly to the proof of (2), $\langle x, y \rangle$ and $\langle \gamma(a_1), \gamma(a_2) \rangle$ are in the same component of $X \times_\pi X$. Using (1), we see that $\langle \gamma(a_1), \gamma(a_2) \rangle$ is in the same component as $\langle z, z \rangle$ for some $z \in \gamma([a_1, a_2])$. Applying Claim 3.5, we conclude that $x - y$ and 0 are in the same component of $(X - X) \cap \{0\} \times R^{n-1}$. □

We now return to the proof of Lemma 3.4. So, X is a definably connected subset of R^n containing 0, and we want to show that for some m, the convex hull of X is contained in $X(m)$.

We will use induction on n. If $n = 1$ then X is already convex. So, we assume that the result is true for $X \subseteq R^n$ and prove it for $X \subseteq R^{n+1}$. We take $x, y \in X$ and first want to show that for some m the line segment $[x, y]$ (i.e the line connecting x and y in R^{n+1}) is contained in $X(m)$.

Using a linear automorphism of R^n, we may assume that $\pi(x) = \pi(y) = 0$. Since X is definably connected, there exists a definable curve $\gamma : [0, 1] \to X$

connecting x and y. Let $\Gamma \subseteq X$ be the image of γ and again let $\gamma_1 = \pi \circ \gamma$.

Notation: For $f : [0,1] \to R$ continuous, let $k = k(f)$ be the minimal natural number so that there are $0 = a_0 < a_1 < \cdots < a_k = 1$ and f is either constant or strictly monotone on $[a_i, a_{i+1}]$.

We consider the map $\gamma_1 : [0,1] \to R$ and prove the result by sub-induction on $k(\gamma_1)$.

Assume first that $k(\gamma_1) = 1$, namely that γ_1 is constant on $[0,1]$. In this case, Γ is contained in $\{0\} \times R^n$, so we can work in R^n and use the inductive hypothesis to conclude that the line segment $[x,y]$ is contained in $\Gamma(m)$ for some m. Clearly, $\Gamma(m) \subseteq X(m)$ so we are done.

Assume then that $k(\gamma_1) > 1$, so γ_1 is not constant. Without loss of generality, γ_1 takes some positive value on $(0,1)$, so let $a \in (0,1)$ be a point where γ_1 takes its maximum value in $[0,1]$.

Case 1 Assume first that γ_1 is not locally constant at a.

Then there are $a_1 < a < a_2$ such that γ_1 is increasing on (a_1, a), decreasing on (a, a_2), $\gamma_1(a_1) = \gamma_1(a_2)$, and furthermore, either a_1 or a_2 are local minimum for γ_1. Indeed, we take $a_1' < a$ to be the minimum of all points t such that γ_1 is increasing on (t, a), take $a_2' > a$ be the maximum of all $t > a$ such that γ_1 is decreasing on (a, t). (In this case, a_1' and a_2' are local minima for γ_1). We then compare $\gamma_1(a_1')$ and $\gamma_1(a_2')$. If $\gamma_1(a_1') > \gamma_1(a_2')$ then we take $a_1 := a_1'$ and take a_2 to be the unique element of the interval (a, a_2') such that $\gamma_1(a_2) = \gamma_1(a_1)$. Otherwise, we do the opposite.

Let $x_1 = \gamma(a_1)$ and $x_2 = \gamma(a_2)$. Consider now the curve Γ' which is the image of $[a_1, a_2]$ under γ. By Claim 3.6 (2), $x_2 - x_1$ and 0 are in the same connected component of $(\Gamma' - \Gamma') \cap \{0\} \times R^n$. But then, we can view this component as living in R^n, so by inductive hypothesis there exists m such that the line segment connecting 0 and $x_2 - x_1$ is contained in $(\Gamma' - \Gamma')(m)$. By adding x_1 to both sides, we see that the line segment connecting x_1 and x_2 is contained in $(X - X)(m+1)$. Hence, after replacing X with $X(m)$, we can also replace the original curve Γ with a new curve Γ'', in which the piece $\gamma([a_1, a_2])$ was replaced by a linear segment all of whose points project to the same point $\pi(x_1)$. Let $\gamma'' : [0,1] \to X$ be the map whose image is Γ'' (so $\gamma'' = \gamma$ everywhere, except on $[a_1, a_2]$, in which the image is linear and γ_1'' is constant). Because a_1 or a_2 is a local minimum of γ_1, it is easy to see that $k(\gamma_1'') = k(\gamma_1) - 1$. By sub-inductive hypothesis, the line connecting x and y is contained in some $X(m')$.

Case 2 Assume that γ_1 is locally constant at a.

So, there are $a_1' \leq a \leq a_2'$ such that γ_1 is constant on $[a_1', a_2']$ and this is a maximal such interval. As in Case 1, we can find $a_1 < a_1'$ and $a_2 > a_2'$ such that γ_1 is increasing on $[a_1, a_1']$, decreasing on $[a_2', a_2]$, $\gamma_1(a_1) = \gamma_1(a_2)$ and furthermore, either a_1 or a_2 is a local minimum of γ_1.

Let Γ' be the piece of Γ connecting $\gamma(a_1)$ and $\gamma(a_2)$. Then, by Claim 3.6(3), the points $\gamma(a_2) - \gamma(a_1)$ and 0 are in the same component of $(\Gamma' - \Gamma') \cap \{0\} \times R^n$. Again, by inductive hypothesis, the line segment connecting 0 and $\gamma(a_2) - \gamma(a_1)$ is contained in $(\Gamma' - \Gamma')(m)$ for some m, so the line segment connecting $\gamma(a_1)$ and $\gamma(a_2)$ is contained in $X(m+1)$. As in Case (1), we can replace Γ by Γ'', in which the piece $\gamma([a_1, a_2])$ is replaced by the line segment connecting $\gamma(a_1)$ and $\gamma(a_2)$. Again, the map $\gamma'' : [0,1] \to X$ whose image is Γ'' now satisfies $k(\gamma_1'') = k(\gamma_1) - 2$ (because we replaced three pieces by one). By sub-inductive hypothesis, the line connecting x and y is in some $X(m')$.

We therefore showed that for every $x, y \in X$, there exists m such that the line segment $[x, y]$ is contained in $X(m)$. To see that we can find a uniform m for all $x, y \in X$, we use logical compactness (writing a type $p(x, y)$, which says that the line segment $[x, y]$ is not contained in any $X(m)$). This ends the proof of Lemma 3.4. □

Question It is interesting to ask what is the required m in the above result. The argument suggests that it depends on the possible number of "twistings" of the curve connecting two points in X. But maybe this is just an effect of the proof and one can find uniform such m which depends only on the ambient R^n.

Next, we show that $\mathcal{U} \subseteq R^n$ can be generated by a sum of intervals in linearly independent one-dimensional spaces. By Lemma 3.4 we can assume that it is generated by a definably connected convex set $X \ni 0$. In particular, $\mathcal{U}$ is convex. Since $\mathcal{U}$ is closed, we may replace X by its closure, which is still convex, and assume that X is closed. We may also assume that $-X = X$ (otherwise we replace it with $X - X$).

We prove the result by induction on n. When $\mathcal{U}$ is a subset of R then any convex subset of R is an interval (possibly equaling the whole of R) so we are immediately done.

We now consider the case $\mathcal{U} \subseteq R^{n+1}$.

Assume first that X is bounded. Consider all line segments contained in X and let J_0 be such segment of maximal length (it exists by o-minimality and the fact that X is closed). Since we work in a field we may assume that J_0 is parallel to the x_{n+1}-coordinate and furthermore that $0 \in J_0$ divides it exactly into two equal parts. We can therefore write $J_0 = (-a_{k+1}, a_{k+1})$. Let $\pi(X)$ be the projection onto the first n coordinates. By induction, there are linearly independent 1-dimensional spaces $R_1, \dots, R_k \subseteq R^n$, and in each R_i an interval $I_i = (-a_i, a_i)$ (with a_i possibly ∞) such that the sum $Y = I_1 + \cdots + I_k$ generates the same group as $\pi(X)$. In particular, there is an $m \in \mathbb{N}$ such that $Y \subseteq \pi(X)(m)$. Our goal is to show that $Y + J_0$ generates the group $\mathcal{U}$. It is thus sufficient to prove the following:

Claim. $X \subseteq Y + J_0 \subseteq X(2m)$.

Proof. Consider $\langle \bar{x}, y \rangle \in X$, with $\bar{x} \in \pi(X)$. Note that $|y| \leq a_{k+1}/2$, because if $y > a_{k+1}/2$ then the length of the line segment connecting $\langle \bar{x}, y \rangle$ to 0 is then greater than $a_{k+1}/2$. Because X is symmetric, the point $\langle -\bar{x}, -y\rangle$ is also in X and thus the line segment connecting $\langle -\bar{x}, -y\rangle$ and $\langle \bar{x}, y\rangle$ is longer than $a_{k+1} = |J_0|$, contradiction. We therefore showed that $y \leq a_{k+1}/2$ and hence

$$\langle \bar{x}, y\rangle \in \{\langle \bar{x}, 0\rangle\} + J_0 \subseteq \pi(X) + J_0 \subseteq Y + J_0.$$

For the opposite inclusion, take $\langle \bar{x}, y\rangle \in Y + J_0$. Since $Y \subseteq \pi(X)(m) = \pi(X(m))$, there exists $y' \in R$, such that $\langle \bar{x}, y'\rangle \in X(m)$. Because $max\{|y| : \langle \bar{x}, y\rangle \in X\} = a_{k+1}/2$, we have $|y'| \leq ma_{k+1}/2$. But then

$$\langle \bar{x}, y\rangle \in X(m) + mJ_0 \subseteq X(2m).$$

This ends the proof of the claim and the case where the generating set X is bounded. □

In the general case, we first find a definable subgroup H such that $\mathcal{U}/H$ is generated by a definably compact set. Since all definable subgroups of R^n are R-vector spaces, the group H is linear. Without loss of generality, $H = R^k$, for $k \leq n$, identified with the first k coordinates. Let $\pi_1 : \mathcal{U} \to R^{n-k}$ be the projection onto the last $n-k$ coordinates and let $\mathcal{V} = \pi_1(\mathcal{U})$. We claim that $\mathcal{U} = H + \mathcal{V}$.

Indeed, assume that $\langle \bar{x}, \bar{y}\rangle \in \mathcal{U}$. Since $\mathcal{U}$ is convex, the line segments which connect $\langle \bar{x}, \bar{y}\rangle$ to arbitrary large points in R^k belong to $\mathcal{U}$. Hence we can approach every point on the affine space $R^k \times \{\bar{y}\}$ by points inside $\mathcal{U}$. Since $\mathcal{U}$ is closed, we have that $H + \{\langle \bar{0}, \bar{y}\rangle\}$ is contained in $\mathcal{U}$. This shows that $H + \mathcal{V}$ is contained in $\mathcal{U}$. The converse is immediate. This ends the proof that $\mathcal{U}$ is generated by a sum of intervals in linearly independent one-dimensional spaces.

Our final goal is to show that $Y_0 = I_1 + \cdots + I_k + J_0$ is generic in $\mathcal{U}$. We have $I_i = (-a_i, a_i)$ and $J_0 = (-a_{k+1}, a_{k+1})$. If we let $\mathcal{V}_i$ be the 1-dimensional group generated by $(-a_i, a_i)$ then we have $\mathcal{U} = \mathcal{V}_1 + \cdots + \mathcal{V}_{k+1}$. Each $(-a_i, a_i)$ is generic in $\mathcal{V}_i$ so it is easy to verify that Y_0 is generic in $\mathcal{U}$. This ends the proof of Theorem 3.1. □

As noted in the above proof, $\mathcal{U}$ is convex in R^n. This immediately implies that $\mathcal{U}$ is divisible. In [4], we prove more generally that Conjecture A implies that every connected definably generated abelian group is divisible.

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Part 5

Structure theorems in tame expansions of o-minimal structures by a dense set

STRUCTURE THEOREMS IN TAME EXPANSIONS OF O-MINIMAL STRUCTURES BY A DENSE SET

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Abstract. We study sets and groups definable in tame expansions of o-minimal structures. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be an expansion of an o-minimal $\mathcal{L}$-structure $\mathcal{M}$ by a dense set P, such that three tameness conditions hold. We prove a structure theorem for definable sets and functions in analogy with the influential cell decomposition theorem known for o-minimal structures. The structure theorem advances the state-of-the-art in all known examples of $\widetilde{\mathcal{M}}$, as it achieves a decomposition of definable sets into *unions* of 'cones', instead of only boolean combinations of them. We also develop the right dimension theory in the tame setting. Applications include: (i) the dimension of a definable set coincides with a suitable pregeometric dimension, and it is invariant under definable bijections, (ii) every definable map is given by an $\mathcal{L}$-definable map off a subset of its domain of smaller dimension, and (iii) around generic elements of a definable group, the group operation is given by an $\mathcal{L}$-definable map.

1. Introduction

Definable groups in models of first-order theories have been at the core of model theory for at least a period of three decades (see, for example, [5, 33, 41]) and have been crucially used in important applications of model theory to other areas of mathematics (such as in [28]). An indispensable tool in their analysis has been a structure theorem for the definable sets and types: analyzability of types and the existence of a rank in the stable category, and a cell decomposition theorem and the associated topological dimension in the o-minimal setting. In this paper we establish a structure theorem for definable sets and functions in tame expansions of o-minimal structures, introduce and analyze the relevant notion of dimension and establish a local theorem for definable groups in this setting. Our structure theorem is inspired by a cone decomposition theorem known for semi-bounded o-minimal structures ([15, 17, 34]), which was also vitally used in the analysis of definable groups therein ([21]). The structure theorem has opened the way to other applications of the tame setting, beyond the study of definable groups, such as the point counting theorems in [19].

Let us briefly discuss the tame setting. O-minimal structures were introduced and first studied by van den Dries [10] and Knight-Pillay-Steinhorn [32, 40] and have since provided a rigid framework to study real algebraic and analytic geometry. They have enjoyed a wide spectrum of applications reaching out even to number theory

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and Diophantine geometry (such as in Pila's solution of certain cases of the André-Oort Conjecture [37]). However, o-minimality can only be used to model phenomena that are at least locally finite, or more precisely, objects that have only finitely many connected components. Tame expansions of o-minimal structures can further model phenomena that escape from the o-minimal context, but yet exhibit tame geometric behavior. They have recently seen significant growth ([1, 3, 6, 8, 12, 14, 25, 31]) and are by now divided into two important classes of structures: those where every open definable set is already definable in the o-minimal reduct and those where an infinite discrete set is definable. We establish our cone decomposition theorem in the former category. In the second category, a relevant structure theorem has already been obtained in [44], benefiting largely by the presence of definable choice in that setting (absent here).

We now fix our setting and describe the results of this paper. Let $\mathcal{M}$ be an o-minimal expansion of an ordered group with underlying language $\mathcal{L}$. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ be an expansion of $\mathcal{M}$ by a dense set P so that certain tameness conditions hold (those are listed in Section 2.1). For example, $\widetilde{\mathcal{M}}$ can be a dense pair ([12]), or P can be an independent set ([9]) or a multiplicative group with the Mann Property ([14]). To establish our structure theorem below, we introduce a new invariant for definable sets, the 'large dimension', which turns out to coincide with the combinatorial dimension coming from a pregeometry in [3]. These results are in the spirit of some standard and recent literature. In an o-minimal structure, the cell decomposition theorem ([13, 32]) is used to show that the associated 'topological dimension' equals the combinatorial dimension coming from the dcl-pregeometry ([38]). In a semi-bounded structure, the cone decomposition theorem ([15, 17, 34]) is used to show that the associated 'long dimension' equals the dimension coming from the short closure pregeometry ([17]). In both settings, the equivalence of the two dimensions has proven extremely powerful in many occasions and in particular in the analysis of definable groups (see, for example, [17, 21, 22, 39]). Here, we apply the strategy from the semi-bounded setting to that of tame expansions of o-minimal structures and establish the analogous results in $\widetilde{\mathcal{M}}$.

In Sections 2 and 3 we include some preliminaries and do preparatory work for what follows. In Section 4, we introduce the notions of a *cone* and *large dimension*. Although the definitions appear to be rather technical, we show in subsequent work that they are in fact optimal (see Section 5.2, Question 5.14 and [18]). In Section 5, we prove the following theorem.

Structure Theorem (5.1).

(1) *Let $X \subseteq M^n$ be an A-definable set. Then X is a finite union of A-definable cones.*
(2) *Let $f : X \to M$ be an A-definable function. Then there is a finite collection $\mathcal{C}$ of A-definable cones, whose union is X and such that f is fiber $\mathcal{L}_A$-definable with respect to each cone in $\mathcal{C}$.*

We then conclude that the large dimension is invariant under definable bijections (Corollary 5.3). The above Structure Theorem is a substantial improvement of the 'near-model completeness' results established in known cases (such as [1, 12, 14]) in that it achieves a decomposition of definable sets into *unions* (instead of boolean combinations) of cones. It also includes definable maps $f : M^n \to M$ for any n (instead of only $n = 1$). To illustrate the last point, let us consider the following example of a map for $n = 1$ from [12]. Consider a dense pair $\langle \mathcal{M}, P\rangle$ of real closed fields and let $\alpha \notin P$. So $\mathcal{M}$ could be the real field, P the field of real algebraic numbers, and $\alpha = \pi$. Let $f : M \to M$ be the definable map given by

$$f(x) = \begin{cases} r & \text{if } x = r + \alpha s \text{ for some (unique) } r, s \in P \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that the graph of f is dense in M^2, and hence f is not as tame as an $\mathcal{L}$-definable map. However, [12, Theorem 3] establishes that every definable map $f : M \to M$ is given by an $\mathcal{L}$-definable map off a small set (here, the $\mathcal{L}$-definable map is 0 and the small set is $P + \alpha P$). A far reaching application of our structure theorem is the following generalization of this phenomenon.

Theorem 5.7. *Every A-definable map $f : M^n \to M$ is given by an $\mathcal{L}_{A\cup P}$-definable map off a set of large dimension $< n$.*

We expect that this theorem will be useful in the future and already manifest one of its immediate corollaries here. Namely, we answer a question by Dolich-Miller-Steinhorn [9]: in dense pairs, the graph of a *$\emptyset$-definable* unary function is nowhere dense (Proposition 5.8).

In Section 6, we compare the large dimension of a definable set to the scl-dimension coming from [3]. In [3], the authors work under three similar tameness conditions on $\widetilde{\mathcal{M}}$ and prove that the *small closure* operator scl defines a pregeometry under further assumptions on $\mathcal{M}$ ([3, Corollary 77]). Here, we observe that those further assumptions are in fact unnecessary (Corollary 6.4) and derive the equivalence of the two dimensions (Proposition 6.9), always.

In Section 7, we exploit this equivalence and set forth the analysis of groups definable in $\widetilde{\mathcal{M}}$. Indeed, making use of desirable properties of 'scl-generic' elements (Fact 6.13), we achieve the following result.

Local theorem for definable groups (7.6). *Let $G = \langle G, * \rangle$ be a definable group of large dimension k. Then for every* scl*-generic element a in G, there is a $2k$-cone $C \subseteq G \times G$, whose topological closure contains (a, a), and on which the operation*

$$(x, y) \mapsto x * a^{-1} * y$$

is given by an $\mathcal{L}$-definable map.

We note that an analogous local theorem for semi-bounded groups was proved in [17, Theorem 6.3] and was then vitally used in the global analysis of semi-bounded groups in [21]. We expect that the present local theorem will be as crucial in forthcoming analysis of definable groups in $\widetilde{\mathcal{M}}$, and we list a series of open questions in the end of Section 7. The ultimate goal would be to understand definable groups in terms of $\mathcal{L}$-definable groups and small groups (Conjecture 7.8). Note that $\mathcal{L}$-definable groups have been exhaustively studied and are well-understood, some of the main results being proved in [7, 16, 21, 22, 29, 30, 35].

We next indicate some of the key aspects of this paper. Both the definition of the large dimension, as well as that of a cone, are based on the notion of a *supercone* given in Section 4, which in its turn is based on the notion of a *large* subset of M coming from [3] or [14]. Namely, a supercone J in M^n is defined, recursively on n, as a union of a specific family of large fibers over a supercone in M^{n-1}. The large dimension of a definable set X is then the maximum k such that a supercone from M^k can be *embedded* into X. The nature of this embedding is crucial: while the definition of the large dimension is given via a strong notion of embedding, proving its invariance under definable bijections in Corollary 5.3 requires an equivalent definition via a weaker notion of embedding. We establish that equivalence in Corollary 4.22.

Let us now describe the main idea behind the proof of the Structure Theorem in Section 5 that also explains the role of large dimension in it and motivates all the preparatory work done in Sections 3 and 4. The notion of a large/small set is defined in Section 2 and that of a k-cone in Section 4. Roughly speaking, a k-cone is a set of the form

$$h\left(\bigcup_{g\in S}\{g\}\times J_g\right),$$

where h is an $\mathcal{L}$-definable continuous map with each $h(g,-)$ injective, $S \subseteq M^m$ is a small set, and $\{J_g\}_{g\in S}$ a definable family of supercones in M^k. The proof of the Structure Theorem runs by simultaneous induction on n for three statements, Theorem 5.1 (1) - (3). For (1), in the inductive step, let $X \subseteq M^{n+1}$. By the inductive hypothesis, we may assume that the projection $\pi(X)$ onto the first n coordinates is a k-cone, and by definability of smallness (Remark 3.4(a)), we may separate two cases. If all fibers of X above $\pi(X)$ are large, then we can simply follow the definition of a cone and, using $(2)_n$ and Lemma 4.10, we conclude that X is a $k+1$-cone. If all fibers of X above $\pi(X)$ are small, then we first need to turn X into a small union of ($\mathcal{L}$-definable images of) subsets $J_g \subseteq \pi(X)$ as above. This is achieved using Lemma 3.7 and it is illustrated in Example 3.9. Unfortunately, the sets J_g obtained are not necessarily supercones, but we can remedy the situation by applying a uniform version of $(1)_{n-1}$, namely $(3)_{n-1}$. We derive $(3)_n$ from $(1)_n$ using a standard compactness argument. We derive $(2)_n$ from $(1)_n$ by first applying Corollary 3.27 to obtain $\mathcal{L}$-definability of f outside a subset of $\pi(X)$ of *smaller large dimension.* We then conclude it by sub-induction on large dimension.

In Section 5.2, we explore the optimality of our Structure Theorem. We prove that a stronger version where the notion of a cone is strengthened by requiring that h is injective on $\bigcup_{g\in S}\{g\}\times J_g$ is not possible. This is essentially due to the lack of definable choice in our setting (see, for example, [8, Section 5.5]). In Section 5.3, however, we isolate a key 'choice property' that implies a strengthened version of Lemma 3.7 (see Lemma 5.11), which in turn guarantees a Strong Structure Theorem (5.12). This study suggests a new line of research where the behavior of $\mathcal{L}$-definable maps on small sets is pending to be explored. A list of open questions is included, whereas further optimality results are established in subsequent work [18].

It is an important feature of this work that we keep track of all parameters. If X is an A-definable set then, by Lemma 2.5 below, its closure is $\mathcal{L}_{A\cup P}$-definable. However, our Structure Theorem establishes that every A-definable set is a finite union of A-definable sets (the cones) whose closures are actually $\mathcal{L}_A$-definable. We warn the reader that we make a slight abuse of terminology in the interests of keeping the text succinct: an A-definable cone will be assumed to have its closure $\mathcal{L}_A$-definable; see Section 4.1 for more details.

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2. The setting

Throughout this paper, we fix an o-minimal theory T expanding the theory of ordered abelian groups with a distinguished positive element 1. We also fix the language $\mathcal{L}$ of T and $\mathcal{L}(P)$ the language $\mathcal{L}$ augmented by a unary predicate symbol P. Let $\widetilde{T}$ be an $\mathcal{L}(P)$-theory expanding T. If $\mathcal{M} = \langle M, <, +, \dots \rangle \models T$, then $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ denotes an expansion of $\mathcal{M}$ that models $\widetilde{T}$. By 'A-definable' we mean 'definable in $\widetilde{\mathcal{M}}$ with parameters from A'. By '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{M}$ with parameters from A'. We omit the index A if we do not want to specify the parameters.

For a subset $X \subseteq M$, we write $\mathrm{dcl}(X)$ for the definable closure of X in $\mathcal{M}$, and $\mathrm{dcl}_{\mathcal{L}(P)}(X)$ for the definable closure of X in $\widetilde{\mathcal{M}}$. By the o-minimality of T, the operation that maps $X \subseteq M$ to $\mathrm{dcl}(X)$ is a pregeometry on M. For an $\mathcal{L}$-definable set $X \subseteq M^n$, we denote by $\dim(X)$ the corresponding pregeometric dimension.

The following definition is taken essentially from [14].

Definition 2.1. Let $X \subseteq M^n$ be a definable set. We call X *large* if there is some m and an $\mathcal{L}$-definable function $f : M^{nm} \to M$ such that $f(X^m)$ contains an open interval in M. We call X *small* if it is not large.

Note that if $X \subseteq M$ is small and I an interval in M, then $I \setminus X$ is large (with a proof identical to that of [3, Lemma 20]). We will use this observation throughout this paper. In Lemma 3.11 and Corollary 3.12 below we prove that smallness is equivalent to P-internality, in the usual sense of geometric stability theory.

Definition 2.2. If $X, Z \subseteq M^n$ are definable, we say that X is *small in Z* if $X \cap Z$ is small. We say that X is *co-small in Z* if $Z \setminus X$ is small.

2.1. **Assumptions.** We assume that $\widetilde{T}$ satisfies the following three tameness conditions: for every model $\widetilde{\mathcal{M}} \models \widetilde{T}$,

(I) P is small.
(II) (Near model-completeness) Every A-definable set $X \subseteq M^n$ is a boolean combination of sets of the form
$$\{x \in M^n : \exists z \in P^m \varphi(x, z)\},$$
where $\varphi(x, z)$ is an $\mathcal{L}_A$-formula.
(III) (Open definable sets are $\mathcal{L}$-definable) For every parameter set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^s$, its topological closure $cl(V) \subseteq M^s$ is $\mathcal{L}_A$-definable.

From now on, and unless stated otherwise, $\widetilde{T}$ satisfies Assumptions (I)-(III) and $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ is a sufficiently saturated model of $\widetilde{T}$.

Remark 2.3. (i) Assumptions (I)-(III) are analogous to Assumptions (1)-(3) from [3, Theorem 3]. Here, however, we insist on having some control on the defining parameters. Moreover, an easy argument shows that under our assumptions, (3) from [3, Theorem 3] holds, but without the additional condition that the set S mentioned there be $\emptyset$-definable.

(ii) Assumption (III) indeed guarantees that open definable sets are $\mathcal{L}$-definable, see Lemma 2.5 below.

(iii) We do not know whether assumptions (I) and (III) imply (II).

Notation-terminology. The topological closure of a set $X \subseteq M^n$ is denoted by $cl(X)$. If $X, Y \subseteq M$ and $b = (b_1, \dots, b_n)$, we sometimes write $X \cup b$ or Xb for $X \cup \{b_1, \dots, b_n\}$, and XY for $X \cup Y$. If $\varphi(x, y)$ is an $\mathcal{L}(P)$-formula and $a \in M^n$, then we write $\varphi(M^m, a)$ for

$$\{b \in M^m \ : \ \widetilde{\mathcal{M}} \models \varphi(b, a)\}.$$

Similarly, given any subset $X \subseteq M^m \times M^n$ and $a \in M^n$, we write X_a for

$$\{b \in M^m \ : \ (b, a) \in X\}.$$

For convenience, we sometimes write $f(t, X)$ for $f(\{t\} \times X)$. If $m \leq n$, then $\pi_m : M^n \to M^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1}, unless stated otherwise. By an open box in M^k, or a k-box, we mean a set $I_1 \times \cdots \times I_k \subseteq M^k$, where each $I_j \subseteq M$ is an open interval. By dimension of an $\mathcal{L}$-definable set we mean its usual o-minimal dimension, and the notions of dcl-independence, dcl-rank and dcl-generics are the usual notions attached to the dcl-pregeometry (see, for example, [39]). A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g \in S} \{g\} \times J_g$ is definable, disjoint if every two elements of it are disjoint, and small if S is small. We often identify $\mathcal{J}$ with $\bigcup_{g \in S} \{g\} \times J_g$. If for each $t \in T$, $\mathcal{J}_t = \{J_{g,t}\}_{g \in S_t}$ is a family of sets, we call $\{\mathcal{J}_t\}_{t \in T}$ definable if $\bigcup_{t \in T, g \in S_t} \{(g, t)\} \times J_{g,t}$ is definable.

Our examples are often given for structures over the reals (such as Example 4.20 and the counterexample in Section 5.2). But they can easily be adopted to the current, saturated setting, by moving to an elementary extension.

2.2. **Examples.**

Dense pairs. The first example we wish to consider is dense pairs of o-minimal structures. A *dense pair* $\langle \mathcal{M}, \mathcal{N} \rangle$ is a pair of models of T such that $\mathcal{N} \neq \mathcal{M}$, but $\mathcal{N}$ is dense in $\mathcal{M}$. Let $\widetilde{T} = T^d$ be the theory of dense pairs in the language $\mathcal{L}(P)$. By [12], T^d is complete and every model of T^d satisfies (I) and (II) ([12, Lemma 4.1] and [12, Theorem 1], respectively).

It is left to explain why (III) holds in dense pairs. Here we apply [6, Corollary 3.1]. Let A be a parameter set such that $A \setminus N$ is dcl-independent over N. Set

$$D := \{a \in M \ : \ a \text{ is dcl-independent over } N \cup A\}.$$

It is easy to see that D and A satisfy Assumptions (1) and (2) of [6, Corollary 3.1]. It is left to show that also the third assumption of that corollary holds. Towards that goal, recall the following notation from [12]. Given $\mathcal{M}, \mathcal{N}, \mathcal{O}, \mathcal{Q} \models T$ with $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{Q}$ and $\mathcal{M} \subseteq \mathcal{O} \subseteq \mathcal{Q}$, we say that $\mathcal{N}$ and $\mathcal{O}$ are *free over* $\mathcal{M}$ (in $\mathcal{Q}$) if every subset $Y \subseteq N$ that is dcl-independent over $\mathcal{M}$ is also dcl-independent over $\mathcal{O}$.

Proposition 2.4. *Let $a \in D$. Then the $\mathcal{L}(P)$-type of a over A is implied by the $\mathcal{L}$-type over A and the fact that $a \in D$.*

Proof. Let $\langle \mathcal{M}, \mathcal{N} \rangle \models T^d$ be κ-saturated, where $\kappa > |T|$. Let Γ be the set of all isomorphisms $i : \langle \mathcal{M}_1, \mathcal{N}_1 \rangle \to \langle \mathcal{M}_2, \mathcal{N}_2 \rangle$ between substructures of $\langle \mathcal{M}, \mathcal{N} \rangle$ such that $|M_1| < \kappa, |M_2| < \kappa$, $\mathcal{M}_1$ and $\mathcal{N}$ are free over $\mathcal{N}_1$ and $\mathcal{M}_2$ and $\mathcal{N}$ are free over $\mathcal{N}_2$. By [12, Claim on p. 67], Γ has the back-and-forth property. Let $a, b \in D$ such that a and b satisfy the same $\mathcal{L}$-type over A. Then there is an $\mathcal{L}$-isomorphism

$$i : \mathrm{dcl}(a \cup A) \to \mathrm{dcl}(b \cup A).$$

Since both a and b are dcl-independent over $N \cup A$, the isomorphism expands to an isomorphisms

$$i : \big\langle \operatorname{dcl}(a \cup A), \operatorname{dcl}(A) \cap \mathcal{N} \big\rangle \to \big\langle \operatorname{dcl}(b \cup A), \operatorname{dcl}(A) \cap \mathcal{N} \big\rangle$$

of substructures of $\langle \mathcal{M}, \mathcal{N} \rangle$. Since $a \cup (A \setminus N)$ is dcl-independent over N, $\operatorname{dcl}(a \cup A)$ and $\mathcal{N}$ are free over $\operatorname{dcl}(N) \cap \mathcal{N}$. By the same argument $\operatorname{dcl}(b \cup A)$ and $\mathcal{N}$ are free over $\operatorname{dcl}(A) \cap \mathcal{N}$. Hence $i \in \Gamma$. Since Γ is a back-and-forth system, a and b satisfy the same $\mathcal{L}(P)$-type over A. ☐

Groups with the Mann property. Let Γ be a dense subgroup of $\mathbb{R}_{>0}$ that has the *Mann property*, that is for every $a_1, \dots, a_n \in \mathbb{Q}^\times$, there are finitely many $(\gamma_1, \dots, \gamma_n) \in \Gamma^n$ such that $a_1\gamma_1 + \cdots + a_n\gamma_n = 1$ and $\sum_{i \in I} a_i\gamma_i \neq 0$ for every nonempty subset I of $\{1, \dots, n\}$. Every multiplicative subgroup of finite rank in $\mathbb{R}_{>0}$ has the Mann property, see [23].

We assume that for every prime number p, the subgroup of p-th powers in Γ has finite index in Γ. Let $\mathcal{L}$ be the language of ordered rings augmented by a constant symbol for each $\gamma \in \Gamma$. Let T be the theory of $\langle \mathbb{R}, (\gamma)_{\gamma \in \Gamma} \rangle$ in that language and let $\widetilde{T} = T(\Gamma)$ be the theory of $\langle \mathbb{R}, (\gamma)_{\gamma \in \Gamma}, \Gamma \rangle$ in the language $\mathcal{L}(P)$. By [14, Theorem 7.5], every model of $T(\Gamma)$ satisfies (II). A proof that every model satisfies (I) is in [25, Proposition 3.5].

Again, we show that (III) follows from [6, Corollary 3.1]. Let $\langle \mathcal{M}, P \rangle \models T(\Gamma)$. Let A for every parameter set A such that $A \setminus P$ is dcl-independent over P. Set

$$D := \{a \in M \ : \ a \text{ is dcl-independent over } P \cup A\}.$$

One can check easily that assumptions (1) and (2) of [6, Corollary 3.1] follow from the o-minimality of T. Finally it is easy to see that almost the same proof as for Proposition 2.4, just using the back-and-forth system in the proof of Theorem 7.1 in [14] instead of [12, Claim on p. 67], shows that assumption (3) of [6, Corollary 3.1] is satisfied as well.

There are several other closely related examples. In [27] proper o-minimal expansions $\mathcal{R}$ of the real field and finite rank subgroups Γ of $\mathbb{R}_{>0}$ are constructed such that the structure $(\mathcal{R}, \Gamma)$ satisfies Assumptions (I)-(III). Indeed, the fact that these structures satisfy Assumptions (I) and (II) is immediate from results in [27]. Assumption (III) follows by the same argument as above. In [1, 25] certain expansions of the real field by subgroups of either the unit circle or an elliptic curve are studied. One can easily show using the above argument that these structures satisfy Assumptions (I)-(III) after adjusting their statements for the fact that P now lies in a 1-dimensional semialgebraic set in $\mathbb{R}^2$. Since no significant new argument is involved, we leave it to the reader to verify that our main results also hold in this slightly more general setting.

Independent sets. Let $\widetilde{T} = T^{\text{indep}}$ be an $\mathcal{L}(P)$-theory extending T by axioms stating that P is dense and dcl-independent. By [9], T^{indep} is complete and every model of T^{indep} satisfies (I) and (II) by [9, 2.1] and [9, 2.9], respectively. As usual, we show that (III) follows from [6, Corollary 3.1]. Let $\langle \mathcal{M}, P \rangle \models T^{\text{indep}}$. Let A be a parameter set such that $A \setminus P$ is dcl-independent over P. Set

$$D := \{a \in M \ : \ a \text{ is dcl-independent over } P \cup A\}.$$

From the o-minimality of T, assumptions (1) and (2) of [6, Corollary 3.1] follow easily as above. By [9, 2.12], assumption (3) of [6, Corollary 3.1] holds as well.

Non-examples.

(1) By Assumption (III), P must be dense in a finite union of open intervals and points. Indeed, the closure of P has to be $\mathcal{L}$-definable. Therefore, tame expansions of $\mathcal{M}$ by discrete sets, such as $\langle \mathbb{R}, 2^{\mathbb{Z}} \rangle$, do not belong to this setting.

(2) We do not know whether the theory of every expansion $\langle \mathcal{M}, P \rangle$ of an o-minimal structure $\mathcal{M}$ with o-minimal open core [8, 31] satisfies Assumptions (II) or (III). Assumption (I) does not hold in case P is a generic predicate.

(3) If $\widetilde{\mathcal{M}} = \langle M, <, +, \mathcal{P} \rangle$ is semi-bounded, that is, a pure ordered group expanded by the structure of a real closed field $\mathcal{P} = \langle P, \oplus, \otimes \rangle$ on some bounded open interval $P \subseteq M$, then Assumptions (II) and (III) hold by [17], but (I) does not.

2.3. **$\mathcal{L}$-definability.** In general, an $\mathcal{L}$-definable set X which is also A-definable need not be $\mathcal{L}_A$-definable. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields, and $x, y \in M \setminus P$ such that there are (unique) $g, h \in P$ with $x = g + hy$. Then $\{g\}$ is $\mathcal{L}$-definable and $\{x, y\}$-definable, but in general not $\mathcal{L}_{\{x,y\}}$-definable. The following lemma, however, implies, in particular, that every such X is always $\mathcal{L}_{A \cup P}$-definable.

Lemma 2.5. *Let $X \subseteq M^n$ be an A-definable set. Then there is a finite $B \subseteq A$ such that X is $B \cup P$-definable and B is* dcl*-independent over P. Hence, by Assumption (III), $cl(X)$ is $\mathcal{L}_{A \cup P}$-definable. In particular, if X is closed (or open), then it is $\mathcal{L}_{A \cup P}$-definable.*

Proof. Without loss of generality, we can assume that A is finite. Let $B \subseteq A$ be a maximal subset of A that is dcl-independent over P. Suppose $B = \{b_1, \dots, b_k\}$. Hence for every $a \in A \setminus B$, there are $g_a \in P^l$ and an $\mathcal{L}_\emptyset$-definable map $h : M^{l+k} \to M$ such that $h(g_a, b_1, \dots, b_k) = a$. Set $H = \{g_a : a \in A \setminus B\}$. Since X is A-definable, it is also $B \cup H$-definable. □

A positive answer to the following open question would give better control to the set of parameters (see also after Corollary 3.23 below).

Question 2.6. *For X as above, are there finite $B \subseteq A$ and $H \subseteq P \cap \mathrm{dcl}_{\mathcal{L}(P)}(A)$, such that X is $B \cup H$-definable and B is* dcl*-independent over P?*

By [9, 2.26], Question 2.6 admits a positive answer when $\widetilde{T} = T^{\text{indep}}$. However, we do not know the answer even when $\widetilde{T} = T^d$.

The reader might wonder whether for every definable subset X of P^l there is an $\mathcal{L}$-definable set $Y \subseteq M^n$ such that $X = Y \cap P^n$. While this is true for dense pairs by [12, Theorem 2(2)], this fails in examples arising from groups with the Mann property (see [3, Proposition 57]).

Although all our known examples that satisfy Assumptions (I)-(III) have NIP (see [2, 26]), the following question stands open.

Question 2.7. *Do Assumptions (I)-(III) imply that $\widetilde{T}$ has NIP?*

2.4. **Basic facts for $\mathcal{L}$-definable and small sets.** We include some basic facts that will be used in the sequel.

Fact 2.8. *Let $f : X \subseteq M^m \to M^n$ be a finite-to-one $\mathcal{L}$-definable function. Then there is a finite partition $X = X_1 \cup \dots \cup X_k$ into definable sets such that each $f_{\restriction X_i}$ is injective.*

Proof. Standard. □

Fact 2.9. *Let $f : A \subseteq M^m \to M^n$ be an $\mathcal{L}$-definable function. Let*

$$X_f = \{a \in A : f^{-1}(f(a)) \text{ is finite}\}.$$

Then $\dim f(A \setminus X_f) < \dim A$.

Proof. Let $R = f(A\setminus X_f)$. By definition of X_f, for every $r \in R$, $f^{-1}(r)$ has dimension > 0. Since $A \setminus X_f$ equals the disjoint union $\bigcup_{r\in R} f^{-1}(r)$, we have by standard properties of dimension:

$$\dim(A \setminus X_f) \geq \min_r \dim f^{-1}(r) + \dim R.$$

Hence, $\dim A \geq 1 + \dim R$ and $\dim R < \dim A$. □

Fact 2.10. *If $X, Z, I \subseteq M^m$ are definable sets, and X is co-small in Z, then $X \cap I$ is co-small in $Z \cap I$.*

Proof. Immediate from the definitions. □

3. Small sets

In this section we establish properties of small sets that will be important in the proof of the Structure Theorem. The two most crucial results are Lemma 3.7 and Corollary 3.27 below.

3.1. **Families of small sets and P-boundness.**

With the exception of Lemma 3.7 below, the results of this section were either established in [3] or are minor improvements of results in [3]. Since the assumptions in [3] differ from ours, we reprove the results here. Most of the proofs are direct adjustments from those in [3], but are included for the convenience of the reader. They often involve induction on formulas whose base step deals with a 'basic' set defined next.

Definition 3.1. A subset $X \subseteq M^n$ is called *basic over A* if it is of the form

$$\bigcup_{g\in P^m} \varphi(M, g),$$

for some $\mathcal{L}_A$-formula. We say X is *basic* if it is basic over some parameter set A.

Note that by Assumption (II) every definable set is a boolean combination of basic sets.

Lemma 3.2. *Let $p \in \mathbb{N}$. For $j = 1, \dots, p$, let $\{S_{1,j,t}\}_{t\in M^l}, \{S_{2,j,t}\}_{t\in M^l}$ be A-definable families of subsets of P^n. Let $f_1, \dots, f_p, h_1, \dots, h_p : M^{n+l} \to M$ be A-definable functions. Then there are A-definable families $\{Q_{j,t}\}_{t\in M^l}, \{R_{j,t}\}_{t\in M^l}$ of subsets of P^n, for $j = 1, \dots, p$, such that for every $t \in M^l$,*

$$\bigcup_j f_j(S_{1,j,t}, t) \cap \bigcup_j h_j(S_{2,j,t}, t) = \bigcup_j f_j(Q_{j,t}, t),$$

$$\Big(M \setminus \bigcup_j f_j(S_{1,j,t}, t)\Big) \cup \bigcup_j h_j(S_{2,j,t}, t) = M \setminus \bigcup_j f_j(R_{j,t}, t).$$

Proof. Set

$$Q_{j,t} := \{g \in S_{1,j,t} : \bigvee_{i=1}^{p} \exists g' \in S_{2,i,t}\ h_i(g', t) = f_j(g, t)\}$$

and

$$R_{j,t} := \{g \in S_{1,j,t} : \bigwedge_{i=1}^{p} \forall g' \in S_{2,i,t}\ h_i(g',t) \neq f_j(g,t)\}.$$

□

Lemma 3.3. *Let $\{X_t\}_{t\in M^l}$ be an A-definable family of subsets of M. Then there are $m, n, p \in \mathbb{N}$ and for each $i = 1, \ldots, m$ there are*

- *an A-definable family $\{S_{i,j,t}\}_{t\in M^l}$ of subsets of P^n, for each $j = 1, \ldots, p$,*
- *$\mathcal{L}_A$-definable functions $h_{i,1}, \ldots, h_{i,p} : M^{n+l} \to M$,*
- *an A-definable function $a_i : M^l \to M \cup \{\infty\}$,*

such that for $t \in M^l$,

(i) *$-\infty = a_0(t) \le a_1(t) \le \cdots \le a_m(t) = \infty$ is a decomposition of M, and*
(ii) *one of the following holds:*
  (a) *$[a_{i-1}(t), a_i(t)] \cap X_t = V_{i,t}$ or*
  (b) *$[a_{i-1}(t), a_i(t)] \cap X_t = (M \setminus V_{i,t}) \cap [a_{i-1}(t), a_i(t)]$,*
  where $V_{i,t} = \bigcup_j h_{i,j}(S_{i,j,t}, t)$.

Proof. First consider a definable family of basic sets, say $(\mathbb{D}_t)_{t\in M^l}$, that is a definable family of the form

$$\mathbb{D}_t = \bigcup_{g\in P^n} \varphi(M, g, t),$$

where $\varphi(x, y, z)$ is an $\mathcal{L}_A$-formula and $t \in M^l$. By cell decomposition, there are two finite sets J_1, J_2, $\mathcal{L}_A$-definable cells $(Y_{1,j})_{j\in J_1}$ and $(Y_{2,j})_{j\in J_2}$ in M^{n+l} and $\mathcal{L}_A$-definable functions $(f_{1,j})_{j\in J_1}$, $(f_{2,j})_{j\in J_2}$ and $(f_{3,j})_{j\in J_2}$ from M^{n+l} to M such that

$$\mathbb{D}_t = \bigcup_{j\in J_1} f_{1,j}(Y_{1,j,t} \cap P^n, t) \cup \bigcup_{j\in J_2} \bigcup_{g\in Y_{2,j,t}\cap P^n} \big(f_{2,j}(g,t), f_{3,j}(g,t)\big).$$

Without loss of generality, we can assume that $|J_1| = |J_2|$. Set $p := |J_1|$ and assume that $J_1 = J_2 = \{1, \ldots, p\}$. Set $U_t := \bigcup_{j\in J_2} \bigcup_{g\in Y_{2,j,t}\cap P^n} \big(f_{2,j}(g,t), f_{3,j}(g,t)\big)$. Note that U_t is open. By Assumption (III), U_t is a finite union of open intervals. Since finitely many intervals only have finitely many endpoints and U_t is At-definable, the endpoints of the intervals of U_t are At-definable. Let V_t be the topological closure of $\bigcup_{j\in J_1} f_{1,j}(Y_{1,j,t} \cap P^n, t)$. By Assumption (III) again, V_t is $\mathcal{L}$-definable. Hence it is a finite union of intervals and points. Since there are only finitely many endpoints and V_t is At-definable, these endpoints are At-definable. Hence we have a decomposition of M

$$-\infty = a_0(t) \le a_1(t) \le \cdots \le a_m(t) = \infty$$

such that either

- $(a_{i-1}(t), a_i(t)) \cap \mathbb{D}_t = (a_{i-1}(t), a_i(t))$ or
- $(a_{i-1}(t), a_i(t)) \cap \mathbb{D}_t = (a_{i-1}(t), a_i(t)) \cap \bigcup_{j\in J_1} f_{1,j}(Y_{1,j,t} \cap P^n, t)$.

In the first case set $S_{i,j,t} := \emptyset$ and set $h_{i,j}(x, y) = 0$ for all $(x, y) \in M^{n+l}$. In the second case set

$$S_{i,j,t} := \{g \in Y_{1,j,t} \cap P^n \ : \ f_{1,j}(g,t) \in (a_{i-1}(t), a_i(t))\},$$

and set $h_{i,j} = f_{1,j}$. By compactness, we can find an $m \in \mathbb{N}$ that works for every $t \in M^l$. Hence (i)-(ii) holds for $(\mathbb{D}_t)_{t\in M^n}$.

By Assumption (II) it is enough to check that if the statement of the Lemma holds for two definable $(X_t)_{t\in M^l}$ and $(Z_t)_{t\in M^l}$, then it also holds for $(M \setminus X_t)_{t\in M^l}$ and $(X_t \cup Z_t)_{t\in M^l}$. So suppose that the statement holds for $(X_t)_{t\in M^l}$ and $(Z_t)_{t\in M^l}$.

It is immediate that the conclusion holds for $(M \setminus X_t)_{t\in M^l}$ as well. It is easy to check that Lemma 3.2 implies that the conclusion also holds for $(X_t \cup Z_t)_{t\in M^l}$. □

Remark 3.4. The sets $V_{i,t}$ above are small, since P is small (Assumption (I)). Hence:

(a) the set
$$\{t \in M^n \ : \ X_t \cap [a_{i-1}(t), a_i(t)] \text{ is small}\}$$
is equal to
$$\{t \in M^n \ : \ X_t \cap [a_{i-1}(t), a_i(t)] = V_{i,t}\}.$$
Hence, it is A-definable. In particular, the set of all $t \in M^n$ such that X_t is small is A-definable.

(b) the set of $(t, a_i(t))$ for which X_t is small in $(a_{i-1}(t), a_i(t))$ is A-definable.

We will make use of the following consequence of Lemma 3.3.

Corollary 3.5. *Let $\{X_t\}_{t\in I}$ be an A-definable family of subsets of M, where each $X_t \subseteq M$ is small and $I \subseteq M^n$. Then there are $m \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $h_j : V_j \subseteq M^{m+n} \to M$ and A-definable families $\{S_{j,t}\}_{t\in I}$ of sets $S_{j,t} \subseteq P^m$, $j = 1, \dots, p$, such that for every $t \in I$, $X_t = \bigcup_j h_j(S_{j,t}, t)$.*

Proof. Without requiring the continuity of the h_j's, the statement is immediate from Lemma 3.3. Now, to get the continuity, apply the cell decomposition theorem for o-minimal structures to get, for each j, cells $V_{j,1}, \dots, V_{j,s(j)}$ such that h_j is continuous on each $V_{j,i}$. Let $S'_{j,i,t} := S_{j,t} \cap V_{j,i} \subseteq P^m$. We have
$$X_t = \bigcup_{j,i} h_j(S'_{j,i,t}, t),$$
as required. □

The following example shows that in the last corollary the set $S_{j,t}$ has to depend on t.

Example 3.6. Let $\widetilde{\mathcal{M}} \models T^d$. For every $a \in M^{>0}$, let $X_a = P \cap (0, a)$, and
$$X = \bigcup_{a\in M^{>0}} \{a\} \times X_a.$$
Let h_j and $S_{j,a}$ be as in Corollary 3.5, and assume towards a contradiction that all $S_{j,a}$'s equal some S_j. So for every $a \in M^{>0}$,
$$(*) \qquad (0, a) \cap P = \bigcup_j h_j(S_j, a).$$
Take $p \in S_j$. By o-minimality, $h_j(p, -)$ is eventually continuous close to 0. Since $h_j(p, M^{>0}) \subseteq P$ by $(*)$ and P is codense in M, $h_j(p, -)$ is eventually constant close to 0. That is, there is $a_p > 0$ and $c_p \in P$, such that for every $0 < a < a_p$, $h_j(p, a) = c_p$. Thus, if $0 < a < c_p$, we have $h_j(p, a) = c_p \notin (0, a) \cap P$, a contradiction.

We now derive a few corollaries of the above results. The next lemma shows how to turn a family $X = \{X_a\}_{a\in C}$ of small sets into a small family of subsets Z_g of C. This will be a crucial step in the proof of the Structure Theorem. There, we will further need to replace Z_{ig} by "cones", which are defined in Section 4.

Lemma 3.7. *Let $X = \bigcup_{a\in C}\{a\} \times X_a$ be A-definable where each $X_a \subseteq M$ is small, non-empty, and $C \subseteq M^n$. Then there are $l, m \in \mathbb{N}$, and for each $i = 1, \dots, l$,*

- *an $\mathcal{L}_A$-definable continuous function $h_i : V_i \subseteq M^{m+n} \to M^{n+1}$,*

- *an A-definable small set $S_i \subseteq M^m$, and*
- *an A-definable set $Z_i \subseteq S_i \times C$ contained in V_i,*

such that for

$$U_i = h_i\left(\bigcup_{g\in S_i}\{g\}\times Z_{ig}\right)$$

we have

(1) *$X = U_1 \cup \dots \cup U_l$ is a disjoint union,*
(2) *for every i and $g \in S$, $h_i(g,-) : V_{ig} \subseteq M^n \to M^{n+1}$ is injective,*
(3) *$C = \bigcup_{i,g} Z_{ig}$.*

Proof. We first observe that there are $m, p \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $h_i : V_i \subseteq M^{m+n} \to M$ and A-definable families Y_i of small sets $Y_{ia} \subseteq P^m$, $i = 1, \dots, p$, such that for every $a \in I$,

(1) $X_a = \bigcup_i h_i(Y_{ia}, a)$
(2) $\{h_i(Y_{ia}, a)\}_{i=1,\dots,p}$ are disjoint.

Indeed, this follows from Corollary 3.5; for (2), recursively replace Y_{ia}, $1 < i \le p$, with the set consisting of all $z \in Y_{ia}$ such that $h_i(z,a) \notin h_j(Y_{ja}, a)$, $0 < j < i$. We now have:

$$X = \bigcup_{a\in C}\{a\}\times X_a = \bigcup_i \bigcup_{a\in C}\{a\}\times h_i(Y_{ia}, a).$$

For every i, let $S_i = P^m$. For every i and $g \in P^m$, let

$$U_i = \bigcup_{a\in C}\{a\}\times h_i(Y_{ia}, a),$$

which are also disjoint, and

$$Z_{ig} = \{a \in C : g \in Y_{ia}\}.$$

Since h_i and $\{Y_{ia}\}_{a\in C}$ are A-definable, so are U_i and $\{Z_{ig}\}_{g\in S_i}$. We have $C = \bigcup_{i,g} Z_{ig}$. Consider now the $\mathcal{L}_A$-definable continuous map $\hat{h}_i : V_i \subseteq M^{m+n} \to M^{n+1}$ with

$$\hat{h}_i(g,a) = (a, h_i(g,a)).$$

Then

$$U_i = \hat{h}_i\left(\bigcup_{g\in S_i}\{g\}\times Z_{ig}\right)$$

works. □

Remark 3.8. As the last proof shows, in fact we obtain $S_i = P^m$. We decided, however, to keep the current formulation because the proof can then be adopted in similar situations (such as in Lemma 5.11 below). Had we kept the stronger formulation ($S_i = P^m$), what follows would result to a Structure Theorem 5.1 where in Definition 4.3 of a cone we could require $S \subseteq P^m$. However, we recover this information anyway, see Remark 4.5(7).

Let us illustrate Lemma 3.7 with an example.

Example 3.9. Let $\widetilde{\mathcal{M}} \models T^d$. For every $a \in M^{>0}$, let $X_a = P \cap (0, a)$, and

$$X = \bigcup_{a\in M^{>0}}\{a\}\times X_a.$$

Then we can turn X into a small union of ($\mathcal{L}$-definable images of) large subsets of M, as follows. For every $g \in P$, let

$$J_g = \{a \in M : a > g\}.$$

Then

$$X = h\left(\bigcup_{g\in P}\{g\}\times J_g\right),$$

where $h : M^2 \to M^2$ switches the coordinates, $h(x, y) = (y, x)$. In this case, X is in fact seen to be 1-cone (according to Definition 4.3 below).

We now turn to examine better the notion of smallness.

Definition 3.10. A set $X \subseteq M^n$ is *P-bound* over A, if there is an $\mathcal{L}_A$-definable function $f : M^m \to M^n$ such that $X \subseteq f(P^m)$. We omit A if we do not want to specify the parameters.

Lemma 3.11. *An A-definable set is small if and only if it is P-bound over A.*

Proof. Since P is small, it follows immediately that every P-bound set is small. For the other direction, observe first that, by Corollary 3.5, every A-definable small subset of M is P-bound over A. Now let $X \subseteq M^n$ be A-definable, and let $\pi_i : M^n \to M$ be the projection onto the i-th coordinate. If X is small, so is $\pi_i(X)$ for $i = 1, \dots, n$. Since each A-definable small subset of M is P-bound over A, so is $\pi_i(X)$. Hence $\prod_{i=1}^n \pi_i(X)$ is P-bound over A and so is $X \subseteq \prod_{i=1}^n \pi_i(X)$. □

We show that in the definition of largeness and P-boundedness, we can replace $\mathcal{L}$-definability by definability. Recall from geometric stability theory that given two definable sets $X \subseteq M^n$ and $Y \subseteq M^k$, X is called *Y-internal* over A if there is an A-definable $f : M^{mk} \to M^n$ such that $X \subseteq f(Y^m)$.

Corollary 3.12. *Let X be a definable set.*

(1) *X is P-bound over A if and only if it is P-internal over A.*
(2) *X is large if and only if an open interval is X-internal.*

Proof. By Lemma 3.11, Definition 2.1 and Assumption (I), it is easy to see that (1) implies (2). For (1), let $F : M^k \to M^n$ be A-definable such that $X \subseteq F(P^k)$. Without loss of generality, we may assume that $A \setminus P$ is dcl-independent over P. For each $g \in P^k$, the singleton $\{F(g)\}$ equals its topological closure. Since $F(g)$ is definable over $A \cup g$ and $(A \cup g) \setminus P$ is dcl-independent over P, we get by Assumption (III) that $\{F(g)\}$ is $\mathcal{L}_{A\cup g}$-definable. Hence, by compactness, there are finitely many $\mathcal{L}_A$-functions $F_1, \dots, F_l$ such that for all $g \in P^k$, $F(g) = F_i(g)$ for some i. Hence

$$F(P^k) \subseteq \bigcup_i F_i(P^k).$$

However, the right hand side is P-bound over A, and hence so is $F(P^k)$. □

The following is then immediate.

Corollary 3.13. *Let $f : X \to M^n$ be a definable injective function. Then X is small if and only if $f(X)$ is small.*

A stronger version of the Corollary 3.13 is provided by the invariance result in Corollary 5.3 below. Here are three more corollaries of Lemma 3.11.

Corollary 3.14. *Let $Y \subseteq M^m$ be small and let $(X_t)_{t\in Y}$ be a definable family of small sets of M^n. Then $\bigcup_{t\in Y} X_t$ is small.*

Proof. By Lemma 3.11 and compactness, there is a definable family of $\mathcal{L}$-definable functions $(f_t)_{t\in Y}$ such that $X_t \subseteq f_t(P^k)$ for each $t \in Y$. Again by Lemma 3.11, there is also an $\mathcal{L}$-definable function $g : P^l \to M^m$ such that $Y \subseteq g(P^l)$. Set $h : M^{k+l} \to M^n$ be the function that takes (x, y) to $f_{g(y)}(x)$. Then $\bigcup_{t\in Y} X_t \subseteq h(P^{k+l})$ and hence is P-bound. □

Corollary 3.15. *The union and cartesian product of finitely many small sets is small.*

Proof. Immediate from Lemma 3.11, Corollary 3.14 and the definitions. □

In the case of dense pairs, we obtain the following interesting result.

Corollary 3.16. *Assume $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ is a dense pair. Then every $\emptyset$-definable small set $X \subseteq M^n$ is contained in P^n.*

Proof. By Corollary 3.11, there is an $\mathcal{L}_\emptyset$-definable $f : M^m \to M^n$, such that $X \subseteq f(P^m)$. By [12, Lemma 3.1], $X \subseteq P^n$. □

3.2. **Definable functions outside small sets.** In this section we analyze the behavior of definable functions outside small, or rather *low*, sets. Note that Assumption (II) is not used in this section.

Definition 3.17. We denote by $I_n(A) \subseteq M^n$ the set of all tuples $a = (a_1, \dots, a_n) \in M^n$ that are dcl-independent over $P \cup A$.

Remark 3.18. (1) Note that $I_n(A)$ is $\mathcal{L}(P)_A$-type definable. Indeed, $a \in I_n(A)$ if and only if for all $0 \le i < n$, $m, l \in \mathbb{N}$ and $\mathcal{L}_A$-$(l + i)$-formula $\varphi(x, y)$, a satisfies:

$$\forall g \in P^l \text{ [if } \varphi(g, a_1, \dots, a_{i-1}, -) \text{ has } m \text{ realizations, then } \models \neg\varphi(g, a_1, \dots, a_i)].$$

(2) It is obvious that $I_n(A) = I_n(A \cup P)$ and $I_n(B) \supseteq I_n(A)$ for $B \subseteq A$.

Lemma 3.19. *Let $A \subseteq M$ that $A \setminus P$ is* dcl*-independent over P, and let $\varphi(x, y, z)$ be an $\mathcal{L}(P)_A$-formula. Then there are $\mathcal{L}_A$-formulas $\psi_1(x, y, z), \dots, \psi_k(x, y, z)$ such that for all $a \in I_m(A)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ with*

$$cl(\varphi(a, b, M^l)) = \psi_i(a, b, M^l).$$

Proof. Let $a = (a_1, \dots, a_m) \in I_m(A)$ and $b = (b_1, \dots, b_n) \in P^n$. It follows that

$$(A \cup \{a_1, \dots, a_m\} \cup \{b_1, \dots, b_n\}) \setminus P$$

is dcl-independent over P. Since $I_m(A)$ is $\mathcal{L}(P)_A$-type definable and P is definable, the statement of the lemma follows from compactness and Assumption (III). □

Proposition 3.20. *Let $F : M^m \times M^n \to M$ be A-definable. Then there are $\mathcal{L}_{A\cup P}$-definable continuous functions $F_i : Z_i \subseteq M^m \times M^n \to M$, $i = 1, \dots, k$, such that for all $a \in I_m(A)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ with $(a, b) \in Z_i$ and*

$$F(a, b) = F_i(a, b).$$

Moreover, if A is dcl*-independent over P, then the F_i's can be chosen to be $\mathcal{L}_A$-definable.*

Proof. By Lemma 2.5, there is a finite $B \subseteq A$ such that B is dcl-independent over P and F is $B \cup P$-definable. So $(B \cup P) \setminus P$ is also dcl-independent over P. Let $\varphi(x, y, z)$ be an $\mathcal{L}(P)_{B\cup P}$-formula that defines the graph of F. Hence by Lemma 3.19 there are $\mathcal{L}_{B\cup P}$-formulas $\psi_1(x, y, z), \dots, \psi_k(x, y, z)$ such that for all $a \in I_m(B)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ with

$$cl(\varphi(a, b, M)) = \psi_i(a, b, M).$$

Since $\varphi(a, b, M)$ is a single point, we have $\varphi(a, b, M) = \psi_i(a, b, M)$. Define $F_i : M^{m+n} \to M$ such that $F_i(a, b)$ is the unique $c \in M$ with $\psi_i(a, b, c)$ if such c exists, and 0 otherwise. Since ψ_i is an $\mathcal{L}_{B\cup P}$-formula, F_i is $\mathcal{L}_{B\cup P}$-definable. Thus we have $\mathcal{L}_{A\cup P}$-definable functions $F_1, \dots, F_k : M^{m+n} \to M$, such that for all $a \in I_m(A)$ and $b \in P^n$ there is $i \in \{1, \dots, k\}$ such that $F(a, b) = F_i(a, b)$. Using cell decomposition in o-minimal structures, we can find an $\mathcal{L}_{A\cup P}$-cell decomposition $C_1, \dots, C_l$ of M^{m+n} such that each F_i is continuous on each C_j. The conclusion of the lemma now holds with the kl-many functions of the form $F_i|_{C_j}$, where $i = 1, \dots, k$ and $j = 1, \dots, l$.

For the 'moreover' clause, if $A \setminus P$ is dcl-independent over P, we need not replace A by $B \cup P$ in the above proof, which then shows that no further parameters from P are needed. □

Corollary 3.21. *Let $F : P^n \to M$ be A-definable. Then there are $t \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous functions $F_i : Z_i \subseteq M^{t+n} \to M$ with Z_i a cell, $i = 1, \dots, k$, and $u \in P^t$, such that for all $b \in P^n$ there is $i \in \{1, \dots, k\}$ with $(u, b) \in Z_i$ and*

$$F(b) = F_i(u, b).$$

Proof. By Proposition 3.20 there are $\mathcal{L}_{A\cup P}$-definable continuous function $H_i : Y_i \subseteq M^n \to M$, $i = 1, \dots, k$, such that for every $b \in P^n$ there is $i \in \{1, \dots, k\}$ with $b \in Y_i$ and $F(b) = H_i(b)$. Now take $u \in P^t$ such that each H_i is $\mathcal{L}_{Au}$-definable. For $i = 1, \dots, k$, pick an $\mathcal{L}_A$-definable function $F_i : Z_i \subseteq M^{t+n} \to M$ such that $(Z_i)_u = Y_i$ and $F_i(u, b) = H_i(b)$ for each $b \in Y_i$. By applying cell decomposition to M^{t+n}, we may further assume that each F_i is continuous and Z_i is a cell. □

A slightly weaker version of Corollary 3.21 is known for dense pairs [12, Theorem 3(3)].

Definition 3.22. We call $X \subseteq M^n$, $n > 0$, *low over B* if there is $i \in \{1, \dots, n\}$ and $\mathcal{L}_B$-definable function $f : M^{n-1} \times M^l \to M$ such that

$$X = \{(a_1, \dots, a_n) \in M^n \ : \ \exists g \in P^l \ f(a_{-i}, g) = a_i\},$$

where $a_{-i} = (a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_n)$.

Note that if a set $X \subseteq M$ is low, then it is small and co-dense in M. Generalizations of this statement are obtained in Lemmas 4.14 and 4.31 below.

Corollary 3.23. *Let $F : M^n \to M$ be A-definable. Then there are $k, m, t \in \mathbb{N}$ and*

- *sets $X_j \subseteq M^n$ low over A, $j = 1, \dots, k$,*
- *$\mathcal{L}_A$-definable continuous functions $F_i : Z_i \subseteq M^{t+n} \to M$, $i = 1, \dots, m$,*
- *$u \in P^t$,*

such that for every $a \in M^n \setminus \bigcup_{j=1}^k X_j$, there is $i \in \{1, \dots, m\}$ with $a \in Z_i$ and

$$F(a) = F_i(u, a).$$

Proof. Note that $a \notin I_n(A)$ if and only if there are $i \in \{1, \ldots, n\}$, an $\mathcal{L}_A$-definable function $f : M^{l+(n-1)} \to M$ and $g \in P^l$, such that $f(a_{-i}, g) = a_i$. Hence $a \notin I_n(A)$ if and only if there is X low over A such that $a \in X$. By compactness and Proposition 3.20, there are $k, m \in \mathbb{N}$ and

- $\mathcal{L}_{A\cup P}$-definable functions $H_i : Y_i \subseteq M^n \to M$, $i = 1, \ldots, m$
- sets $X_j \in M^n$ low over A, $j = 1, \ldots, k$,

such that for every $a \in M^n \setminus (\bigcup_{j=1}^k X_j)$ there is $i \in \{1, \ldots, m\}$ with $a \in Y_i$ and $F(a) = H_i(a)$. Now take $u \in P^t$ such that each H_i is $\mathcal{L}_{Au}$-definable, and continue as in the proof of Corollary 3.21. □

Remark 3.24. It is natural to ask whether the extra parameter $u \in P^t$ in Corollary 3.23 can be chosen to be in $\operatorname{dcl}_{\mathcal{L}(P)}(A)$. When the answer to Question 2.6 is positive, then the same proof gives that u is $\mathcal{L}_{A\cup H}$-definable, for some $H \subseteq P \cap \operatorname{dcl}_{\mathcal{L}(P)}(A)$. So in particular, this holds when $\tilde{T} = T^{\text{indep}}$ (independent set). When $\tilde{T} = T^d$ (dense pairs), we do not know the answer.

Remark 3.25. If $A \setminus P$ is dcl-independent over P, then using the 'moreover' clause of Proposition 3.20, we can see that in Corollaries 3.21 and 3.23, we obtain $t = 0$ and u be the empty tuple.

Since low subsets of M are small, we can easily get the following corollary of 3.23. This corollary is already known for $\widetilde{T} = T^d$ by [12], with the aforementioned control in parameters also established in [45, Lemma 5]. We omit its proof since it is in fact a special case of Theorem 5.7(2) below.

Corollary 3.26. *Let $f : M \to M$ be A-definable. Then f agrees off some small set with an $\mathcal{L}_{A\cup P}$-definable function $F : M \to M$.*

The Structure Theorem below is intended, among others, to generalize this corollary to arbitrary definable maps $f : X \subseteq M^n \to M$ (see Theorem 5.7(2)). For the moment, using compactness, we directly get the following uniform version of Corollary 3.23.

Corollary 3.27. *Let $f : Z \times M^n \subseteq M^{m+n} \to M$ be an A-definable map. Then there are $p, t \in \mathbb{N}$ and for each $i = 1, \ldots, p$ there are*

- *an A-definable family $\{X_z^i\}_{z\in Z}$ of low subsets of M^n,*
- *an $\mathcal{L}_A$-definable continuous function $f_i : Z_i \subseteq M^m \times P^t \times M^n \to M$,*

such that for all $z \in Z$ there is $u \in P^t$ such that for all $a \in M^n \setminus \bigcup_i X_z^i$, there is $i \in \{1, \ldots, p\}$ with

$$f(z, a) = f_i(z, u, a).$$

Proof. The corollary follows easily from compactness and Corollary 3.23. □

4. Cones and large dimension

In this section, we introduce and analyze the two main objects of the paper, cones and large dimension.

4.1. **Cones.** As mentioned in the introduction, the notion of a cone is based on that of a supercone, which in its turn generalizes the notion of being co-small in an interval. Both notions, supercones and cones, are unions of specific families of sets, which not only are definable, but they are so in a very uniform way. The definitions appear to be quite technical in the beginning, but as it turns out they are in fact optimal in several ways (see Section 5.2, Question 5.14 and [18]).

Definition 4.1 (Supercones)**.** We define recursively the notion of a *supercone* $J \subseteq M^k$, $k \geq 0$, as follows:

- $M^0 = \{0\}$ is a supercone.
- A definable set $J \subseteq M^{n+1}$ is a supercone if $\pi(J) \subseteq M^n$ is a supercone and there are $\mathcal{L}$-definable continuous $h_1, h_2 : M^n \to M \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it.

Abusing terminology, we say that a supercone J is *A-definable* if J is an A-definable set and its closure is $\mathcal{L}_A$-definable.

Note that, for $k > 0$, the interior U of $cl(J)$ is an open cell, and for every $a \in \pi(J)$, J_a is contained in U_a and it is co-small in it.

We remind the reader that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g\in S}$ with $\bigcup_{g\in S}\{g\} \times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

Definition 4.2 (Uniform families of supercones)**.** Let $\mathcal{J} = \bigcup_{g\in S}\{g\} \times J_g \subseteq M^{m+k}$ be a definable family of supercones. We call $\mathcal{J}$ *uniform* if there is a cell $V \subseteq M^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \leq k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a V a *shell* for $\mathcal{J}$. Abusing terminology, we call a uniform family *A-definable*, if it is an A-definable family of sets and has an $\mathcal{L}_A$-definable shell.

A shell for $\mathcal{J}$ need not be unique. It is, however, canonical in the sense of Lemma 4.9 below. Note also that if $\mathcal{J}$ is uniform, then so is each projection $\pi_{m+j}(\mathcal{J})$.

Definition 4.3 (Cones)**.** A set $C \subseteq M^n$ is a *k-cone*, $k \geq 0$, if there are a definable small $S \subseteq M^m$, a uniform family $\mathcal{J} = \{J_g\}_{g\in S}$ of supercones in M^k, and an $\mathcal{L}$-definable continuous function $h : V \subseteq M^{m+k} \to M^n$, where V is a shell for $\mathcal{J}$, such that

(1) $C = h(\mathcal{J})$, and
(2) for every $g \in S$, $h(g, -) : V_g \subseteq M^k \to M^n$ is injective.

A *cone* is a k-cone for some k. Abusing terminology, we call a cone $h(\mathcal{J})$ *A-definable* if h is $\mathcal{L}_A$-definable and $\mathcal{J}$ is A-definable.

Definition 4.4 (Fiber $\mathcal{L}$-definable maps)**.** Let $C = h(\mathcal{J}) \subseteq M^n$ be a k-cone with $\mathcal{J} \subseteq M^{m+k}$, and $f : D \to M$ a definable function with $C \subseteq D$. We say that f is *fiber $\mathcal{L}$-definable with respect to C* if there is an $\mathcal{L}$-definable continuous function $F : V \subseteq M^{m+k} \to M$, where V is a shell for $\mathcal{J}$, such that

- $(f \circ h)(x) = F(x)$, for all $x \in \mathcal{J}$.

We call f *fiber $\mathcal{L}_A$-definable with respect to C* if F is $\mathcal{L}_A$-definable.

Remark 4.5.

(1) If $J \subseteq M^n$ is a supercone, then $\pi_m(J)$ is a supercone, and for every $t \in \pi_m(J)$, J_t is a supercone with closure $cl(J)_t$.
(2) Let $\{X_t\}_{t\in Z}$ be an A-definable family of subsets of M^n, $\{U_t\}_{t\in Z}$ an $\mathcal{L}_A$-definable family of subsets of M^n, and $\{C_t\}_{t\in Z}$ an A-definable family of cones in M^n. Using Remark 3.4(a), it is not hard to see that the sets
    - $\{t \in Z \ : \ X_t \text{ is a supercone with closure } cl(U_t)\}$
    - $\{t \in Z \ : X_t \text{ is a cone}\}$

are both A-definable.

(3) The 0-cones are exactly the small sets. Low subsets of M^n (Definition 3.22) are $n-1$ cones, but not every $(n-1)$-cone is low.

(4) The terminology of f being fiber $\mathcal{L}_A$-definable with respect to $C = h(\mathcal{J})$ is justified by the fact that, in that case, for every $g \in \pi(\mathcal{J})$, f agrees on $h(g, J_g)$ with an $\mathcal{L}_{Ag}$-definable map; namely $F \circ h(g, -)^{-1}$. But we require further that the family of these $\mathcal{L}_{Ag}$-definable maps is actually $\mathcal{L}_A$-definable and continuous. We illustrate this last point with Example 4.6 below. The same example also shows that the notion of being fiber $\mathcal{L}$-definable depends on h and $\mathcal{J}$.

(5) It is easy to see that if $C = h(\mathcal{J})$ is an A-definable k-cone and $f : C \to M$ fiber $\mathcal{L}_A$-definable with respect to C, then the graph of f is an A-definable k-cone. We will not make use of this fact.

(6) The closure of an A-definable cone $h(\mathcal{J})$ is $\mathcal{L}_A$-definable. Indeed, if $h : V \subseteq M^{m+k} \to M^n$ is as in Definition 4.3, then it is easy to check that $cl(h(\mathcal{J})) = cl(h(cl(\mathcal{J}) \cap V))$.

(7) We may replace S by a definable subset of P^l in the definition of a cone C. Indeed, let $h : V \subseteq M^{m+k} \to M^n$ be as in that definition. Since S is P-bound, there is an $\mathcal{L}$-definable $f : M^l \to M^m$ with $f(P^l) \supseteq S$. By partitioning C into finitely many cones, we may assume that for some cell $Z \subseteq M^l$, $f : Z \to \pi(V)$ is continuous and $S \subseteq f(Z \cap P^l)$. So we may replace S by $S' := f^{-1}(S) \cap Z \subseteq P^l$, and h by $H : \bigcup_{g \in Z} \{g\} \times V_{f(g)} \to M^n$ with $H(x, y) = h(f(x), y)$. We decided, however, to keep the current definition because we can then adopt it in similar situations (such as Theorem 5.12 below). See also Remark 3.8.

Example 4.6. Consider a dense pair $\langle \mathcal{M}, P \rangle$ of real closed fields and let $S = P + aP$ for some $a \notin P$. The following map is taken from [12]. Let $f : S \to M$ be the a-definable map given by $f(x) = r$, where $x = r + sa$ for some (unique) $r, s \in P$. Then, clearly, for every $x = r + sa \in S$, $f(x, -) : M^0 \to M$ agrees with the $\mathcal{L}_r$-definable map H_r map given by $H_r(x, -) = r$. However, the family of maps H_r is not $\mathcal{L}$-definable. Now re-write S as the a-definable cone

$$S \times M = h(P^2)$$

where $h(p, q) = p + aq$, and let $F : M^2 \to M$ be the projection onto the first coordinate. Then, for every $(p, q) \in P^2$, we have

$$f(p + aq) = p = F(p, q),$$

witnessing that f is fiber $\mathcal{L}_a$-definable with respect to $h(P^2)$.

We next observe several easy consequences of the definitions that will be used in the proof of the Structure Theorem. The first lemma draws a connection between cones and the dcl-rank over tuples over P. Further results of this sort will be explored in Section 6.

Lemma 4.7. *Let $a \in M^n$ and $A \subseteq M$. Then*

$$\text{dcl-rank}(a/AP) = \min\{k \in \mathbb{N} : a \textit{ is contained in an } A\textit{-definable } k\textit{-cone}\}.$$

Proof. $(\leq)$. This follows easily from the definition of a k-cone.
$(\geq)$. Let $a = (a_1, \dots, a_n)$ and set $k = \text{dcl-rank}(a/AP)$. We will find an A-definable k-cone that contains a. Without loss of generality, we may assume that

dcl-rank$((a_1,\dots,a_k)/AP) = k$. Hence there are an $\mathcal{L}_A$-definable $Z \subseteq M^{l+n}$ and $s \in P^l$ such that $(s,a) \in Z$ and $\dim Z_s = k$. By cell decomposition in o-minimal structures, there are an $\mathcal{L}_A$-definable cell $X \subseteq M^{l+k}$ and a continuous $\mathcal{L}_A$-definable function $h : X \to M^n$ such that

- $\{(x, h(x,y)) : (x,y) \in X\} \subseteq Z$
- $(s, a_1, \dots, a_k) \in X$ and $h(s, a_1, \dots, a_k) = a$,
- X_y is an open cell and $h(y, -)$ is injective for each $y \in \pi_l(X)$.

In particular, $(X_y)_{y\in\pi_l(X)}$ is a uniform family of supercones with closure $cl(X)$. Thus

$$h\left(\bigcup_{g\in P^l\cap\pi_l(X)} \{g\} \times X_g\right)$$

is a k-cone containing a. □

Lemma 4.8. *Let C be an A-definable 0-cone in M^n and $f : C \to M$ be A-definable. Then there is a finite collection $\mathcal{C}$ of A-definable 0-cones whose union is C and such that f is fiber $\mathcal{L}$-definable with respect to each cone in $\mathcal{C}$.*

Proof. Let S be A-definable small and $h : Z \subseteq M^m \to M^n$ be $\mathcal{L}_A$-definable and continuous such that $h(S) = C$. We may assume that $S \subseteq P^l$, for some l. Indeed, since S is P-bound over A, one can easily see that S is a finite union of sets $\sigma(S')$, where $S' \subseteq P^l$ is A-definable and $\sigma : W \to M^m$ is an $\mathcal{L}_A$-definable map. So C is a finite union of 0-cones of the form $h \circ \sigma(S')$.

Now, by Corollary 3.21 there are $k, t \in \mathbb{N}$ and, for $i = 1, \dots, k$, an $\mathcal{L}_A$-definable continuous function $F_i : Z_i \subseteq P^t \times M^l \to M$ with Z_i a cell, and $s \in P^t$, such that for all $g \in S$ there is $i \in \{1, \dots, k\}$ with $(f \circ h)(g) = F_i(s, g)$. Now set

$$S_i := \{(s, g) \in P^t \times S \ : \ (s, g) \in Z_i, (f \circ h)(g) = F_i(s, g)\}.$$

Set $\tau : M^t \times Z \to M^n$ to map (x, y) to $h(y)$. Then $\tau(S_i)$ is an A-definable 0-cone and f is fiber $\mathcal{L}_A$-definable with respect to $\tau(S_i)$. Moreover, $C = \bigcup_{i=1}^k \tau(S_i)$. □

Our next goal is to prove Lemmas 4.10 and 4.12 below, which will be used in the proof of the Structure Theorem $(1)_n$, Cases I and II, respectively. First, a lemma about shells.

Lemma 4.9. *Let $\mathcal{J} \subseteq M^{m+k}$ be an A-definable uniform family of supercones with an $\mathcal{L}_A$-definable shell V. Assume that $Z \subseteq M^{m+k}$ is an $\mathcal{L}_A$-definable cell containing $\mathcal{J}$. Then there are disjoint A-definable uniform families of supercones $\mathcal{J}_1, \dots, \mathcal{J}_n$ such that*

$$\mathcal{J} = \mathcal{J}_1 \cup \dots \cup \mathcal{J}_n,$$

and each $\mathcal{J}_i$ has an $\mathcal{L}_A$-definable shell $V_i \subseteq V \cap Z$.

Proof. First observe that for every $g \in \pi(\mathcal{J})$, $V_g \subseteq Z_g$. Indeed, $cl(V_g) = cl(\mathcal{J}_g) \subseteq cl(Z_g)$. Since V_g is an open cell, and Z_g is a cell too, this implies that $V_g \subseteq Z_g$.

Now let

$$D = \{g \in M^k : (V \cap Z)_g \text{ is an open cell}\}.$$

This set is $\mathcal{L}_A$-definable. Moreover, since for every $g \in \pi(\mathcal{J})$, $(V \cap Z)_g = V_g$, we obtain $\pi(\mathcal{J}) \subseteq D$. Let

$$D = D_1 \cup \dots \cup D_n$$

be a partition of D into $\mathcal{L}_A$-definable cells, and, for each i,

$$\mathcal{J}_i = \mathcal{J} \cap (D_i \times M^k)$$

and

$$Z_i = (V \cap Z) \cap (D_i \times M^k).$$

Since both V, Z are cells and $D_i \subseteq D$, it is not hard to see that each Z_i is a cell. It clearly also contains $\mathcal{J}_i$. Finally, for every $g \in D_i$ and $0 < j \leq k$, we have

$$cl(\pi_{m+j}(\mathcal{J}_i)_g) = cl(\pi_{m+j}(V)_g) = cl(\pi_{m+j}(V \cap Z)_g) = cl(\pi_{m+j}(Z_i)_g),$$

showing that Z_i is a shell for $\mathcal{J}_i$. □

We now prove that a suitable family of large subsets of M ranging over a k-cone gives rise to a $k+1$-cone.

Lemma 4.10. *Let $C \subseteq M^n$ be an A-definable k-cone, let $\{X_a\}_{a \in C}$ be an A-definable family of subsets of M. Assume that $h_1, h_2 : C \to M \cup \{\pm\infty\}$ are fiber $\mathcal{L}_A$-definable with respect to C, and such that for all $a \in C$, X_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it. Then $\bigcup_{a \in C} \{a\} \times X_a$ is a finite disjoint union of A-definable $k+1$-cones.*

Proof. Suppose that $C = h(\mathcal{J})$ for some uniform family $\mathcal{J} = \{J_g\}_{g \in S}$ of supercones in M^k with shell V and $\mathcal{L}_A$-definable continuous $h : U \subseteq M^{m+k} \to M^n$, where U is a cell containing $\mathcal{J}$. By the assumption on h_1 and h_2, there are $\mathcal{L}_A$-definable continuous functions $H_1, H_2 : Z \subseteq M^{m+k} \to M$, where Z is a cell containing $\mathcal{J}$, such that for every $g \in S$ and $t \in J_g$,

$$H_1(g,t) = h_1(h(g,t)) \text{ and } H_2(g,t) = h_2(h(g,t)).$$

By Lemma 4.9, we may assume that $V \subseteq U \cap Z$. Now set

$$V' := \{(g,t,x) \ : \ (g,t) \in V,\ H_1(g,t) < x < H_2(g,t)\}$$

and

$$J'_g := \bigcup_{t \in J_g} \{t\} \times X_{h(g,t)}.$$

It is easy to check that $\{J'_g\}_{g \in S}$ is a uniform family of supercones in M^{k+1} with closure $cl(V')$. Let $\tau : V \times M \to M^{n+1}$ map $((g,t),x)$ to $(h(g,t),x)$. For each $g \in S$ the function $\tau(g,-)$ is injective, because so is $h(g,-)$. Thus

$$\bigcup_{a \in C} \{a\} \times X_a = \tau\left(\bigcup_{g \in S} \{g\} \times J'_g\right)$$

is a $k+1$-cone. □

The proof of the Structure Theorem will run in parallel with its own uniform version (see Theorem 5.1(3) below), which prompts the following definition.

Definition 4.11 (Uniform families of cones)**.** Let $\mathcal{C} := \{C_t\}_{t \in X \subseteq M^m}$ be a definable family of k-cones in M^n. We call $\mathcal{C}$ *uniform* if there are

- an $\mathcal{L}$-definable continuous function $h : Z \subseteq M^{m+l+k} \to M^n$,
- a definable family $\{S_t\}_{t \in X}$ of small subsets of M^l,
- a uniform definable family of supercones $Y = \{Y_{t,g}\}_{t \in X, g \in S_t}$ in M^k

such that $Y \subseteq Z$ and

(i) $h(t,g,-) : Z_{t,g} \subseteq M^k \to M^n$ is injective for each $g \in S_t$,
(ii) $C_t = h\left(\{t\} \times \left(\bigcup_{g \in S_t} \{g\} \times Y_{t,g}\right)\right)$.

Abusing terminology, we call $\mathcal{C}$ *A-definable* if it is an A-definable family of sets, h is $\mathcal{L}_A$-definable, and $\{S_t\}_{t \in X}$ and $\{Y_{t,g}\}_{t \in X, g \in S_t}$ are A-definable.

We now prove that the union of a small uniform family of k-cones under a suitable map results again in a k-cone.

Lemma 4.12. *Let $\{C_t\}_{t\in K}$ be an A-definable uniform family of k-cones in M^n, with $K \subseteq M^m$ small, and let $\tau : W \subseteq M^{m+n} \to M^p$ be an $\mathcal{L}_A$-definable continuous map such that for each $t \in K$*

- $\{t\} \times C_t \subseteq W$,
- $\tau(t, -) : M^n \to M^p$ *is injective.*

Then $\tau\left(\bigcup_{t\in K}\{t\} \times C_t\right)$ is an A-definable k-cone in M^p.

Proof. Let $h : Z \subseteq M^{m+l+k} \to M^n$ be an $\mathcal{L}_A$-definable continuous function, $\{S_t\}_{t\in K}$ an A-definable family of small subsets of M^l, and $\{Y_{t,g}\}_{t\in K, g\in S_t}$ an A-definable family of supercones that witness that $\{C_t\}_{t\in K}$ is a uniform family of k-cones. Let $\sigma : Z \subseteq M^{m+l+k} \to M^p$ be defined by $\sigma(t, g, a) := \tau(t, h(t, g, a))$. We see directly that $\sigma(t, g, -)$ is injective, since $\tau(t, -)$ and $h(t, g, -)$ are injective. Note also that σ is $\mathcal{L}_A$-definable and continuous, since both h and τ are. Set

$$S := \bigcup_{t\in K}\{t\} \times S_t.$$

It is then straightforward to check that

$$\tau\left(\bigcup_{t\in K}\{t\} \times C_t\right) = \sigma\left(\bigcup_{(t,g)\in S}\{(t, g)\} \times Y_{t,g}\right)$$

is the desired k-cone. □

The following lemma will be used in the last step of the proof of the Structure Theorem, $(1)_n \Rightarrow (3)_n$. It follows easily from Definition 4.2 and the next observations. Let $X \subseteq M^{m+n}$ be a set. Then for every $0 < j \leq n$ and $g \in \pi_m(X)$, we have

$$\pi_{m+j}(X)_g = \pi_j(X_g).$$

Let $X, Y \subseteq M^n$ and $0 < j \leq n$. Then

$$cl(X) = cl(Y) \Rightarrow cl(\pi_j(X)) = cl(\pi_j(Y)).$$

Indeed, $\pi_j(X) \subseteq \pi_j(cl(Y)) \subseteq cl(\pi_j(Y))$.

Lemma 4.13. *Let $U \subseteq M^{m+l+k}$ be an A-definable cell. Let*

$$\mathcal{K} = \{J_{t,g}\}_{t\in Y,\, g\in S_t}$$

be an A-definable family of supercones $J_{t,g} \subseteq M^k$, where $Y \subseteq M^m$ and $S_t \subseteq M^l$. Assume that for every $0 < j \leq k$, $t \in Y$ and $g \in S_t$,

$$cl(J_{t,g}) = cl(U_{t,g}). \tag{1}$$

Then U is a shell for $\mathcal{K}$. In particular, $\mathcal{K}$ is an A-definable uniform family of supercones.

Proof. For every $t \in Y$ and $g \in S_t$, we have

$$cl(\mathcal{J}_{t,g}) = cl(U_{t,g}) \Rightarrow cl(\pi_j(\mathcal{J}_{t,g})) = cl(\pi_j(U_{t,g})) \Rightarrow$$
$$\Rightarrow cl(\pi_{m+l+j}(\mathcal{J})_{t,g}) = cl(\pi_{m+l+j}(U)_{t,g}),$$

as required. □

We finally include two lemmas that will be useful in the discussion of 'large dimension' in Section 4.3 below.

Lemma 4.14. *Let $J \subseteq M^n$, $n > 0$, be a supercone and $X \subseteq M^n$ a low set. Then $J \setminus X$ contains a supercone.*

Proof. Easy, following the definitions, by induction on n. □

Lemma 4.15. *Let $J \subseteq M^n$ be a supercone and $\{X_s\}_{s\in S}$ a small definable family of subsets of M^n such that $J = \bigcup_{s\in S} X_s$. Then some X_s contains a supercone in M^n.*

Proof. By induction on n. If $n = 0$, it is obvious. If $n > 0$, for every $s \in S$, let

$$Y_s := \{t \in \pi(X_s) : \text{ the fiber } (X_s)_t \text{ is large}\}.$$

By Remark 3.4(a), $\{Y_s\}_{s\in S}$ is a definable family of sets. By Corollary 3.14, we have $\pi(J) = \bigcup_{s\in S} Y_s$. By Inductive Hypothesis, some Y_s contains a supercone K. Since for every $t \in K$, $(X_s)_t$ is large, Remark 3.4(b) provides us with definable functions $h_1, h_2 : M^{n-1} \to M \cup \{\pm\infty\}$ such that for every $t \in \pi(X_s)$, $(X_s)_t$ is co-small in $(h_1(t), h_2(t))$. By Corollary 3.23, there are finitely many low sets in M^{n-1} off whose union h_1, h_2 are both $\mathcal{L}$-definable and continuous. Hence, by repeated use of Lemma 4.14, we obtain a supercone K' contained in K on which h_1, h_2 are both $\mathcal{L}$-definable. Therefore, the set

$$\bigcup_{t\in K'} \{t\} \times (X_s)_t \cap (h_1(t), h_2(t))$$

is a supercone contained in X_s. □

4.2. **$\mathcal{L}$-definable functions on supercones.** The goal of this section (Proposition 4.19(1) below) is to show that a supercone from M^m cannot be 'embedded' into M^n, for $n < m$. This will make meaningful the notion of 'large dimension' we introduce in Section 4.3.

Lemma 4.16. *Let $J \subseteq M^n$ be an A-definable supercone and $S \subseteq cl(J)$ an open $\mathcal{L}_A$-definable cell. Then $S \cap J$ is an A-definable supercone with closure $cl(S)$.*

Proof. We work by induction on n. For $n = 0$ it is obvious. Assume we know the statement for subsets of M^k, $k < n$, and let $J \subseteq M^n$ be a supercone and $S \subseteq cl(J)$ be an open $\mathcal{L}_A$-definable cell. Since $\pi(S) \subseteq \pi(cl(J)) \subseteq cl(\pi(J))$, the inductive hypothesis gives that $\pi(S) \cap \pi(J)$ is an A-definable supercone $K \subseteq M^{n-1}$ with closure $cl(\pi(S))$. Since for every $t \in K$, J_t is co-small in $cl(J)_t$, we have that $(S \cap J)_t = S_t \cap J_t$ is co-small in S_t. Hence $S \cap J = \bigcup_{t\in K}\{t\} \times (S \cap J)_t$ is a supercone with closure $cl(S)$. □

Lemma 4.17. *Let $K \subseteq M^{n+1}$ be a supercone. Then $cl(K) \setminus K$ is a finite union of sets of the form*

$$\bigcup_{g\in P^m} h(g, Z_g),$$

where $Z \subseteq P^m \times M^n$ is definable, $h : M^{m+n} \to M^{n+1}$ is $\mathcal{L}$-definable and each $h(g, -) : Z_g \to M^{n+1}$ is injective.

Proof. By induction on n. Denote $U = cl(K)$. For $n = 0$, this is clear since $U \setminus K$ is a small set and can be written as $h(P^m)$ with h as above. Now assume we know the statement for $k < n$, let $K \subseteq M^{n+1}$ be as above. We have:

$$U \setminus K = \left(\bigcup_{t\in\pi(K)} \{t\} \times (U_t \setminus K_t)\right) \cup \left(\bigcup_{t\in\pi(U)\setminus\pi(K)} \{t\} \times U_t\right). \tag{2}$$

By inductive hypothesis the second part is a finite union of sets of the form

$$T = \bigcup_{t\in X} \{t\} \times U_t,$$

where $X = \bigcup_{g\in P^m} h(g, Z_g)$, for suitable h. Observe that then

$$T = \bigcup_{g\in P^m} h'(g, W_g),$$

where $W_g = \bigcup_{v\in Z_g} \{v\} \times U_{h(g,u)}$ and $h'(g, v, u) = (h(g, v), u)$, as required.

The first part of the union in (2) is of the right form, as it follows immediately by applying Lemma 3.7. □

Before proving Proposition 4.19, we illustrate it with an example.

Example 4.18. Consider the function $f : M^2 \to M$ with $f(x_1, x_2) = x_1 + x_2$. Let $J_1 = M \setminus P$ and for all $t \in J_1$, $J_t = J_1 \cap (t, \infty)$. Let $J = \bigcup_{t\in J_1} \{t\} \times J_t$. We will show that $f_{\restriction J}$ is not injective. The proof is inspired by an example in [3, page 5]. Assume towards a contradiction that $f_{\restriction J}$ is injective. Pick any two distinct $t_0 > t \in J$. Since $f_{\restriction J}$ is injective, for every $b \in t_0 + J_{t_0}$, we have $b \notin t + J_t$. But $b \in t + cl(J_t)$, so $b \in t + P$. Since this holds for every $b \in t_0 + J_{t_0}$, we have that $t_0 + J_{t_0} \subseteq t + P$, which is a contradiction, since a large set cannot be contained in a small one.

Proposition 4.19. *Let $f : M^m \to M^n$ be an $\mathcal{L}$-definable function and $J \subseteq M^m$ a supercone, such that $f_{\restriction J}$ is injective. Then*

(1) *$m \le n$.*
(2) *there is an $\mathcal{L}$-definable $X \subseteq cl(J)$ such that $\dim(cl(J) \setminus X) < m$ and $f_{\restriction X}$ is finite-to-one. Namely, $X = X_f \cap cl(J)$, with notation from Fact 2.9.*
(3) *If $K \subseteq M^n$ is another supercone and $f : cl(J) \to cl(K)$ is injective, then $f(J) \cap K \neq \emptyset$.*

In particular, by (2), there is an open $\mathcal{L}$-definable $X \subseteq cl(J)$ such that $f_{\restriction X}$ is injective.

Proof. The last clause follows from Fact 2.8.

We write $(1)_m$ - $(3)_m$ for the above statements, and prove them simultaneously by induction on m. Statement $(1)_1$ is clear. Let $m \ge 1$.

$\mathbf{(1)_m} \Rightarrow \mathbf{(2)_m}$. Denote

$$X_f = \{a \in cl(J) : f^{-1}(f(a)) \text{ is finite}\}.$$

We claim that $\dim(cl(J) \setminus X_f) < m$. Assume not. Let $I \subseteq cl(J) \setminus X_f$ be an open box. By Lemma 4.16, $I \cap J$ contains a supercone $K \subseteq M^m$. By Fact 2.9, $f(I)$ has dimension $l < m$. In particular, $f(I)$ is in definable bijection with a subset of M^l via the restriction of an $\mathcal{L}$-definable map $h : M^n \to M^l$. Consider now $g = h \circ f : M^m \to M^l$. Then g is $\mathcal{L}$-definable and injective on K. We have contradicted $(1)_m$.

$\mathbf{(1)_m} \Rightarrow \mathbf{(3)_m}$. Let $K \subseteq M^m$ be a supercone and assume that $f : cl(J) \to cl(K)$ is injective. Suppose now for a contradiction that $f(J) \subseteq cl(K) \setminus K$. By Lemma 4.17 and Corollary 3.15, $cl(K) \setminus K$ is contained in the union of a small definable family of sets each of the form $h(g, Z_g)$ (for finitely many h's), with each $Z_g \subseteq M^{m-1}$ and each $h(g, -) : Z_g \to M^m$ being $\mathcal{L}$-definable and injective. In particular, J is the union of a small definable family of sets of the form $f^{-1}h(g, Z_g) \cap J$. By Lemma 4.15, one of those sets must contain a supercone $L \subseteq M^n$. By Lemma 4.16, $T :=$ interior of $cl(L)) \cap J$

is a supercone in M^m. But then the map $F = h(g,-)^{-1} \circ f : cl(T) \to M^{m-1}$ is an $\mathcal{L}$-definable map that is injective on T, contradicting $(1)_m$.

$\mathbf{(2)_m} \,\&\, \mathbf{(3)_m} \Rightarrow \mathbf{(1)_{m+1}}$. Let $f : M^{m+1} \to M^n$ be an $\mathcal{L}$-definable function and $J \subseteq M^{m+1}$ a supercone with closure V such that $f_{\restriction J}$ is injective. Assume towards a contradiction that $m \geq n$. Let $J_1 = \pi_1(J)$ be the projection of J onto the first coordinate, and $V_1 = \pi_1(V)$. By $(2)_m$, for every $t \in J_1$, there is an open box $X_t \subseteq Y_t$ on which $f(t,-)$ is injective. By cell decomposition in o-minimal structures, and since J_1 is dense in V_1, there is an open cell $U \subseteq V$, such that for every $t \in \pi_1(U)$, $f(t,-)$ is injective on U_t. By Lemma 4.16, $U \cap J$ is a supercone with closure $cl(U)$. We may thus replace J by $U \cap J$, and V by $cl(U)$, and assume from now on that for every $t \in V_1$, $f(t,-)$ is injective on V_t.

Claim 1. *There is an open interval $I_1 \subseteq V_1$ and an open box $I \subseteq M^n$, such that for every $t \in I_1$, $I \subseteq f(t, V_t)$.*

Proof of Claim 1. Since for every $t \in V_1$, $f(t,-)$ is injective on V_t, it follows that the dimension of the $\mathcal{L}$-definable set

$$Z = \bigcup_{t \in V_1} \{t\} \times f(t, V_t)$$

is $n+1$. By cell decomposition, there is an open interval $I_1 \subseteq V_1$ and an open box $I \subseteq M^n$ such that $I_1 \times I \subseteq Z$. In particular, for all $t \in I_1$, $I \subseteq f(t, V_t)$. ☐

By Claim 1, we can pick two distinct $t_0, t \in J_1$ such that

$$I \subseteq f(t_0, V_{t_0}) \cap f(t, V_t)$$

has dimension n. Since $f_{\restriction J}$ is injective, for any $b \in I \cap f(t_0, J_{t_0})$, we have $b \notin f(t, J_t)$, and hence $b \in f(t, V_t \setminus J_t)$. Since this holds for every $b \in I \cap f(t_0, J_{t_0})$, we have that

$$I \cap f(t_0, J_{t_0}) \subseteq f(t, V_t \setminus J_t).$$

Claim 2. *There is a supercone $T \subseteq V_{t_0}$ such that $f(t_0, T) \subseteq I \cap f(t_0, J_{t_0})$.*

Proof. Denote $f_{t_0}(-) = f(t_0, -)$. So f_{t_0} is injective on V_{t_0}. Since $I \subseteq f(t_0, V_{t_0})$, we have $f_{t_0}^{-1}(I) \subseteq V_{t_0}$. Let $I' \subseteq f_{t_0}^{-1}(I)$ be an open cell. By Lemma 4.16, $T := I' \cap J_{t_0}$ is a supercone, as required. ☐

We conclude that the map $f(t,-)^{-1} \circ f(t_0,-) : V_{t_0} \to V_t$ is an injective $\mathcal{L}$-definable map that maps T into $V_t \setminus J_t$, contradicting $(3)_m$. ☐

We show with an example that the assumption on J being a supercone (and not just satisfying $\dim(cl(J)) = m$) is necessary.

Example 4.20. Let f be the function from Example 4.6. The usual projection map $\pi : \mathbb{R}^2 \to \mathbb{R}$ is injective on $Graph(f)$ but of course not injective on any open subset of $cl(Graph(f)) = \mathbb{R}^2$.

The next definition and corollary will be useful when we discuss the notion of large dimension in Section 4.3.

Definition 4.21. Let $f : M^k \to M^n$ be an $\mathcal{L}$-definable map, $J \subseteq M^k$ a supercone and $X \subseteq M^n$ a definable set. We say that

- f is a *strong embedding of J into X* if f is injective and $f(J) \subseteq X$.
- f is a *weak embedding of J into X* if $f_{\restriction J}$ is injective and $f(J) \subseteq X$.

Corollary 4.22. *Let $X \subseteq M^n$ be a definable set. The following are equivalent:*

(1) *there is a weak embedding of a supercone $J \subseteq M^k$ into X.*
(2) *there is a supercone $K \subseteq M^k$ and an $\mathcal{L}$-definable $f : M^k \to M^n$, injective on $cl(K)$, with $f(K) \subseteq X$.*
(3) *there is a strong embedding of a supercone $L \subseteq M^k$ into X.*

Proof. (3)⇒(1) is obvious.

(1)⇒(2). Let $f : M^k \to M^n$ be an $\mathcal{L}$-definable map, injective on J, with $f(J) \subseteq X$. By Proposition 4.19, there is an open definable $S \subseteq cl(J)$ such that $f_{\restriction cl(S)}$ is injective. By Lemma 4.16, $J \cap S$ contains a supercone K.

(2)⇒(3). Let $S \subseteq cl(K)$ be open so that $f_{\restriction S}$ can be extended to an injective $\mathcal{L}$-definable map $F : M^k \to M^n$. By Lemma 4.16 again, $S \cap K$ contains a supercone L. □

4.3. **Large dimension.** We introduce an invariant for every definable set X which tends to measure 'how large' X is. This invariant will be used in the inductive proof of the Structure Theorem in Section 5.

Definition 4.23. Let $X \subseteq M^n$ be definable. If $X \neq \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k-cone. Equivalently, it is the maximum $k \in \mathbb{N}$ such that there is a strong embedding of a supercone $J \subseteq M^k$ into X. We also define the large dimension of the empty set to be $-\infty$. We denote the large dimension of X by $\text{ldim}(X)$.

Clearly, the large dimension of a subset of M^n is bounded by n. In view of Corollary 4.22, the large dimension of X is the maximum $k \in \mathbb{N}$ such that there is a weak embedding of a supercone $J \subseteq M^k$ into X. In Section 6, we will prove that the large dimension equals the 'scl-dimension' arising from a relevant pregeometry in [3]. Here we establish some of its basic properties. The first lemma is obvious.

Lemma 4.24. *For every definable $X, Y \subseteq M^n$, if $X \subseteq Y$, then* $\text{ldim}(X) \leq \text{ldim}(Y)$.

Lemma 4.25. *Let $\{Z_s\}_{s\in S}$ be a small definable family of sets. Then*

$$\text{ldim}\left(\bigcup_{s\in S} Z_s\right) = \max \text{ldim} Z_s.$$

Proof. ($\leq$). Assume $f : M^n \to M^m$ is an $\mathcal{L}$-definable injective map, $J \subseteq M^n$ is a supercone, and $f(J) \subseteq \bigcup_{s\in S} Z_s$. We show that for some $s \in S$, $\text{ldim}(Z_s) \geq n$. For every $s \in S$, let $X_s := f^{-1}(Z_s)$. Then $\{X_s \cap J\}_{s\in S}$ is a definable family of subsets of M^n that cover J, and by Lemma 4.15, one of them must contain a supercone $K \subseteq M^n$. Since $f(K) \subseteq Z_s$, we have that $\text{ldim}(Z_s) \geq n$.

($\geq$). This is clear. □

In particular, we obtain the following standard property that holds for any good notion of dimension.

Corollary 4.26. *Let $X_1, \dots, X_l$ be definable sets. Then*

$$\text{ldim}(X_1 \cup \cdots \cup X_l) = \max\{\text{ldim}(X_1), \dots, \text{ldim}(X_l)\}.$$

About supercones and cones we have:

Corollary 4.27. *If $C \subseteq M^n$ is a k-cone, then* $\text{ldim}(C) = k$.

Proof. By Lemma 4.25 and the definition of a cone it suffices to show that every supercone in M^k has large dimension k. But this is clear. □

Corollary 4.28. *Let $n > 0$ and $J \subseteq M^n$ be a supercone. Then* $\operatorname{ldim}(cl(J) \setminus J) < n$.

Proof. Immediate from Proposition 4.19(3) and the definitions. □

Lemma 4.29. *Let $X \subseteq M^{n+1}$ be a definable set, such that for every $t \in \pi(X)$, X_t is small. Then* $\operatorname{ldim}(X) = \operatorname{ldim}(\pi(X))$.

Proof. Let U_i, S_i, h_i and Z_{ig} be as in Lemma 3.7. In particular,

$$U_i = h_i\left(\bigcup_{g\in S_i}\{g\}\times Z_{ig}\right). \tag{3}$$

($\geq$). By Lemma 3.7(3), we have $\pi(X) = \bigcup_{i,g} Z_{ig}$. By Lemma 4.25, for some i, g, we have $\operatorname{ldim}(Z_{ig}) = \operatorname{ldim}(\pi(X))$. By Equation (3) and Lemma 3.7(1), we obtain

$$\operatorname{ldim}(Z_{ig}) \leq \operatorname{ldim}(U_i) \leq \operatorname{ldim}(X).$$

($\leq$). By Corollary 4.26, $\operatorname{ldim}(X) = \max_i \operatorname{ldim}(U_i)$. By Equation (3), Lemma 3.7(2) and Lemma 4.25, for every i, $\operatorname{ldim}(U_i) = \max_g \operatorname{ldim}(Z_{ig})$. But $Z_{ig} \subseteq \pi(X)$, so $\operatorname{ldim}(X) \leq \operatorname{ldim}(\pi(X))$. □

Corollary 4.30. *Let $X \subseteq M^n$ be a definable set. Then* $\operatorname{ldim}(X) = 0$ *if and only if X is small.*

Proof. Right-to-left is immediate from the definitions of a small set and large dimension. For the left-to-right, we use induction on n. If $n = 1$, the statement is clear by Lemma 3.3. Assume we know the statement for all $l \leq n$ and let $X \subseteq M^{n+1}$.

Claim. *The projection of X onto any of its coordinates is small.*

Proof of Claim. Without loss of generality we may just prove that the projection $\pi(X)$ onto the first n coordinates is small. Since $\operatorname{ldim}(X) = 0$, using Lemma 3.3, we see that for every $t \in \pi(X)$, X_t is small. By Lemma 4.29, $\operatorname{ldim}(\pi(X)) = \operatorname{ldim}(X) = 0$. By Inductive Hypothesis, $\pi(X)$ is small. □

Since X is contained in the product of its coordinate projections, it is again small. □

In Definition 3.22, we introduced low sets. We are now able to determine their large dimension.

Lemma 4.31. *Let $X \subseteq M^n$ be a low definable set. Then* $\operatorname{ldim}(X) = n - 1$.

Proof. By Remark 4.5(3) and Corollary 4.27. □

Remark 4.32. We observe that the converse of Lemma 4.31 does not hold, even if we allow finite unions of low definable sets. For example, let $X := (M \setminus P) \times P$. One can see that X is a 1-cone. Suppose X is the finite union of low sets. Then the image of X under at least one of the coordinate projections has interior. But the images of X under the two coordinate projections are $M \setminus P$ and P. Neither of these two sets has nonempty interior.

5. Structure theorem

We are now ready to prove the main result of this paper, which consists of statements (1) and (2) below. The proof runs by simultaneous induction along with statement (3). The latter is a uniform version of (1).

Theorem 5.1 (Structure Theorem)**.**

(1) *Let $X \subseteq M^n$ be an A-definable set. Then X is a finite union of A-definable cones.*
(2) *Let $f : X \subseteq M^n \to M$ be an A-definable function. Then there is a finite collection $\mathcal{C}$ of A-definable cones whose union is X and such that f is fiber $\mathcal{L}_A$-definable with respect to each $C \in \mathcal{C}$.*
(3) *Let $\{X_t\}_{t\in M^m}$ be an A-definable family of subsets of M^n. Then there is $p \in \mathbb{N}$ and for every $i \in \{1, \dots, p\}$,*
   - *an A-definable subset $Y_i \subseteq M^m$,*
   - *$k_i \in \mathbb{N}$,*
   - *an A-definable uniform family of k_i-cones $\{C^i_t\}_{t\in Y_i}$,*

   such that for all $t \in M^m$

$$X_t = \bigcup \left\{C^i_t : t \in Y_i\right\}.$$

Proof. We write $(1)_n$ - $(3)_n$ for the above statements. We will now show by induction on n that $(1)_n$ - $(3)_n$ hold. Statements $(1)_0$ - $(3)_0$ are trivial. Suppose now that $n > 0$ and $(1)_l$ - $(3)_l$ hold for every $l < n$. It is left to show $(1)_n$ - $(3)_n$.

$(\mathbf{1})_\mathbf{n}$. Let $X \subseteq M^n$. By Remark 3.4(b), we may assume that there are A-definable $h_1, h_2 : M^{n-1} \to M \cup \{\pm\infty\}$ such that for every $a \in \pi(X)$, X_a is contained in $(h_1(a), h_2(a))$, and it is either small in it for all $a \in \pi(X)$, or co-small in it for all $a \in \pi(X)$. We handle the two cases separately.

Case I: For every $a \in \pi(X)$, X_a is co-small in $(h_1(a), h_2(a))$.
By $(2)_{n-1}$, we may assume that $\pi(X)$ is an A-definable cone, such that h_1, h_2 are fiber $\mathcal{L}_A$-definable with respect to it. By Lemma 4.10, X is a finite union of A-definable cones.

Case II: For every $a \in \pi(X)$, X_a is small in $(h_1(a), h_2(a))$.
By Lemma 3.7, we may assume that there are an $\mathcal{L}_A$-definable continuous function $h : Y \subseteq M^{m+n-1} \to M^n$, and A-definable small set $S \subseteq M^m$, and an A-definable family $\{Z_g\}_{g\in S}$ with $Z_g \subseteq \pi(X)$ such that

- $X = h\left(\bigcup_{g\in S}\{g\} \times Z_g\right)$, and
- for every $g \in S$, $h(g, -) : M^{n-1} \to M^n$ is injective.

By $(3)_{n-1}$, there is $p \in \mathbb{N}$ and for every $i \in \{1, \dots, p\}$,

- an A-definable subset $Y_i \subseteq S$,
- $k_i \in \mathbb{N}$,
- an A-definable uniform family of k_i-cones $\{C^i_g\}_{g\in Y_i}$,

such that for all $g \in S$,

$$Z_g = \bigcup \left\{C^j_g : g \in Y_j\right\}.$$

By Lemma 4.12, we have that for each $j \in \{1, \dots, p\}$,

$$h\left(\bigcup_{g\in Y_j} \{g\} \times C_g^j\right)$$

is an A-definable k_j-cone. Thus X is a finite union of A-definable cones.

$(\mathbf{1})_\mathbf{n} \Rightarrow (\mathbf{2})_\mathbf{n}$. Let $f : X \subseteq M^n \to M$ be an A-definable function. We prove $(2)_n$ by sub-induction on $\operatorname{ldim}(X)$. Suppose first that $\operatorname{ldim}(X) = 0$. By $(1)_n$ we can assume that X is a 0-cone. By Lemma 4.8 we can find a finite collection $\mathcal{C}$ of A-definable cones whose union is X and such that f is fiber $\mathcal{L}_A$-definable with respect to each $C \in \mathcal{C}$. So we can now assume that $\operatorname{ldim}(X) = k > 0$ and $(2)_n$ holds for all definable functions whose domain has ldim $< k$. By $(1)_n$, we may assume $X \subseteq M^n$ is an A-definable k-cone, say $X = h(\mathcal{J})$. Let $S = \pi(\mathcal{J})$. We now apply Corollary 3.27 to $f \circ h : \bigcup_{g\in S}\{g\} \times J_g \to M$ to get $p, t \in \mathbb{N}$ and for every $i \in \{1, \dots, p\}$

- an A-definable family $\{X_g \subseteq M^k\}_{g\in S}$ with $\operatorname{ldim}(X_g) < k$,
- an $\mathcal{L}_A$-definable continuous function $f^i : Z_i \subseteq M^{l+t+k} \to M$

such that for every $g \in S$ there is $u \in P^t$ such that

(A) for all $a \in J_g \setminus X_g$ there is $i \in \{1, \dots, p\}$ such that $(f \circ h)(g, a) = f^i(g, u, a)$.

We denote the set of all pairs $(g, u) \in S \times P^t$ that satisfy (A) by K. For each $i \in \{1, \dots, p\}$ we define for $(g, u) \in K$,

$$B^i_{g,u} = \{a \in J_g \setminus X_g \ : \ (f \circ h)(g, a) = f^i(g, u, a)\}.$$

Note that for $g \in S$,

$$\bigcup_{u\in K_g} \left(J_g \setminus \bigcup_i B^i_{g,u}\right) \subseteq X_g.$$

Therefore

$$\operatorname{ldim}\left(\bigcup_{u\in K_g} \left(J_g \setminus \bigcup_i B^i_{g,u}\right)\right) < k.$$

Since $h(g, -)$ is injective on J_g and $\mathcal{L}$-definable,

$$\operatorname{ldim} h\left(\{g\} \times \bigcup_{u\in K_g} (J_g \setminus \bigcup_i B^i_{g,u})\right) < k.$$

By Lemma 4.25

$$\operatorname{ldim} h\left(\bigcup_{g\in S} \{g\} \times \left(\bigcup_{u\in K_g} (J_g \setminus \bigcup_i B^i_{g,u})\right)\right) < k.$$

By sub-induction hypothesis, it is only left to show that the restriction of f to each

$$h\left(\bigcup_{g\in S} \{g\} \times \bigcup_{u\in K_g} B^i_{g,u}\right)$$

satisfies the conclusion of $(2)_n$. Let $i \in \{1, \dots, p\}$. Let $h' : M^{s+t+k} \to M^n$ map (g, u, a) to $h(g, a)$. Then

$$h\left(\bigcup_{g\in S} \{g\} \times \bigcup_{u\in K_g} B^i_{g,u}\right) = h'\left(\bigcup_{(g,u)\in K} \{(g, u)\} \times B^i_{g,u}\right).$$

By $(3)_{n-1}$, there is $q \in \mathbb{N}$ such that for every $j \in \{1, \dots, q\}$ there are an A-definable subset K^j of K, $k_j \in \{0, \dots, n\}$ and an A-definable uniform family of k_j-cones $\{Y^j_{g,u}\}_{(g,u) \in K^j}$ such that for each $(g,u) \in K$

$$B^i_{g,u} = \bigcup \left\{ Y^j_{g,u} : (g,u) \in K^j \right\}.$$

By Lemma 4.12, we have that for each $j \in \{1, \dots, q\}$

$$h' \left(\bigcup_{(g,u) \in K^j} \{(g,u)\} \times (Y^j_{g,u}) \right)$$

is an A-definable k_j-cone $h'(\mathcal{Y}^j)$, where $\mathcal{Y}^j$ denotes the inside family. Since

$$(f \circ h')(g, u, -) = (f \circ h)(g, -) = f^j(g, u, -)$$

on $Y^j_{g,u}$, we have that f is fiber $\mathcal{L}_A$-definable with respect to $h'(\mathcal{Y}^j)$.

$\mathbf{(1)_n} \Rightarrow \mathbf{(3)_n}$. This is by a standard (but lengthy) compactness argument, which we include for completeness. Let $\{X_t\}_{t \in M^m}$ be an A-definable family of subsets of M^n. Suppose that $(3)_n$ fails. Then for every finite collection $\{C^1_t\}_{t \in Y_1}, \dots, \{C^p_t\}_{t \in Y_p}$ of A-definable uniform families of cones, there are $t \in M^m$ and $z \in M^n$ such that

$$z \in X_t \setminus \left(\bigcup_{i=1}^{p} C^i_t \right).$$

Since $\widetilde{\mathcal{M}}$ is sufficiently saturated, there is $x \in M^m$ and $z \in X_x$ such that for every A-definable uniform family of cones $\{C_t\}_{t \in Y}$ either $x \notin Y$ or $z \notin C_x$. For the rest of the proof, we fix this x and z. By $(1)_n$ there is an Ax-definable k-cone $E \subseteq X_x$ with $z \in E$. This is not yet a contradiction, because we do not have a uniform family of cones such that E is one element of this family. Let $k' = \text{dcl-rank}(z/AxP)$. By Lemma 4.7, there is an Ax-definable k'-cone E' such that $z \in E'$. By $(1)_n$, there is an Ax-definable cone $F \subseteq E \cap E'$ such that $z \in F$. By Lemma 4.7, F is a k'-cone. Therefore we can assume that $F = E$ and $k = k'$. It is left to show that there is an A-definable uniform family of $\{C_t\}_{t \in Y}$ such that

(I) $C_t \subseteq X_t$ for each $t \in Y$,
(II) $x \in Y$ and $E = C_x$.

Let $\mathcal{J} = \{J_g\}_{g \in S}$ be an Ax-definable uniform family of supercones in M^k, and $h : Z \subseteq M^{l+k} \to M^n$ an $\mathcal{L}_{Ax}$-definable map, such that $E = h(\mathcal{J})$. Fix an $s \in S$ and $y \in J_s$ such that $h(s,y) = z$.

Pick an $\mathcal{L}_A$-definable function $h' : Z' \subseteq M^{m+l+k} \to M^n$ such that $h'(x, -, -) = h$. Thus in particular, $Z'_x = Z$. Let $U \subseteq M^{m+l+k}$ be an $\mathcal{L}_A$-definable cell such that h' is continuous on U and $(x, s, y) \in U$. Since $\text{dcl-rank}(z/AxP) = k$ we have that $\dim U_{x,s} = k$. By Lemma 4.16, $J_s \cap U_{x,s}$ is a supercone with closure $cl(U_{x,s}) = cl(U_x)_s$. We now take

- an A-definable family $\{S_t\}_{t \in M^m}$ of small subsets of M^l,
- an A-definable family of $\{J'_{t,g}\}_{t \in M^m, g \in S_t}$ of subsets of M^k,

such that $S_x = S$ and $J'_{x,g} = J_g$ for all $g \in S$. Note that we make no further claims about the objects just defined, in particular we do not claim that they directly give rise to a family of cones satisfying (I) and (II). Let

$$S'_t := \{g \in S_t \ : \ J'_{t,g} \cap U_{t,g} \text{ is a supercone with closure } cl(U_{t,g})\}.$$

By Remark 4.5(2), $(S'_t)_{t\in Y}$ is an A-definable family. Let $Y' \subseteq M^m$ be the set of all $t \in Y$ such that $S'_t \neq \emptyset$ and

$$h'\left(t, \bigcup_{g\in S'_t} \{g\} \times (J'_{t,g} \cap U_{t,g})\right) \subseteq X_t.$$

This set is A-definable. It is not hard to check that $s \in S'_x$ and hence $x \in Y'$. Denote

$$\mathcal{K} = \{J'_{t,g} \cap U_{t,g}\}_{t\in Y', g\in S'_t}.$$

By Lemma 4.13, $\mathcal{K}$ is an A-definable uniform family of supercones and

$$\left\{h'\left(t, \bigcup_{g\in S'_t} \{g\} \times (J'_{t,g} \cap U_{t,g})\right)\right\}_{t\in Y'}$$

is an A-definable uniform family of k-cones satisfying (I) and (II). □

Remark 5.2.

(1) The proof of the Structure Theorem uses our standing assumption that $\widetilde{\mathcal{M}}$ is sufficiently saturated. However, by Remark 4.5(2), the Structure Theorem holds for any $\widetilde{\mathcal{M}} \models \widetilde{T}$.
(2) Using a standard compactness argument, the reader can verify that the following uniform version of (2) easily follows (from (2)): let $\{X_t\}_{t\in M^m}$ be an A-definable family of subsets of M^n and $\{f_t : X_t \to M\}_{t\in M^m}$ an A-definable family of maps. Then the conclusion of (3) holds with every f_t being fiber $\mathcal{L}_{At}$-definable with respect to C^i_t.
(3) We do not know whether we can have disjointness of the cones in the Structure Theorem. However, under one additional assumption, we do obtain it; see Theorem 5.12 below.

5.1. **Corollaries of the Structure Theorem.** We collect a few important corollaries of the Structure Theorem. The main result we are aiming for is Theorem 5.7, a generalization of Corollary 3.26. We start with showing the invariance of the large dimension under definable bijections. Recall from Section 4.3 that that the large dimension of a definable set $X \subseteq M^n$ is the maximum $k \in \mathbb{N}$ such that there is a weak embedding of a supercone $J \subseteq M^k$ into X.

Corollary 5.3 (Invariance of large dimension)**.** *Let $f : X \to M^n$ be a definable injective function. Then* $\operatorname{ldim}(X) = \operatorname{ldim} f(X)$.

Proof. Assume that $k \leq \operatorname{ldim}(X)$. It suffices to show $k \leq \operatorname{ldim} f(X)$. By the Structure Theorem, X is the union of finitely many cones such that f is fiber $\mathcal{L}$-definable with respect to each of them. By Corollary 4.26, one of them, say $h(\mathcal{J})$ must be a k-cone. Pick any $g \in \pi(\mathcal{J})$. Then $(f \circ h)(g, -) : J \to M^n$ agrees with an $\mathcal{L}$-definable map on J and it is injective. Therefore, $k \leq \operatorname{ldim} f(X)$. □

The following is an easy consequence of Structure Theorem (3).

Corollary 5.4. *Let $D \subseteq M^m \times M^n$ an A-definable set. Then D is a finite union of A-definable sets of the form*

$$\bigcup_{t\in\Gamma} \{t\} \times C_t,$$

where $\Gamma \subseteq M^m$ is an A-definable cone and there is k such that $\{C_t\}_{t\in\Gamma}$ is an A-definable uniform family of k-cones in M^n.

Proof. Left to the reader. □

We now establish certain desirable properties of large dimension.

Corollary 5.5. *Let $X \subseteq M^{m+n}$ be an A-definable set and let $\pi_m(X)$ be its projection onto the first m coordinates. Then*

(1) *For every $k \in \mathbb{N}$, the set of all $t \in \pi_m(X)$ such that $\mathrm{ldim}(X_t) = k$ is A-definable.*
(2) *Assume that for every $t \in \pi_m(X)$, $\mathrm{ldim}(X_t) = k$. Then*

$$\mathrm{ldim}(X) = \mathrm{ldim}(\pi_m(X)) + k.$$

Proof. We observe that by [11, Proposition 1.4], we only need to prove both statements for $n = 1$. Statement (1) is then immediate by Lemma 3.3 and Remark 3.4(a).

(2). For $k = 0$, this is by Lemma 4.29. For $k = 1$, assume that $\mathrm{ldim}(\pi_m(X)) = l$. By Structure Theorem (1), $\pi_m(X)$ is the finite union of cones $J_1, \ldots, J_p$. Assume that J_i is a k_i-cone. By Lemma 4.10, $T_i = J_i \times M$ is a finite union of $k_i + 1$-cones, and by Corollary 4.27, each of them has large dimension $k_i + 1$. Since X is contained in $T_1 \cup \cdots \cup T_p$, it follows from Corollary 4.26 that $\mathrm{ldim}(X) \leq \max_i k_i + 1 = l + 1$.

On the other hand, let C be an l-cone contained in $\pi_m(X)$. By Remark 3.4(b), there are definable $h_1, h_2 : M^m \to M$ such that for every $t \in \pi_m(X)$, X_t is co-small in $(h_1(t), h_2(t))$. By Structure Theorem (2), $\pi_m(X)$ contains an l-cone C' on which $h_1, h_2 : M^m \to M$ are both fiber $\mathcal{L}$-definable. By Lemma 4.10, it follows that X contains an $l + 1$-cone. □

Lemma 5.6. *Let $J_1, J_2 \subseteq M^k$ be two supercones and $h_1 : Z_1 \to M^n$, $h_2 : Z_2 \to M^n$ two $\mathcal{L}$-definable continuous injective maps, where Z_i is the interior of $cl(J_i)$, $i = 1, 2$. Then*

$$\dim\Big(h_1(Z_1) \cap h_2(Z_2)\Big) = k \quad \Longrightarrow \quad \mathrm{ldim}\Big(h_1(J_1) \cap h_2(J_2)\Big) = k.$$

Proof. Let

$$K_1 = h_1^{-1}\big(h_1(Z_1) \cap h_2(Z_2)\big).$$

Then $K_1 \subseteq Z_1$ and $\dim(K_1) = k$. By Lemma 4.16, $K_1 \cap J_1$ contains a supercone J. Since $J \subseteq K_1$, we have

$$h_2^{-1}h_1(J) \subseteq Z_2.$$

Observe that $h_2^{-1}h_1(J)$ has large dimension k and it is contained in the union of $Z_2 \setminus J_2$ and J_2. By Corollary 4.28, $Z_2 \setminus J_2$ has large dimension $< k$. Hence

$$\mathrm{ldim}\big(h_2^{-1}h_1(J) \cap J_2\big) = k.$$

Then $\mathrm{ldim}\left(h_2(h_2^{-1}h_1(J) \cap J_2)\right) = k$. We observe

$$h_2(h_2^{-1}h_1(J) \cap J_2) \subseteq h_1(J) \cap h_2(J_2) \subseteq h_1(J_1) \cap h_2(J_2),$$

proving that $h_1(J_1) \cap h_2(J_2)$ has large dimension k. □

We are now ready to prove the main result of this section. Statement (2) below is a higher dimensional analogue of Corollary 3.26. To our knowledge, it has not been known even in the special case of dense pairs of o-minimal structures.

Theorem 5.7.

(1) *Let $X \subseteq M^n$ be A-definable. Then there are disjoint AP-definable supercones $J_1, \dots, J_p \subseteq X$ such that*

$$\operatorname{ldim}\left(X \setminus \bigcup_{i=1}^{p} J_i\right) < n.$$

(2) *Every A-definable map $f : M^n \to M$ is given by an $\mathcal{L}_{AP}$-definable map $F : M^n \to M$ off an AP-definable set of large dimension $< n$.*

Moreover, if $A \setminus P$ is dcl*-independent over P, then in both statements the parameters from P can be omitted.*

Proof. We again denote the above two statements by $(1)_n$ and $(2)_n$, and proceed by simultaneous induction on n. For $n = 0$, they are both trivial. Suppose now that $n > 0$ and $(1)_l$ and $(2)_l$ hold for every $l < n$. It is left to show $(1)_n$ and $(2)_n$.

$\mathbf{(1)_n}$: Let $X \subseteq M^n$ and $\pi : M^n \to M^{n-1}$ be the usual projection onto the first $n-1$ coordinates. By Corollary 5.5, the set

$$\{t \in X \ : \ \operatorname{ldim}(X_{\pi(t)}) = 0\}$$

is A-definable and has ldim $< n$. Therefore, we can reduce to the case that $\dim X_a = 1$ for all $a \in \pi(X)$. By Remark 3.4(b), we may further assume that there are A-definable functions $h_1, h_2 : M^{n-1} \to M \cup \{\pm\infty\}$ such that for every $a \in \pi(X)$, X_a is co-small and contained in $(h_1(a), h_2(a))$. By $(2)_{n-1}$ there are $\mathcal{L}_{AP}$-definable functions $H_1, H_2 : M^{n-1} \to M$ and an AP-definable set $Z \subseteq M^{n-1}$ such that $\operatorname{ldim}(Z) < n-1$ and $H_1 = h_1$ and $H_2 = h_2$ on $M^{n-1} \setminus Z$. By $(1)_{n-1}$ there are disjoint AP-definable supercones $J_1, \dots, J_p$ of M^{n-1} such that $J_i \subseteq \pi(X) \setminus Z$,

$$(*) \qquad \operatorname{ldim}\left((\pi(X) \setminus Z) \setminus \bigcup_{i=1}^{p} J_i\right) < n-1.$$

By Lemma 4.16 and cell decomposition in o-minimal structures, we can assume that h_1, h_2 are continuous on the interior of each $cl(J_i)$. Then each $K_i := \bigcup_{t \in J_i} \{t\} \times X_t$ is an AP-definable supercone. It follows immediately from Corollary 5.5 and $(*)$ that $\operatorname{ldim}(X \setminus \bigcup_{i=1}^{p} K_i) < n$, and that $K_1, \dots, K_p$ are disjoint.

$\mathbf{(1)_n} \Rightarrow \mathbf{(2)_n}$: Let $f : M^n \to M$ be A-definable. By Corollaries 3.23 and 4.31, there are $m \in \mathbb{N}$ and

- an A-definable set $Z \subseteq M^n$ with $\operatorname{ldim}(Z) < n$,
- $\mathcal{L}_{A \cup P}$-definable functions $f_i : Z_i \to M$ for $i = 1, \dots, m$,

such that for each $a \in M^n \setminus Z$ there is $i \in \{1, \dots, m\}$ such that $a \in Z_i$ and $f(a) = f_i(a)$. Set

$$X_i := \{a \in M^n \ : \ f(a) = f_i(a) \wedge f(a) \neq f_j(a) \text{ for } j < i\}.$$

Note that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\operatorname{ldim}(M^n \setminus \bigcup_{i=1}^{m} X_i) < n$. By $(1)_n$, for each $i = 1, \dots, m$, there are AP-definable supercones $J_{ik} \subseteq X_i$, $k = 1, \dots, p_i$, such that $\operatorname{ldim}(X_i \setminus \bigcup_{k=1}^{p_i} J_{ik}) < n$. Note that $J_{ik} \cap J_{jl} = \emptyset$ for $i, j \in \{1, \dots, m\}$ with $i \neq j$ and $k = 1, \dots, p_i$, $l = 1, \dots, p_j$. Denote by V_{ik} the interior of $cl(J_{ik})$. By Lemma 5.6, for such i, j, k and l, $V_{ik} \cap V_{jl}$ has dimension $< n$, and hence, since V_{ik} and V_{jl} are open, empty. Thus define $F : M^n \to M$ to map $x \in V_{ik}$ to $f_i(a)$ and $x \notin \bigcup_i \bigcup_k V_{ik}$ to 0. Note that this function is well-defined and $\mathcal{L}_{AP}$-definable, since all f_i and V_{ik} are. Moreover, F agrees with f outside a set of large dimension $< n$; namely $X \setminus \bigcup_{i=1}^{m} \bigcup_{k=1}^{p_i} J_{ik}$.

The 'moreover' clause follows from the above proof and Remark 3.25. □

We expect that Theorem 5.7 will find many applications in the future, and illustrate one here in the case of dense pairs. Namely, we answer the following question from Dolich-Miller-Steinhorn [9, page 702]: in dense pairs, is the graph of every $\emptyset$-definable unary map nowhere dense? This property is known to fail if we allow parameters, as the example in Introduction shows. In [8] the above authors isolate this property and examine it in the context of structures with o-minimal open core.

Proposition 5.8. *Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ be a dense pair. Then the graph of every $\emptyset$-definable map $f : X \subseteq M \to M$ is nowhere dense.*

Proof. By Theorem 5.7, f agrees off a $\emptyset$-definable small set $S \subseteq X$ with an $\mathcal{L}_\emptyset$-definable function F. Clearly, the graph of $f_{\restriction X\setminus S}$ is nowhere dense. We therefore only need to prove that the graph of $f_{\restriction S}$ is nowhere dense. By Lemma 3.16, $S \subseteq P$. By [12, Lemma 3.1], $f(S) \subseteq P$. By [12, Theorem 3(3)], f is piecewise given by $\mathcal{L}$-definable functions, and hence its graph is nowhere dense. □

5.2. **Optimality of the Structure Theorem.** In this section, we prove that our Structure Theorem is in a certain sense optimal.

Definition 5.9. A *strong cone* is a cone $h(\mathcal{J})$ which, in addition to the properties of Definition 4.3, satisfies:

- $h : \mathcal{J} \to M^n$ is injective.

By *Strong Structure Theorem* we mean the Structure Theorem where cones are replaced everywhere by strong cones. Below we give a counterexample to the Strong Structure Theorem and in the next section we point out a 'choice property' that implies it. We will need the following lemma.

Lemma 5.10. *Let $J \subseteq M^n$ be a supercone and $S \subseteq M^m$ small. Assume that $f : Z \subseteq M^n \to M^m$ is an $\mathcal{L}$-definable continuous map with $J \subseteq Z$ that satisfies $f(J) \subseteq S$. Then $f_{\restriction J}$ is constant.*

Proof. We work by induction on n. For $n = 0$, the statement is trivial. Now let $n > 1$ and assume we know the statement for all $J \subseteq M^k$ with $k < n$. Let $J \subseteq M^n$ and $f : Z \to S$ be as in the statement with $f(J) \subseteq S$. For every $t \in \pi_1(J)$, by inductive hypothesis applied to $f(t, -) : Z_t \to M^m$, there is unique $c_t \in S$ so that $f(\{t\} \times J_t) = \{c_t\}$. Since f is continuous, and by definition of a supercone, for every $t \in \pi_1(Z)$, there is also unique $c_t \in S$ so that $f(\{t\} \times Z_t) = \{c_t\}$. We let $h : \pi_1(Z) \to M^m$ be the map given by $t \mapsto c_t$. If f is not constant on J, there must be an interval $I \subseteq \pi_1(Z)$ on which h is injective. But $I \cap \pi_1(J) \subseteq M$ is a supercone by Lemma 4.16, and $h(I \cap \pi_1(J)) \subseteq S$, a contradiction. Therefore, f is constant on J. □

Counterexample to the Strong Structure Theorem. We consider two closely related o-minimal structures: $\mathcal{M} = \langle \mathbb{R}, <, +, 1, x \mapsto \pi x_{\restriction[0,1]}\rangle$ and its expansion $\mathcal{M}' = \langle \mathbb{R}, <, +, 1, x \mapsto \pi x\rangle$. It is well-known that $\mathcal{M}$ does not define unrestricted multiplication by π and that the theory of $\mathcal{M}'$ is the theory of ordered $\mathbb{Q}(\pi)$-vector spaces. We denote the language of $\mathcal{M}$ by $\mathcal{L}$ and the language of $\mathcal{M}'$ by $\mathcal{L}'$.

We now set $P := \mathrm{dcl}_{\mathcal{L}}(\emptyset)$. We first observe that $P = \mathbb{Q}(\pi) = \mathrm{dcl}_{\mathcal{L}'}(\emptyset)$. Indeed, since π is $\mathcal{L}_\emptyset$-definable, it is easy to see that $\mathbb{Q}(\pi) \subseteq P$. Note that $\mathbb{Q}(\pi)$ is a $\mathbb{Q}(\pi)$-vector space and therefore a model of the theory of $\mathcal{M}'$. Thus $\mathrm{dcl}_{\mathcal{L}'}(\emptyset) \subseteq \mathbb{Q}(\pi)$.

Since $P = \mathbb{Q}(\pi) = \mathrm{dcl}_{\mathcal{L}'}(\emptyset)$, $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ is a dense pair of models of the theory of $\mathcal{M}$ and $\langle \mathcal{M}', P\rangle$ is a dense pair of models of the theory of $\mathcal{M}'$. We will now show that the Strong Structure Theorem fails in $\widetilde{\mathcal{M}}$. Being able to work in the two different dense pairs will be crucial. In the following, whenever we say a set is definable without referring to a particular language, we mean definable in $\widetilde{\mathcal{M}}$.

For $t \in M$, we denote by l_t the straight line of slope π that passes through $(t, 0)$. Define

$$U = \bigcup_{g \in P} l_g.$$

We will prove that U is definable but not a finite union of strong cones. By an *endpart of* l_t, we mean $l_t \cap ([a, \infty) \times \mathbb{R})$, for some $a \in \mathbb{R}$.

Claim 1. *U is definable.*

Proof of Claim 1. For every $a \in M$, let $C_a = M \times [a, a+1)$ and $E_a \subseteq C_a \times C_a$ given by:

$$(x, y)E_a(x', y') \;\Leftrightarrow\; y' - y = \pi(x' - x) \text{ and } |x' - x| \leq 1.$$

Thus, if $(x, y) \in l_t \cap C_a$, then $[(x, y)]_{E_a}$ is the segment of l_t that lies in C_a. Define $p_a : C_a \to M^2$ via

$$p_a(x, y) = \text{ the midpoint of } [(x, y)]_{E_a},$$

and let

$$Y_a = p_a(C_a \cap P^2).$$

Clearly, for $t \in P$, we have $l_t \cap P^2 = \{(g, \pi(g - t)) : g \in P\}$, and for $t \notin P$, we have $l_t \cap P^2 = \emptyset$. We claim that

$$U = \bigcup_{a \in M} Y_a,$$

and hence U is definable.

($\subseteq$). Let $(x, y) \in l_t$, $t \in P$. We claim that $(x, y) \in p_a(C_a \cap P^2)$, for $a = y - \frac{1}{2}$. Indeed, (x, y) is the midpoint of $[(x, y)]_{E_a} = l_t \cap C_a$, and hence all we need is to find a point $(g_1, g_2) \in l_t \cap C_a \cap P^2$. Take any $g_2 \in [a, a+1) \cap P$ and let $g_1 = t + \frac{g_2}{\pi} \in P$. Then clearly $(g_1, g_2) \in l_t \cap C_a \cap P^2$ and hence $p_a(g_1, g_2) = (x, y)$.

($\supseteq$). Let $(x, y) = p_a(g_1, g_2) \in p_a(C_a \cap P^2)$. Then $y - g_2 = \pi(x - g_1)$. Hence, for $t = g_1 - \frac{g_2}{\pi}$, we have $(x, y) \in l_t$. □

Claim 2. *U is not a finite union of strong cones.*

Proof of Claim 2. First we observe that $\mathrm{ldim}(U) = 1$. Indeed, U contains infinite $\mathcal{L}$-definable sets, so $\mathrm{ldim}(U) \geq 1$. It cannot be $\mathrm{ldim}(U) = 2$, by Lemma 4.29 and since each vertical fiber is small (it contains at most one element of each l_t, $t \in P$). Therefore $\mathrm{ldim}(U) = 1$.

Now assume, towards a contradiction, that U is a finite union of strong cones. Let $h(\mathcal{J})$ be one of them, where $\mathcal{J} = \bigcup_{g \in S} \{g\} \times J_g$, and $h : Z \to M^2$. In particular, h is injective on $\mathcal{J}$. In the next two subclaims we make use of the expansion $\mathcal{M}'$ of $\mathcal{M}$ and the dense pair $\langle \mathcal{M}', P\rangle$.

Subclaim 1. *For every $g \in S$, $h(g, Z_g)$ must be contained in a unique l_t.*

Proof of Subclaim. Each of l_t and the family $\{l_t\}_{t \in M}$ is now $\mathcal{L}'$-definable. Consider the $\mathcal{L}'$-definable and continuous map $f : Z_g \to M$ where

$$f(x) = t \;\Leftrightarrow\; h(g, x) \in l_t.$$

By Lemma 5.10 applied to $J = J_g$, $S = P$ and f, it follows that $h(g, J_g)$ must be contained in a unique l_t. By continuity of h, so does $h(g, Z_g)$. □

Subclaim 2. *For every $t \in P$, there are only finitely many $g \in S$ such that $h(g, Z_g) \subseteq l_t$.*

Proof of Subclaim. Assume, towards a contradiction, that for some $t \in P$ there are infinitely many $g \in S$ with $h(g, Z_g) \subseteq l_t$. For each $g \in S$, denote by a_g the infimum of the projection of $h(g, Z_g)$ onto the first coordinate. By injectivity of h, for every two $g_1, g_2 \in S$, we have $h(g_1, J_{g_1}) \cap h(g_2, J_{g_2}) = \emptyset$. By Lemma 5.6, $h(g_1, Z_{g_1}) \cap h(g_2, Z_{g_2})$ is finite (in fact, a singleton). Therefore, the set

$$\{a_g : g \in S \text{ and } h(g, Z_g) \subseteq l_t\}$$

is an infinite discrete $\mathcal{L}'(P)$-definable subset of $\mathbb{R}$, a contradiction. □

Since the subclaims hold for each of the finitely many strong cones, it turns out that for one of them, say $h(\mathcal{J})$, there is some $g \in \pi(\mathcal{J})$ such that $h(g, Z_g)$ contains an endpart of l_0. So some endpart of l_0 is definable in $\widetilde{\mathcal{M}}$. But then its closure, which equals that endpart, is $\mathcal{L}$-definable. It follows easily that the full multiplication $x \mapsto \pi x$ is $\mathcal{L}$-definable, a contradiction. □

5.3. **Future directions.** We now point out a key 'choice property' which guarantees the Strong Structure Theorem. Indeed, together with Corollary 3.5 it implies a strengthened version of Lemma 3.7 below, which is enough.

Choice Property: Let $h : Z \subseteq M^{n+k} \to M^l$ be an $\mathcal{L}_A$-definable continuous map and $S \subseteq M^n$ A-definable and small. Then there are $p, m \in \mathbb{N}$, $\mathcal{L}_A$-definable continuous maps $h_i : Z_i \subseteq M^{m+k} \to M^l$, $Y_i \subseteq M^m$ A-definable and small, and A-definable families $X_i \subseteq M^{m+k}$ with $X_{ia} \subseteq Y_i$, $i = 1, \dots, p$, such that for every $a \in \pi(Z)$,

(1) $h_i(-, a) : X_{ia} \to M^l$ is injective, and
(2) $h(S \cap Z_a, a) = \bigcup_i h_i(X_{ia}, a)$,

where $\pi(Z)$ denotes the projection of Z onto the last k coordinates.

Lemma 5.11. *If $\widetilde{M}$ satisfies the Choice Property, then Lemma 3.7 holds with the additional conclusion that each $h_i : Z_i \to M^{n+1}$ is injective.*

Proof. We first claim that there are $m, p \in \mathbb{N}$, and for each $i = 1, \dots, p$, an $\mathcal{L}_A$-definable continuous function $h_i : Z_i \subseteq M^{m+n} \to M$, an A-definable small set $S_i \subseteq M^m$ and an A-definable family $Y_i \subseteq S_i \times C$, such that for all $a \in I$,

(1) $h_i(-, a) : Y_{ia} \to M$ is injective,
(2) $X_a = \bigcup_i h_i(Y_{ia}, a)$,
(3) $\{h_i(Y_{ia}, a)\}_{i=1,\dots,p}$ are disjoint.

Indeed, apply the Choice Property to each h_i from Corollary 3.5 to get (1) and (2). For (3), recursively replace Y_{ia}, $1 < i \leq l$, with the set consisting of all $z \in Y_{ia}$ such that $h_i(z, a) \notin h_j(Y_{ja}, a)$, $0 < j < i$. We now have:

$$X = \bigcup_{a \in C} \{a\} \times X_a = \bigcup_i \bigcup_{a \in C} \{a\} \times h_i(Y_{ia}, a).$$

From this point on the argument continues identically with the corresponding part of Lemma 3.7, noting in the end that, by (1), each $\hat{h}_i$ turns out to be injective. □

Theorem 5.12. *If $\widetilde{M}$ satisfies the Choice Property, then the Structure Theorem holds with cones replaced by strong cones. Moreover, the unions of cones in Structure Theorem are disjoint.*

Proof. The reader can check that Lemmas 4.10 and 4.12 hold with cones replaced everywhere by strong cones, with identical proofs. Moreover, the Choice Property for $k = 0$ implies that every 0-cone is a finite union of strong 0-cones, and hence it is easy to obtain Lemma 4.8 with strong 0-cones in place of 0-cones, as well. It is then a (rather lengthy) routine to check that the proof of the current statement is, again, identical with that of the Structure Theorem, with cones replaced everywhere by strong cones and with the further condition that the unions of cones can be taken to be disjoint. In the proof, Lemma 3.7 has to be replaced by Lemma 5.11 in order to get strong cones and not just cones. The injectivity of the h_i's in Lemma 5.11 guarantees the disjointness of the cones. We leave the details to the reader. □

The counterexample to the Strong Structure Theorem relies on a somewhat unnatural condition on $\mathcal{M}$. In [20], we establish the Choice Property for a collection of structures $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$, such as when $\mathcal{M}$ is a real closed field, or when P is a dense independent set. More generally, we can ask the following question.

Question 5.13. *Under what assumptions on $\mathcal{M}$ or $\widetilde{\mathcal{M}}$ does the Choice Property hold?*

There are other ways in which one could try to improve the Structure Theorem. In general, a supercone $J \subseteq M^n$ does not contain a product of supercones in M. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields and $J \subseteq M^2$ with

$$J = \bigcup_{a \in M} \{a\} \times (M \setminus aP).$$

It is natural to ask whether J contains an image of such product under $\mathcal{L}$-definable map. More generally, one could ask the following question.

Question 5.14. *Would the Structure Theorem remain true if we defined:*

(1) *supercones in M^k to be products $J_1 \times \cdots \times J_k$, where each J_i is a supercone in M?*
(2) *k-cones to be of the form $h(S \times J)$? (That is, h and S are as before, but $J_g = J$ in Definition 4.3 is fixed.)*

In subsequent work [18], we refute both questions, showing that our definitions and Structure Theorem are optimal in yet another way.

6. Large dimension versus scl-dimension

In this section we use our Structure Theorem to establish the equality of the large dimension with the 'scl-dimension' arising from a relevant pregeometry in [3]. In Section 7 we use this equality to set forth the analysis of groups definable in $\widetilde{\mathcal{M}}$.

We start by quoting [3, Definition 28], which was given independently from, and in complete analogy with, [17, Definition 5.2].

Definition 6.1. The *small closure* operator $\mathrm{scl} : \mathcal{P}(M) \to \mathcal{P}(M)$ is defined by:

$$a \in \mathrm{scl}(A) \Leftrightarrow \ a \text{ belongs to an } A\text{-definable small set.}$$

In [3] scl was shown to define a pregeometry under certain assumptions (in addition to their basic tameness conditions). We show that in the current context scl always defines a pregeometry. This follows from the first equality below, which is proved using only results from Section 3. In the interests of completeness, we also prove a second equality, using the Structure Theorem. Recall that $\mathrm{dcl}(A)$ denotes the usual definable closure of A in the o-minimal structure $\mathcal{M}$.

Lemma 6.2. $\mathrm{scl}(A) = \mathrm{dcl}(P \cup A) = \mathrm{dcl}_{\mathcal{L}(P)}(P \cup A)$.

Proof. $\mathrm{scl}(A) \subseteq \mathrm{dcl}(P \cup A)$. Let $b \in \mathrm{scl}(A)$. Then there are an $\mathcal{L}(P)$-formula $\varphi(x, y)$ and $a \in A^l$, such that $\varphi(\mathcal{M}, a)$ is small and contains b. Consider the $\emptyset$-definable family $\{\varphi(\mathcal{M}, t)\}_{t \in M^l}$. By Remark 3.4(a), the set I consisting of all $t \in M^l$ such that $\varphi(\mathcal{M}, t)$ is small is $\emptyset$-definable. Of course, I contains a. By Corollary 3.5, there is an $\mathcal{L}_\emptyset$-definable function $h : M^{m+l} \to M$ such that for all $t \in I$, $\varphi(\mathcal{M}, t) \subseteq h(P^m, t)$. Therefore $b \in h(P^m, a)$, and $b \in \mathrm{dcl}(P \cup A)$.

$\mathrm{scl}(A) \supseteq \mathrm{dcl}(P \cup A)$. Let $b \in \mathrm{dcl}(P \cup A)$. Then there is an $\mathcal{L}_\emptyset$-definable $h : M^{m+l} \to M$ and $a \subseteq A^l$ such that $b \in h(P^l, a)$. But the latter set is small, hence $b \in \mathrm{scl}(A)$.

$\mathrm{dcl}(P \cup A) = \mathrm{dcl}_{\mathcal{L}(P)}(P \cup A)$. It suffices to show $\mathrm{dcl}_{\mathcal{L}(P)}(P \cup A) \subseteq \mathrm{dcl}(P \cup A)$. Let $b = f(a)$, where $a \subseteq P \cup A$ and f is $\emptyset$-definable. By Structure Theorem, there is a $\emptyset$-definable cone $h(\mathcal{J})$, where h is $\mathcal{L}_\emptyset$-definable, containing a on which f is fiber $\mathcal{L}_\emptyset$-definable. Denote $S = \pi(\mathcal{J})$. Let $g \in S$ and $t \in J_g$ be so that $a = h(g, t)$. Since $h(g, -) : M^k \to M^n$ is $\mathcal{L}_g$-definable and injective, $t \in \mathrm{dcl}(P \cup A \cup S)$. Moreover, S is P-bound over $\emptyset$ (Lemma 3.11) and hence $t \in \mathrm{dcl}(A \cup P)$. Since $fh(g, -)$ agrees with an $\mathcal{L}_{A \cup P}$-definable map on J_g, it follows that

$$b = f(h(g, t)) \in \mathrm{dcl}(A \cup P).$$

□

Remark 6.3. In general $\mathrm{dcl}(P \cup A) \neq \mathrm{dcl}_{\mathcal{L}(P)}(A)$. For example, let $\langle \mathcal{M}, \mathcal{N} \rangle$ be a dense pair of real closed fields and let $\mathcal{N}_0$ be a real closed subfield of $\mathcal{N}$. Then $\mathrm{dcl}_{\mathcal{L}(P)}(\mathcal{N}_0) = \mathcal{N}_0$ by [12, Lemma 3.2].

The following corollary is then immediate.

Corollary 6.4. *The small closure operator* scl *defines a pregeometry.*

Definition 6.5. Let $A, B \subseteq M$. We say that B is scl-*independent over* A if for all $b \in B$, $b \notin \mathrm{scl}\big(A \cup (B \setminus \{b\})\big)$. A maximal scl-independent subset of B over A is called *a basis for* B *over* A.

By the Exchange property for scl, any two bases for B over A have the same cardinality. This allows us to define the *rank of* B *over* A:

$$\mathrm{rank}(B/A) = \text{ the cardinality of any basis of } B \text{ over } A.$$

In complete analogy with the corresponding fact for *acl* in a pregeometric theory, we can prove:

Lemma 6.6. *If p is a partial type over $A \subseteq M$ and $a \models p$ with* $\operatorname{rank}(a/A) = m$*, then for any set $B \supseteq A$ there is $a' \models p$ such that* $\operatorname{rank}(a'/B) \geq m$*.*

Proof. The proof of the analogous result for the rank coming from *acl* in a pregeometric theory is given, for example, in [24, page 315]. The proof of the present lemma is word-by-word the same with that one, after replacing an 'algebraic formula' by a 'formula defining a small set' in the definition of Φ_B^m ([24, Definition 2.2]) and the notion of 'algebraic independence' by that of 'scl-independence' we have here. □

It follows that the corresponding dimension of partial types and definable sets is well-defined and independent of the choice of the parameter set.

Definition 6.7. Let p be a partial type over $A \subset M$. The scl-*dimension of* p is defined as follows:

$$\text{scl-dim}(p) = \max\{\operatorname{rank}(\bar{a}/A) : \bar{a} \subset M \text{ and } \bar{a} \models p\}.$$

Let X be a definable set. Then the scl-*dimension of* X, denoted by scl-dim(X) is the dimension of its defining formula.

We next prove the equivalence of the scl-dimension and large dimension of a definable set. First, by a standard routine, using the saturation of $\widetilde{\mathcal{M}}$, we observe the following fact about supercones.

Fact 6.8. *Let $J \subseteq M^k$ be an A-definable supercone. Then J contains a tuple of rank k over A.*

Proposition 6.9. *For every definable $X \subseteq M^n$.*

$$\operatorname{ldim}(X) = \text{scl-dim}(X).$$

Proof. We may assume that X is $\emptyset$-definable.

$\leq$. Let $f : M^k \to M^n$ be an $\mathcal{L}$-definable injective function and $J \subseteq M^k$ a supercone, such that $f(J) \subseteq X$. Suppose both f and J are defined over A. We need to show that $f(J)$ contains a tuple b with $\operatorname{rank}(b/\emptyset) \geq k$. By Fact 6.8, J contains a tuple a of rank k over A. Let $b = f(a)$. Since f is injective, we have $a \in \operatorname{dcl}(Ab)$ and $b \in \operatorname{dcl}(Aa)$. In particular, $a \in \mathit{scl}(Ab)$ and $b \in \operatorname{scl}(Aa)$. So a and b have the same rank over A. Hence,

$$\operatorname{rank}(b/\emptyset) \geq \operatorname{rank}(b/A) = \operatorname{rank}(a/A) = k.$$

$\geq$. Let $b \in X$ be a tuple of rank k. By the Structure Theorem, b is contained in some l-cone $C \subseteq X$. We prove that $l \geq k$. Let $C = h(\mathcal{J})$, where $\mathcal{J}$ is a uniform family of supercones in M^l. Suppose $b = h(g, a)$, for some $g \in \pi(\mathcal{J})$ and $a \in J_g$. Since $h(g, -)$ is $\mathcal{L}_g$-definable and injective, we have $a \in \operatorname{dcl}(gb)$ and $b \in \operatorname{dcl}(ga)$. In particular, $a \in \operatorname{scl}(gb)$ and $b \in \operatorname{scl}(ga)$. Hence a and b have the same rank over g. But $a \in J \subseteq M^l$ and, hence,

$$k = \operatorname{rank}(b/g) = \operatorname{rank}(a/g) \leq l.$$

□

We next record several properties of the rank and large dimension, for future reference. By dcl-rank we denote the usual rank associated to dcl.

Lemma 6.10. *For every $a \in M$ and $A \subseteq M$, we have*

(1) $\mathrm{scl}(A \cup P) = \mathrm{scl}(A)$
(2) $\mathrm{rank}(a/AP) = \mathrm{rank}(a/A) = \text{dcl-rank}(a/AP)$.

Proof. Immediate from Lemma 6.2 and the definitions. □

Lemma 6.11. *Let $X, Y, X_1, \dots, X_k$ be definable sets. Then:*
(1) $\mathrm{ldim}(X) \leq \dim(cl(X))$. *Hence, if X is $\mathcal{L}$-definable,* $\mathrm{ldim}X = \dim X$.
(2) $X \subseteq Y \subseteq M^n \Rightarrow \mathrm{ldim}(X) \leq \mathrm{ldim}(Y) \leq n$.
(3) *X is small if and only if* $\mathrm{ldim}(X) = 0$.
(4) *If C is a k-cone, then* $\mathrm{ldim}(C) = k$.
(5) $\mathrm{ldim}(X_1 \cup \dots \cup X_l) = \max\{\mathrm{ldim}(X_1), \dots, \mathrm{ldim}(X_l)\}$.
(6) $\mathrm{ldim}(X \times Y) = \mathrm{ldim}(X) + \mathrm{ldim}(Y)$.

Proof. (1). Assume X is A-definable and let $a \in X$ with $\mathrm{rank}(a/A) = \mathrm{ldim}(X)$. Since $a \in cl(X)$, we have

$$\mathrm{ldim}(X) = \mathrm{rank}(a/A) = \text{dcl-rank}(a/A \cup P) \leq \text{dcl-rank}(a/A) \leq \dim cl(X).$$

Now, if X is $\mathcal{L}$-definable, $\mathrm{ldim}(X) \leq \dim cl(X) = \dim X$. On the other hand, if $\dim X = k$, one can $\mathcal{L}$-definably embed a k-box in X which of course is a k-cone. (2)-(5) were proved in Section 4, and (6) is by virtue of scl defining a pregeometry. □

6.1. scl**-generics.** For a treatment of the classical notion of dcl-generic elements, see, for example, [39]. Here we introduce the corresponding notion for scl.

Definition 6.12. Let $X \subseteq M^n$ be an $A \cup P$-definable set, and let $a \in X$. We say that a is a scl-*generic element of X over A* if it does not belong to any A-definable set of large dimension $< \mathrm{ldim}(X)$. If $A = \emptyset$, we call a a scl-*generic element of X*.

By saturation, scl-*generic elements always exist*. More precisely, every $A \cup P$-definable set X contains an scl-generic element over A. Indeed, by Compactness and Lemma 6.11(5), the collection of all formulas which express that x belongs to X but not to any A-definable set of large dimension $< \mathrm{ldim}(X)$ is consistent.

Two scl-generics are called *independent* if one (each) of them is scl-generic over the other. The facts that scl defines a pregeometry and that the scl-dim agrees with ldim imply:

Fact 6.13. *Let $G = \langle G, * \rangle$ be a $\emptyset$-definable group. If $a, b \in G$ are independent* scl-*generics, then so are a and $a * b^{-1}$.*

Proof. We have

$$\mathrm{rank}(b/a) = \mathrm{rank}(a * b^{-1}/a).$$

So if b is scl-generic over a, then so is $a * b^{-1}$. □

Note that none of the notions 'dcl-generic element' and 'scl-generic element' implies the other, but, by Lemma 6.10, if X is $A \cup P$-definable and $a \in X$, we have:

a is scl-generic over $A \cup P \Leftrightarrow a$ is scl-generic over $A \Leftrightarrow a$ is dcl-generic over $A \cup P$.

7. Definable groups

In this section we obtain our main application of the Structure Theorem. We fix a $\emptyset$-definable group $G = \langle G, *, 0_G \rangle$ with $G \subseteq M^n$ and $\mathrm{ldim}(G) = k$ and prove a local theorem for G: around scl-generic elements the group operation is given by an $\mathcal{L}$-definable map.

A convention on terminology. When we say that $h(J)$ is a k-cone, we mean that there is a k-cone $h'(\mathcal{J})$ and $g \in \pi(\mathcal{J})$, such that $J = J_g$ and $h(-) = h'(g, -)$. We call $h(J)$ $A \cup P$-definable, if $h'(\mathcal{J})$ is A-definable. Likewise, when we say that $\mathcal{T} = \{\tau_t(J_t)\}_{t\in X}$ is a uniform family of k-cones, we mean that there is a uniform family $\mathcal{C} = \{C_t\}_{t\in X}$ of k-cones as in Definition 4.11 and $g \in \bigcap_t S_t$, such that for every $t \in X$, $J_t = Y_{t,g}$ and $\tau_t(-) = h(t, g, -)$. We call $\mathcal{T}$ $A \cup P$-definable if $\mathcal{C}$ is A-definable. We write $\mathcal{T} = \{\tau(J_t)\}_{t\in X}$, if for all t, s, we have $\tau_t = \tau_s$.

Lemma 7.1. *Let $\{C_t = \tau_t(J_t)\}_{t\in\Gamma}$ be a uniform family of k-cones in M^n and $\Gamma \subseteq M^m$ a k'-cone. Then*

$$C = \bigcup_{t\in\Gamma} \{t\} \times C_t$$

is a $k' + k$-cone.

Proof. Assume $\Gamma = \tau(\mathcal{I})$, where $\mathcal{I} = \bigcup_{s\in S}\{s\} \times I_s$ and $S \subseteq M^p$, and for every $t \in \Gamma$,

$$C_t = h(t, g, Y_{t,g}),$$

for some fixed $g \in \bigcap_t S_t$, and h, $\{Y_{t,g}\}_{t\in\Gamma}$ as in Definition 4.11. We define

$$h' : Z \subseteq M^{p+k'+k} \to M^{m+n} : (s, x, y) \mapsto (\tau(s, x), h(\tau(s, x), g, y)),$$

for a suitable Z, and, for every $s \in S$,

$$J_s = \bigcup_{x\in I_s} \{x\} \times Y_{\tau(s,x),g}.$$

The reader can verify that

$$C = h'\left(\bigcup_{s\in S} \{s\} \times J_s\right)$$

is a $k' + k$-cone, as required. □

Lemma 7.2. *Let $h(J)$ be a k-cone, and $\{D_t\}_{t\in\Gamma}$ a definable family of sets, such that for each $t \in \Gamma$, $\operatorname{ldim}(D_t) = k$ and $D_t \subseteq h(J)$. Then there is a uniform definable family of k-cones $\{C_t = h(Y_t)\}_{t\in\Gamma}$ with $C_t \subseteq D_t$.*

Proof. This follows from a uniform version of Theorem 5.7(1), which can be proved easily via a standard compactness argument. Indeed, for every $t \in \Gamma$, let $X_t = h^{-1}(D_t) \subseteq J$. So $\operatorname{ldim}(X_t) = k$. By the uniform Theorem 5.7(1), we can find a uniform family of supercones $Y_t \subseteq X_t$. Then $C_t = \{h(Y_t)\}_{t\in\Gamma}$ is as required. □

Lemma 7.3. *Let $X \subseteq M^n$ be a $\emptyset$-definable set of large dimension k, (a, b) an* scl-*generic element of $X \times X$, and $D \subseteq X \times X$ a $\emptyset$-definable $2k$-cone containing (a, b). Then there is a P-definable uniform family of k-cones $\{E_t = \tau_t(J_t)\}_{t\in T}$, where T is a k-cone containing a, such that $b \in \bigcap_{t\in T} cl(E_t)$ and*

$$(a, b) \in \bigcup_{t\in T} \{t\} \times E_t \subseteq D.$$

Proof. By Corollary 5.4, and since (a, b) is scl-generic of $X \times X$, it is contained in a $\emptyset$-definable set of the form

$$\bigcup_{t\in\Gamma} \{t\} \times C_t \subseteq D,$$

where $\Gamma \subseteq X$ is a cone and there is l such that $\{C_t\}$ is an $\emptyset$-definable uniform family of l-cones contained in X. Write

$$C_t = h\left(\{t\} \times \left(\bigcup_{g \in S_t} \{g\} \times Y_{t,g}\right)\right),$$

as in Definition 4.11 where $h : Z \to M^n$. Since $a \in \Gamma \subseteq X$ and a is a scl-generic element of X, Γ must be a k-cone. Thus there is a supercone $J_0 \subseteq M^k$ and an $\mathcal{L}_P$-definable, continuous and injective map $f : U \subseteq M^k \to M^n$ such that $f(J_0) = \Gamma$. Let $\hat{a} \in M^k$ such that $f(\hat{a}) = a$. Because (a, b) is an scl-generic element of $X \times X$, $\hat{a}$ is scl-generic over b. Since $b \in C_a \subseteq X$ and b is a scl-generic element of X over a, C_a must be a k-cone, and hence $l = k$. Fix $g \in S_a$ such that $b \in h(a, g, Y_{a,g})$. Because $\hat{a}$ is scl-generic over b, there is an open box $B \subseteq M^k$ containing $\hat{a}$ such that $b \in cl(h(f(x), g, Z_{f(x),g}))$ for every $x \in B$. By density of P we can assume that B is $\mathcal{L}_P$-definable. By Lemma 4.16, $J_0 \cap B$ is a supercone. Hence

$$(a, b) \in \bigcup_{t \in f(J_0 \cap B)} \{t\} \times h(t, g, Y_{t,g})$$

and $b \in \bigcap_{t \in f(J_0 \cap B)} cl(h(t, g, Y_{t,g}))$. Set $E_t = h(t, g, Y_{t,g})$. □

Remark 7.4. In general, there is no $\{E_t\}_{t \in T}$ as above so that $b \in \bigcap_{t \in T} E_t$. For example, let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ be a dense pair of real closed fields, $X = M$

$$D = \bigcup_{c \in M} \{c\} \times (M \setminus cP),$$

and (a, b) any element of D.

Corollary 7.5. *Let $X \subseteq M^n$ be a $\emptyset$-definable set of large dimension k. Let (a, b) be an* scl-*generic element of $X \times X$ and $f : X \times X \to X$ a $\emptyset$-definable function. Then there is a P-definable uniform family of k-cones $\{E_t = \tau_t(J_t)\}_{t \in T}$, where T is a k-cone containing a, such that $b \in \bigcap_{t \in T} cl(E_t)$ and f agrees with an $\mathcal{L}_P$-definable continuous map on*

$$E = \bigcup_{t \in T} \{t\} \times E_t.$$

Proof. By the Structure Theorem, there is a $\emptyset$-definable $2k$-cone $D \subseteq G \times G$ that contains (a, b) and such that f agrees with an $\mathcal{L}_P$-definable continuous map on D. The statement then follows from Lemma 7.3. □

We are now ready to prove the local theorem for definable groups.

Theorem 7.6 (Local theorem for definable groups)**.** *Let a be an* scl-*generic element of G. Then there is a $2k$-cone $C \subseteq G \times G$, whose closure contains (a, a), and an $\mathcal{L}$-definable continuous map $F : Z \subseteq M^n \times M^n \to M^n$, such that for every $(x, y) \in C$,*

$$x * a^{-1} * y = F(x, y).$$

Moreover, F is a homeomorphism in each coordinate.

Proof. Let $a_1 \in G$ be scl-generic over a, and let $a_2 = a_1^{-1} * a$. By Fact 6.13, a, a_1, a_2 are pairwise independent. By the Structure Theorem, for $i = 1, 2$, there is a Pa_i-definable k-cone $C_i = h_i(J_i) \subseteq G$ containing a, and $\mathcal{L}_{Pa_i}$-definable continuous $f_i : Z_i \subseteq M^n \to M^n$ such that for every $x \in C_1$,

$$x * a_2^{-1} = f_2(x)$$

and for every $y \in C_2$,

$$a_1^{-1} * y = f_1(y).$$

Observe that $f_2(a) = a_1$ and $f_1(a) = a_2$.

We now look at the independent scl-generic elements a_1 and a_2. By Corollary 7.5, there is a P-definable uniform family of k-cones $\{E_t = \tau_t(J_t)\}_{t\in T}$ in G, where $T \subseteq G$ is a k-cone containing a_1 and $a_2 \in \bigcap_{t\in T} cl(E_t)$, such that $*$ agrees with an $\mathcal{L}_P$-definable continuous map $f : Z \subseteq M^n \times M^n \to M^n$ on

$$E = \bigcup_{t\in T} \{t\} \times E_t.$$

Observe that (a, a_i) is also scl-generic of $G \times G$. Moreover, since a_2 is dcl-generic of G over P, there is an $\mathcal{L}_P$-definable B of dimension k with

$$a_2 \in B \subseteq \bigcap_{t\in T} cl(E_t).$$

Claim. *For every $t \in T$, $f_1^{-1}(E_t) \cap h_1(J_1)$ has large dimension k.*

Proof of Claim. Let $F_t = f_1^{-1}\tau_t$. Since a belongs to the $\mathcal{L}_{Pa_1}$-definable set $f_1^{-1}(B) \cap h_1(cl(J_1))$ and it is scl-generic over a_1, the set

$$f_1^{-1}(B) \cap h_1(cl(J_1)) \subseteq f_1^{-1}(cl(\tau_t(J_t)) \cap h_1(cl(J_1))$$

has dimension k. This implies that $F(cl(J_t)) \cap h_1(cl(J_1))$ has dimension k. By Lemma 5.6, $f_1^{-1}(E_t) \cap h_1(J_1) = F(J_t) \cap h_1(J_1)$ has large dimension k. □

Now, since a belongs to the Pa_2-definable set $f_2^{-1}(T) \cap h_2(J_2)$ and it is scl-generic over a_2, it must also belong to a Pa_2-definable k-cone

$$\Gamma \subseteq f_2^{-1}(T) \cap h_2(J_2).$$

For every $t \in \Gamma$, we let

$$D_t = f_1^{-1}(E_{f_2(t)}) \cap h_1(J_1).$$

By Claim, $\mathrm{ldim}(D_t) = k$. Since every $D_t \subseteq h_1(J_1)$, by Lemma 7.2, we can find a uniform definable family of k-cones

$$C_t = h_1(Y_t) \subseteq D_t,\ t \in \Gamma,$$

where $Y_t \subseteq J_1$ is a supercone in M^k, and $a \in \bigcap_{t\in\Gamma} C_t$. By Lemma 7.1, the set

$$C = \bigcup_{t\in\Gamma} \{t\} \times C_t$$

is a $2k$-cone. We can now conclude as follows. For every $(x, y) \in C$,

$$x * a^{-1} * y = (x * a^{-1} * a_1) * (a_1^{-1} * y) = f_2(x) * f_1(y) = f(f_2(x), f_1(y)).$$

Set

$$F(x, y) = f(f_2(x), f_1(y)) : M^n \times M^n \to M^n.$$

For the "moreover" clause, we need to check that (a) each f_i can be chosen to be a homeomorphism, and (b) f can be chosen to be a homeomorphism in each coordinate. The former fact follows from the scl-genericity of a over each a_i and the injectivity of each $x \mapsto x * a_i^{-1}$, and the latter fact from the scl-genericity of (a_1, a_2) and the injectivity of $*$ in each coordinate. □

Remark 7.7. We observe that we cannot always have $C = C' \times C''$, where C', C'' are k-cones containing a. For example, consider the group $\mathcal{H} = \langle H = [0,1), +\, mod\, 1\rangle$ in the real field, and let $T = \mathbb{Q}^{rc} \cap H$. Now let $g : H \to M$ be the translation $x \mapsto 2 + x$ on T, and identity elsewhere. Let G be the induced group on $(H \setminus T) \cup g(T)$. Clearly, G is definable in $\widetilde{M} = \langle \mathbb{R}, \mathbb{Q}^{rc}\rangle$, and it is easy to verify that the above observation holds for every $a \in G$. Of course, the conclusion of Theorem 7.6 holds for every $a \in H \setminus T$, by letting $\Gamma = H \setminus T$, $C_t = H \setminus (T \cup (T - t))$ and $f = +\, mod\, 1$. Moreover, we can achieve $C = C' \times C'$, but only up to definable isomorphism. It is reasonable to ask whether that is always true, and we include some relevant (in fact, stronger) questions at the end of this section.

We expect that the above local theorem will play a crucial role in forthcoming analysis of groups definable in $\widetilde{M}$. The ultimate goal would be to understand definable groups in terms of $\mathcal{L}$-definable groups and small groups. Motivated by the successful analysis of semi-bounded groups in [21] and the more recent [4], we conjecture the following statement.

Conjecture 7.8. *Let $\langle G, *\rangle$ be a definable group. Then there is a short exact sequence*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{U} & \longrightarrow & K & \longrightarrow & 0 \\ & & & & \big\downarrow{\scriptstyle \tau} & & & & \\ & & & & G & & & & \end{array}$$

where

- *$\mathcal{U}$ is $\bigvee$-definable*
- *$\mathcal{B}$ is $\bigvee$-definable in $\mathcal{L}$ with* $\dim(B) = \operatorname{ldim}(G)$.
- *K is definable and small*
- *$\tau : \mathcal{U} \to G$ is a surjective group homomorphism and*
- *all maps involved are $\bigvee$-definable.*

The conjecture is in a certain sense optimal: we next produce an example of a definable group G which is *not* a direct product of an $\mathcal{L}$-definable group by a small group. Using known examples of $\mathcal{L}$-definable groups B from [36, 43], which are not direct products of one-dimensional subgroups, it would be easy to provide such an G by restricting some of the one-dimensional subgroups of the universal cover of B to the subgroup P (say, in a dense pair). Our example below, however, is not constructed in this way, as it is *not* a subgroup of the examples in [36, 43].

Example 7.9. Let $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle \models T^d$. Let $G = \langle P \times [0,1), \oplus, 0\rangle$, where $x \oplus y = x + y$ mod $(1,1)$; that is,

$$x \oplus y = \begin{cases} x + y, & \text{if } x + y \in P \times [0,1) \\ x + y - (1,1), & \text{otherwise} \end{cases}$$

Then G is clearly not small. But it cannot contain any non-trivial $\mathcal{L}$-definable subgroup. Indeed, by o-minimality, every $\mathcal{L}$-definable subset of $P \times [0,1)$ must be contained in a finite union of fibers $\{g\} \times [0,1)$, $g \in P$. On the other hand, an $\mathcal{L}$-definable subgroup of G is a topological group containing some $\mathcal{L}$-definable neighborhood of 0 and, thus, also every fiber $\{n\} \times [0,1)$, $n \in \mathbb{Z}$.

The reader can verify that for $\mathcal{B} = Fin(M)$, $K = P$, $\mathcal{U} = \mathcal{B} \times K$ and $\tau(x, y) = (x, y)$ mod $(1,1)$, we obtain the diagram of Conjecture 7.8.

Finally, observe that G is a subgroup of the $\mathcal{L}$-definable group B, which *is* the direct product $B = S \times \langle M, +\rangle$, where S has domain $\{(x, x) : 0 \leq x < 1\}$ and operation $(x, y) \mapsto x + y \mod (1, 1)$.

We finish with some open questions which we expect our local theorem to have an impact on.

Question 7.10. *Does G, up to definable isomorphism, contain an $\mathcal{L}$-definable local subgroup (in the sense of* [42, §23 (D)]*) whose dimension equals* ldim$G$*?*

Question 7.11. *Assume* ldim$G = \dim cl(G)$. *Is G, up to definable isomorphism, $\mathcal{L}$-definable?*

If Conjecture 7.8 is true, it would be nice to know what the small groups are.

Question 7.12. *Is every small definable group/set definably isomorphic to a group/set definable in the induced structure on P?*

Question 7.13. *Is G, up to definable isomorphism, a subgroup of an $\mathcal{L}$-definable group (whose dimension might be bigger than* ldim(G)*)?*

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Part 6

Counting algebraic points in expansions of o-minimal structures by a dense set

COUNTING ALGEBRAIC POINTS IN EXPANSIONS OF O-MINIMAL STRUCTURES BY A DENSE SET

PANTELIS E. ELEFTHERIOU

Abstract. The Pila-Wilkie theorem states that if a set $X \subseteq \mathbb{R}^n$ is definable in an o-minimal structure $\mathcal{R}$ and contains 'many' rational points, then it contains an infinite semialgebraic set. In this paper, we extend this theorem to an expansion $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ of $\mathcal{R}$ by a dense set P, which is either an elementary substructure of $\mathcal{R}$, or it is independent, as follows. If X is definable in $\widetilde{\mathcal{R}}$ and contains many rational points, then it is dense in an infinite semialgebraic set. Moreover, it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$, where $\overline{\mathbb{R}}$ is the real field.

1. Introduction

Point counting theorems have recently occupied an important part of model theory, mainly due to their pivotal role in applications of o-minimality to number theory and Diophantine geometry. Arguably, the biggest breakthrough was the Pila-Wilkie theorem [21], which roughly states that if a definable set in an o-minimal structure contains "many" rational points, then it contains an infinite semialgebraic set. Pila employed this result together with the so-called Pila-Zannier strategy to give an unconditional proof of certain cases of the André-Oort Conjecture [20]. An excellent survey on the subject is [22]. Although several strengthenings of these theorems have since been established within the o-minimal setting, the topic remains largely unexplored in more general tame settings. In this paper, we establish the first point counting theorems in tame expansions of o-minimal structures by a dense set.

Recall that, for a set $X \subseteq \mathbb{R}^n$, the *algebraic part* X^{alg} of X is defined as the union of all infinite connected semialgebraic subsets of X. Pila in [20], generalizing [21], proved that if a set X is definable in an o-minimal structure, then $X \setminus X^{alg}$ contains "few" algebraic points of fixed degree (see definitions below and Fact 2.3). This statement immediately fails if one leaves the o-minimal setting. For example, the set $\mathcal{A}$ of algebraic points itself contains many algebraic points, but $\mathcal{A}^{alg} = \emptyset$. However, adding $\mathcal{A}$ as a unary predicate to the language of the real field results in a well-behaved model theoretic structure, and it is desirable to retain point counting theorems in that setting. We achieve this goal by means of the following definition.

Definition 1.1. Let $X \subseteq \mathbb{R}^n$. The *algebraic trace part* of X, denoted by X^{alg}_t, is the union of all traces of infinite connected semialgebraic sets in which X is dense. That is,

$$X^{alg}_t = \bigcup \{X \cap T : T \subseteq \mathbb{R}^n \text{ infinite connected semialgebraic, and } T \subseteq cl(X \cap T)\}.$$

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The density requirement $T \subseteq cl(X \cap T)$ is essential: without it, we would always have $X_t^{alg} = X$, as witnessed by $T = \mathbb{R}^n$.

We first show in Section 2 that the above notion is a natural generalization of the usual notion of the algebraic part of a set, in the following sense.

Proposition 1.2. *Suppose $X \subseteq \mathbb{R}^n$ is definable in an o-minimal expansion of the real field. Then $X^{alg} = X_t^{alg}$.*

Then, in Sections 3 and 4, we establish point counting theorems in two main categories of tame structures that go beyond the o-minimal setting: dense pairs and expansions of o-minimal structures by a dense independent set. Indeed, we prove that if X is a definable set in these settings, then $X \setminus X_t^{alg}$ contains few algebraic points of fixed degree (Theorem 1.3 below). We postpone a discussion about the general tame setting until later in this introduction, as we now proceed to fix our notation and state the precise theorem. Some familiarity with the basic notions of model theory, such as definability and elementary substructures, is assumed. The reader can consult [11, 17, 19]. An example of an elementary substructure of the real field is the field $\mathcal{A}$ of algebraic numbers.

For the rest of this paper, and unless stated otherwise, we fix an o-minimal expansion $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ of the real field $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot \rangle$, and let $\mathcal{L}$ be the language of $\mathcal{R}$. We fix an expansion $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$ of $\mathcal{R}$ by a set $P \subseteq \mathbb{R}$, and let $\mathcal{L}(P) = \mathcal{L} \cup \{P\}$ be the language of $\widetilde{\mathcal{R}}$. By 'A-definable' we mean 'definable in $\widetilde{\mathcal{R}}$ with parameters from A', and by '$\mathcal{L}_A$-definable' we mean 'definable in $\mathcal{R}$ with parameters from A'. We omit the index A if we do not want to specify the parameters. For a subset $A \subseteq \mathbb{R}$, we write $\operatorname{dcl}(A)$ for the definable closure of A in $\mathcal{R}$, and $\operatorname{dcl}_{\mathcal{L}(P)}(A)$ for the definable closure in $\widetilde{\mathcal{R}}$. We call a set $X \subseteq \mathbb{R}$ dcl-*independent over A*, if for every $x \in X$, $x \notin \operatorname{dcl}((X \setminus \{x\}) \cup A)$, and simply dcl-*independent* if it is dcl-independent over $\emptyset$. An example of a dcl-independent set in the real field is a transcendence basis over $\mathbb{Q}$.

Following [19], we define the *(multiplicative) height* $H(\alpha)$ of an algebraic point α as $H(\alpha) = \exp h(\alpha)$, where $h(\alpha)$ is the *absolute logarithmic height* from [6, page 16]. For a set $X \subseteq \mathbb{R}^n$, $k \in \mathbb{Z}^{>0}$ and $T \in \mathbb{R}^{>1}$, we define

$$X(k,T) = \{(\alpha_1, \dots, \alpha_n) \in X : \max_i [\mathbb{Q}(\alpha_i) : \mathbb{Q}] \leq k, \max_i H(\alpha_i) \leq T\}$$

and

$$N_k(X,T) = \#X(k,T).$$

We say that X *has few algebraic points* if for every $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$,

$$N_k(X,T) = O_{X,k,\epsilon}(T^\epsilon).$$

We say that it *has many algebraic points*, otherwise.

Our main result is the following.

Theorem 1.3. *Suppose $\mathcal{R} = \langle \mathbb{R}, <, +, \cdot, \dots \rangle$ is an o-minimal expansion of the real field, and $P \subseteq R$ a dense set such that one of the following two conditions holds:*

(A) *$P \preccurlyeq \mathcal{R}$ is an elementary substructure.*

(B) *P is a* dcl-*independent set.*

Let $X \subseteq \mathbb{R}^n$ be definable in $\widetilde{\mathcal{R}} = \langle \mathcal{R}, P \rangle$. Then $X \setminus X_t^{alg}$ has few algebraic points.

Note that if $\mathcal{R} = \overline{\mathbb{R}}$, Theorem 1.3 is trivial. Indeed, in both cases (A) and (B), if X is a definable set, then $cl(X)$ is $\mathcal{L}$-definable ([14, Section 2]). So, in this case, $cl(X)$ is semialgebraic and hence $X_t^{alg} = X$. In fact, whenever $\widetilde{\mathcal{R}} = \langle \overline{\mathbb{R}}, P \rangle$ satisfies

Assumption III from [14], the conclusion of Theorem 1.3 holds. An example of such $\widetilde{\mathcal{R}}$ is an expansion of the real field by a multiplicative group with the Mann property.

The contrapositive of Theorem 1.3 implies that if a definable set contains many algebraic points, then it is dense in an infinite semialgebraic set. We strengthen this result as follows.

Theorem 1.4. *Let X be as in Theorem 1.3. If X has many algebraic points, then it contains an infinite set Y which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$.*

Note that such X is dense in $cl(Y)$, which is semialgebraic by [14, Section 2].

A few words about the general tame setting are in order. As o-minimality can only be used to model phenomena that are locally finite, many authors have early on sought expansions of o-minimal structures which escape from the o-minimal context, yet preserve the tame geometric behavior on the class of all definable sets. These expansions have recently seen significant growth ([1, 2, 5, 8, 10, 12, 16, 18]) and are by now divided into two important categories of structures: those where every open definable set is already definable in the o-minimal reduct and those where an infinite discrete set is definable. Cases (A) and (B) from Theorem 1.3 belong to the first category. Further examples of this sort can be found in [8] and [14]. Certain point counting theorems in the second category have recently appeared in [7]. In both categories, sharp *cone decomposition theorems* are by now at our disposal ([14] and [23]), in analogy with the cell decomposition theorem known for o-minimal structures.

Expansions $\mathcal{R}$ of type (A) are called *dense pairs* and were first studied by van den Dries in [10], whereas expansions of type (B) were recently introduced by Dolich-Miller-Steinhorn in [9]. These two examples are representative of the first category and are often thought of as "orthogonal" to each other, mainly because in the former case $\mathrm{dcl}(\emptyset) \subseteq P$, whereas in the latter, $\mathrm{dcl}(\emptyset) \cap P = \emptyset$. This orthogonality is vividly reflected in our proof of Theorem 1.3. Indeed, since the set $\mathcal{A}$ of algebraic points is contained in $\mathrm{dcl}(\emptyset)$, we have $\mathcal{A} \subseteq P$ in the case of dense pairs and $\mathcal{A} \cap P = \emptyset$ in the case of dense independent sets. Based on this observation, the proof for (A) becomes almost immediate, assuming facts from [10], whereas the proof for (B) makes an essential use of the aforementioned cone decomposition theorem from [14].

The current work provides an extension of the influential Pila-Wilkie theorem to the above two settings. The next step is, of course, to explore any potential applications to number theory and Diophantine geometry. Even though it is currently unclear whether the exact setting of Theorem 1.3 will yield any, the machinery used in our proofs is also available in other settings, or it may be possible to develop therein. Two far reaching generalizations of our two settings are lovely pairs [3] and H-structures [4], respectively. Those settings can also accommodate structures coming from geometric stability theory, such as pairs of algebraically closed fields, or SU-rank 1 structures, and point counting theorems in them are wildly unknown.

Notation. The topological closure of a set $X \subseteq \mathbb{R}^n$ is denoted by $cl(X)$. If $X, Z \subseteq \mathbb{R}^n$, we call X dense in Z, if $Z \subseteq cl(X \cap Z)$. Given any subset $X \subseteq \mathbb{R}^m \times \mathbb{R}^n$ and $a \in \mathbb{R}^m$, we write X_a for

$$\{b \in \mathbb{R}^n \ : \ (a, b) \in X\}.$$

If $m \leq n$, then $\pi_m : \mathbb{R}^n \to \mathbb{R}^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1}, unless stated otherwise. A family $\mathcal{J} = \{J_g\}_{g \in S}$ of sets is called definable if $\bigcup_{g\in S}\{g\}\times J_g$ is definable. We often identify $\mathcal{J}$ with $\bigcup_{g\in S}\{g\}\times J_g$. If $X, Y \subseteq \mathbb{R}$, we sometimes write XY for $X\cup Y$. By $\mathcal{A}$ we denote the set of real algebraic points. If $M\subseteq\mathbb{R}$, by $M\preccurlyeq\mathcal{R}$ we mean that M is an elementary substructure of $\mathcal{R}$ in the language of $\mathcal{R}$.

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2. The algebraic trace part of a set

In this section, we introduce the notion of the *algebraic trace part* of a set, and prove that it generalizes the notion of the algebraic part of a set definable in an o-minimal structure. We also state a version of Pila's theorem [19], Fact 2.3 below, suitable for our purposes.

The proof of Theorem 1.3, in both cases (A) and (B), is by reducing it to Pila's theorem, Fact 2.3 below. The formulation of that fact involves a refined version of the usual algebraic part of a set, which prompts the following definitions.

Definition 2.1. Let $A\subseteq\mathbb{R}$ be a set. An *A-set* is an infinite connected semialgebraic set definable over A. If it is, in addition, a cell, we call it an *A-cell*.

We are mainly interested in $\mathbb{Q}$-sets. One important observation is that the set $\mathcal{A}$ of algebraic points is dense in every $\mathbb{Q}$-set. This fact will be crucial in the proofs of Lemma 3.2 and Theorem 4.15 below.

Definition 2.2. Let $X\subseteq\mathbb{R}^n$ and $A\subseteq\mathbb{R}$. The *algebraic part of X over A*, denoted by X^{alg_A}, is the union of all A-subsets of X. That is,

$$X^{alg_A}=\bigcup\{T\subseteq X : T \text{ is an } A\text{-set}\}.$$

It is an effect of the proof in [19] that the following statement holds.

Fact 2.3. *Let $X\subseteq\mathbb{R}^n$ be $\mathcal{L}$-definable. Then $X\setminus X^{alg_{\mathbb{Q}}}$ has few algebraic points.*

Let us now also refine Definition 1.1 from the introduction, as follows.

Definition 2.4. Let $X\subseteq\mathbb{R}^n$ and $A\subseteq\mathbb{R}$. The *algebraic trace part of X over A*, denoted by $X_t^{alg_A}$ is the union of all traces of A-sets in which X is dense. That is,

$$X_t^{alg_A}=\bigcup\{X\cap T : T \text{ an } A\text{-set, } X \text{ dense in } T\}$$

Remark 2.5.

(1) An $\mathbb{R}$-set is exactly an infinite connected semialgebraic set. Also, $X^{alg_{\mathbb{R}}}=X^{alg}$ and $X_t^{alg_{\mathbb{R}}}=X_t^{alg}$.

(2) In Theorems 3.3 and 4.15 below, we prove Theorem 1.3 after replacing X_t^{alg} by $X_t^{alg_{\mathbb{Q}}}$. Since the latter set is contained in the former, these are stronger statements.

Remark 2.6. An alternative expression for $X_t^{alg_A}$ is the following:

$$X_t^{alg_A}=\bigcup\{Y\subseteq X : cl(Y) \text{ is an } A\text{-set}\}.$$

$\subseteq$. Let T be an A-set such that X is dense in T. Set $Y=X\cap T\subseteq X$. Then $T\subseteq cl(Y)\subseteq cl(T)$, and hence $cl(Y)=cl(T)$ is an A-set, as required.

$\supseteq$. Let $Y \subseteq X$ such that $cl(Y)$ is an A-set. Set $T = cl(Y)$. Then $Y \subseteq X \cap T$ and $T \subseteq cl(X \cap T)$, as required.

The goal of this section is to prove the following proposition. This result is not essential for the rest of the paper, but we include it here as it provides canonicity of our definitions. Observe also that it is independent of the expansion $\widetilde{\mathcal{R}}$ of $\mathcal{R}$ we consider.

Proposition 2.7. *Let $X \subseteq \mathbb{R}^n$ be an $\mathcal{L}$-definable set. Then*

$$X^{alg} = X_t^{alg}.$$

The main idea for proving ($\supseteq$) is as follows. Let Z be an $\mathbb{R}$-set with $Z \subseteq cl(Z \cap X)$. We need to prove that every point $x \in Z \cap X$ is contained in an $\mathbb{R}$-set W contained in X. If one applies cell decomposition directly to $Z \cap X$, then the resulting cells need not be semialgebraic, as X is not. So we apply cell decomposition only to Z, deriving an $\mathbb{R}$-cell $Z_0 \subseteq Z$ with $x \in cl(Z_0)$ and of maximal dimension. We then show that close enough to x, the set $T = Z_0 \setminus X$ has dimension strictly smaller than $\dim Z_0$. We use Lemma 2.10 to express this fact properly. Finally, by Lemma 2.11, we find an $\mathbb{R}$-set $W_0 \subseteq Z_0 \setminus T$ with $x \in \mathrm{cl}(W_0)$. We set $W = W_0 \cup \{x\}$.

The first lemma asserts that, under certain assumptions, the property of being dense in a set passes to suitable subsets.

Lemma 2.8. *Let $X, Z \subseteq \mathbb{R}^n$ be $\mathcal{L}$-definable sets, with $Z \subseteq cl(Z \cap X)$. Suppose that $Z_0 \subseteq Z$ is a cell with $\dim Z_0 = \dim Z$. Then $Z_0 \subseteq cl(Z_0 \cap X)$.*

Proof. Let $x \in Z_0$, and suppose towards a contradiction that $x \notin cl(Z_0 \cap X)$. Then there is an open box $B \subseteq \mathbb{R}^n$ containing x such that $B \cap Z_0 \cap X = \emptyset$. It follows that for every $x' \in B \cap Z_0$, $x' \notin cl(Z_0 \cap X)$. Since $Z \subseteq cl(Z \cap X)$,

$$B \cap Z_0 \subseteq cl((Z \setminus Z_0) \cap X) \subseteq cl(Z \setminus Z_0)$$

and, hence,

$$B \cap Z_0 \subseteq cl(Z \setminus Z_0) \setminus (Z \setminus Z_0),$$

and thus $\dim(B \cap Z_0) < \dim(Z \setminus Z_0)$. Moreover, since Z_0 is a cell and $B \cap Z_0 \neq \emptyset$, $\dim(Z_0) = \dim(B \cap Z_0)$. All together,

$$\dim(Z_0) < \dim(Z \setminus Z_0) \leq \dim Z,$$

a contradiction. □

We will need a local version of Lemma 2.8. First, a definition.

Definition 2.9. Let $Z \subseteq \mathbb{R}^n$ be an $\mathcal{L}$-definable set and $x \in Z$. The *local dimension of Z at x* is defined to be

$$\dim_x(Z) = \min\{\dim(B \cap Z) : \ B \subseteq \mathbb{R}^n \text{ an open box containing } x\}.$$

Lemma 2.10. *Let $X, Z \subseteq \mathbb{R}^n$ be infinite $\mathcal{L}$-definable sets with $Z \subseteq cl(Z \cap X)$, and $x \in Z$. Suppose $Z_0 \subseteq Z$ is an $\mathbb{R}$-cell with $\dim_x(Z) = \dim Z_0$ and $x \in cl(Z_0)$. Then there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that $B \cap Z_0 \subseteq cl(Z_0 \cap X)$. Moreover, $B \cap Z_0$ is an $\mathbb{R}$-cell.*

Proof. Let $Z \setminus Z_0 = Z_1 \cup \cdots \cup Z_m$ be a decomposition into cells. It is not hard to see from the definition of $\dim_x(Z)$, that there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that for every $1 \leq i \leq m$, if $B \cap Z_i \neq \emptyset$, then $\dim_x(Z) \geq \dim B \cap Z_i$. We may shrink B if needed so that $B \cap Z_0$ becomes an $\mathbb{R}$-cell. Let I be the set of indices $1 \leq i \leq m$ such that $B \cap Z_i \neq \emptyset$. Set

$$Z' := B \cap Z.$$

Since $Z \subseteq cl(Z \cap X)$, we easily obtain that $Z' \subseteq cl(Z' \cap X)$. Moreover, since $x \in cl(Z)$, we have

$$Z' = (B \cap Z_0) \cup \bigcup_{i \in I} (B \cap Z_i),$$

and hence $\dim Z' = \dim(B \cap Z_0)$. Therefore, by Lemma 2.8 (for Z' and $B \cap Z_0 \subseteq Z'$),

$$B \cap Z_0 \subseteq cl(B \cap Z_0 \cap X) \subseteq cl(Z_0 \cap X),$$

as needed. □

We also need the following lemma.

Lemma 2.11. *Let $Z \subseteq \mathbb{R}^n$ be an $\mathbb{R}$-cell, $T \subseteq Z$ a definable set, and $x \in cl(Z) \setminus T$. Suppose that $\dim T < \dim Z$. Then there is an $\mathbb{R}$-set $W \subseteq Z \setminus T$ with $x \in cl(W)$.*

Proof. We work by induction on $n > 0$. For $n = 0$, it is trivial. Let $n > 0$. We split into two cases:

Case I: $\dim Z = n$. Since $\dim T < \dim Z$, it follows easily, by cell decomposition, that there is a line segment $W \subseteq Z$ with initial point x, staying entirely outside T.

Case II: $\dim Z = k < n$. Let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a suitable coordinate projection such that $\pi_{\upharpoonright Z}$ is injective. Then $\pi(Z)$ is an $\mathbb{R}$-cell, $\pi(T) \subseteq \pi(Z)$, $\dim \pi(T) < \dim \pi(Z)$ and $\pi(x) \in cl(\pi(Z))$. By inductive hypothesis, there is an $\mathbb{R}$-set $W_1 \subseteq \pi(Z) \setminus \pi(T)$, such that $\pi(x) \in cl(W_1)$. Let

$$W = \pi^{-1}(W_1) \cap Z.$$

Then W is clearly an $\mathbb{R}$-set with $W \subseteq Z \setminus T$, and it is also easy to check that $x \in cl(W)$. □

We are now ready to prove Proposition 2.7.

Proof of Proposition 2.7. We need to show $X_t^{alg} \subseteq X^{alg}$. Let Z be an $\mathbb{R}$-set with $Z \subseteq cl(Z \cap X)$. We need to prove that every point $x \in Z \cap X$ is contained in an $\mathbb{R}$-set W contained in X. By cell decomposition in the real field, there is a semialgebraic cell $Z_0 \subseteq Z$ over A, such that $\dim_x(Z) = \dim Z_0$ and $x \in cl(Z_0)$. By Lemma 2.10, there is an open box $B \subseteq \mathbb{R}^n$ containing x, such that $B \cap Z_0$ is an $\mathbb{R}$-cell and $B \cap Z_0 \subseteq cl(Z_0 \cap X)$. Let

$$T = (B \cap Z_0) \setminus (Z_0 \cap X) \subseteq cl(Z_0 \cap X) \setminus (Z_0 \cap X).$$

Then

$$\dim T < \dim(Z_0 \cap X) \leq \dim Z_0 = \dim(B \cap Z_0).$$

Also, $x \in Z \setminus T$. Therefore, by Lemma 2.11 (for $Z = B \cap Z_0$), there is an $\mathbb{R}$-set $W_0 \subseteq (B \cap Z_0) \setminus T$ with $x \in cl(W_0)$. But

$$(B \cap Z_0) \setminus T = B \cap Z_0 \cap X,$$

so $W_0 \subseteq X$. Since $x \in cl(W_0)$, the set $W = W_0 \cup \{x\}$ is connected, and hence the desired $\mathbb{R}$-set. □

Remark 2.12. If we specify parameters in Proposition 2.7, then the proposition need not be true. Indeed

$$X^{alg_{\mathbb{Q}}} \neq X_t^{alg_{\mathbb{Q}}}.$$

For example, fix a dcl-independent tuple $a = (a_1, a_2) \in \mathbb{R}^2$, and let

$$X = \mathbb{R}^2 \setminus \{(a_1, y) : y > a_2\}.$$

Then $a \in X \subseteq X_t^{alg_{\mathbb{Q}}}$, since $cl(X) = \mathbb{R}^2$ is a $\mathbb{Q}$-set. However, $a \notin X^{alg_{\mathbb{Q}}}$. Indeed, no open box around a can be contained in X. Hence if $a \in X^{alg_{\mathbb{Q}}}$, there must be some 1-dimensional semialgebraic set over $\emptyset$ that contains a, contradicting the dcl-independence of a. Note that in the proof of Proposition 2.7, unless $x \in \text{dcl}(\emptyset)$, we cannot conclude that W is semialgebraic over $\emptyset$.

We do not know whether $X^{alg_A} = X_t^{alg_A}$ is true if X is A-definable.

Remark 2.13. The proof of Proposition 2.7 uses nothing in particular about the real field. In other words, if we fix an expansion $\widetilde{\mathcal{M}}$ of any real closed field $\mathcal{M}$, and define the notions of X^{alg} and X_t^{alg} in the same way as in the introduction after replacing 'semialgebraic' by '$\mathcal{M}$-definable', and 'connected' by '$\mathcal{M}$-definably connected', then for every $\mathcal{M}$-definable set X, we have $X^{alg} = X_t^{alg}$.

We conclude this section with an easy fact.

Fact 2.14. *Let $X, Y \subseteq \mathbb{R}^n$ be two definable sets.*

(1) *If $X \subseteq Y$, then $X_t^{alg_{\mathbb{Q}}} \subseteq Y_t^{alg_{\mathbb{Q}}}$.*
(2) *(a) If $X \subseteq Y$ and Y has few algebraic points, then so does X.*
(b) If X and Y have few algebraic points, then so does $X \cup Y$.
(3) *If $X \setminus X_t^{alg_{\mathbb{Q}}}$ and $Y \setminus Y_t^{alg_{\mathbb{Q}}}$ have few algebraic points, then so does $(X \cup Y) \setminus (X \cup Y)_t^{alg_{\mathbb{Q}}}$.*

Proof. (1) and (2) are obvious. For (3), we have:

$$(X\cup Y)\setminus(X\cup Y)_t^{alg_{\mathbb{Q}}} \subseteq (X\setminus(X\cup Y)_t^{alg_{\mathbb{Q}}})\cup(Y\setminus(X\cup Y)_t^{alg_{\mathbb{Q}}}) \subseteq (X\setminus X_t^{alg_{\mathbb{Q}}})\cup(Y\setminus Y_t^{alg_{\mathbb{Q}}}),$$

and we are done by (2). □

3. Dense pairs

In this section, we let $\widetilde{\mathcal{R}} = \langle \mathbb{R}, P \rangle$ be a dense pair. As mentioned in the introduction, since $P \preccurlyeq \mathcal{R}$, we have $\mathcal{A} \subseteq \text{dcl}(\emptyset) \subseteq P$. In this setting, Theorem 1.4 has a short and illustrative proof, and we include it first.

Theorem 3.1. *For every definable set X, if X has many algebraic points, then it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$.*

Proof. Since $\mathcal{A} \subseteq P$, $X \cap P^n$ also contains many algebraic points. By [10, Theorem 2], there is an $\mathcal{L}$-definable $Y \subseteq \mathbb{R}^n$, such that $X = Y \cap P^n$. So Y also contains many algebraic points. By Fact 2.3, there is a $\mathbb{Q}$-set $Z \subseteq Y$. Then the set $Z \cap P^n$ is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$ and it is contained in $Y \cap P^n = X$. Since the set of algebraic points $\mathcal{A}^n$ is dense in Z, we have $Z \subseteq cl(Z \cap \mathcal{A}^n) \subseteq cl(Z \cap P^n)$, and hence $Z \cap P^n$ is infinite. □

We now proceed to the proof of Theorem 1.3.

Lemma 3.2. *Let $X = Y \cap P^n$, for some $\mathcal{L}$-definable set $Y \subseteq \mathbb{R}^n$. Then*

$$X \cap Y^{alg_{\mathbb{Q}}} \subseteq X_t^{alg_{\mathbb{Q}}}.$$

Proof. Let $x \in X \cap Y^{alg_{\mathbb{Q}}}$. So x is contained in a $\mathbb{Q}$-set $Z \subseteq Y$. We prove that X is dense in Z. Observe that $Z \cap X = Z \cap P^n$. Since $\mathcal{A}^n \subseteq P^n$, we have

$$Z \subseteq cl(Z \cap \mathcal{A}^n) \subseteq cl(Z \cap P^n) = cl(Z \cap X),$$

and hence X is dense in Z. □

Theorem 3.3. *For every definable set X, $X \setminus X_t^{alg_{\mathbb{Q}}}$ has few algebraic points.*

Proof. Let $k \in \mathbb{Z}^{>0}$ and $\epsilon \in \mathbb{R}^{>0}$. We first observe that if the statement holds for $X \cap P^n$, then it holds for X. Of course, $X \setminus X_t^{alg_{\mathbb{Q}}} \subseteq X \setminus (X \cap P^n)_t^{alg_{\mathbb{Q}}}$. Since $\mathcal{A}^n \subseteq P^n$, the set X has the same algebraic points as $X \cap P^n$, and hence if $(X \cap P^n) \setminus (X \cap P^n)_t^{alg_{\mathbb{Q}}}$ has few algebraic points, then so does $X \setminus (X \cap P^n)_t^{alg_{\mathbb{Q}}}$, and therefore also $X \setminus X_t^{alg_{\mathbb{Q}}}$.

We may thus assume that $X \subseteq P^n$. By [10, Theorem 2], there is an $\mathcal{L}$-definable $Y \subseteq \mathbb{R}^n$, such that $X = Y \cap P^n$. By Fact 2.3, $Y \setminus Y^{alg_{\mathbb{Q}}}$ has few algebraic points. By Lemma 3.2,

$$X \cap Y^{alg_{\mathbb{Q}}} \subseteq X_t^{alg_{\mathbb{Q}}}.$$

Hence

$$X \setminus X_t^{alg_{\mathbb{Q}}} \subseteq X \setminus Y^{alg_{\mathbb{Q}}} \subseteq Y \setminus Y^{alg_{\mathbb{Q}}}$$

has few algebraic points. □

4. Dense independent sets

In this section, $P \subseteq \mathbb{R}$ is a dense dcl-independent set. The proof of Theorem 4.15 runs by induction on the *large dimension* of a definable set X (Definition 4.8), by making use of the *cone decomposition theorem* from [14] (Fact 4.5). As mentioned in the introduction, since P contains no elements in $\text{dcl}(\emptyset)$, we have $P \cap \mathcal{A} = \emptyset$. The base step of the aforementioned induction is to show a generalization of this fact; namely, that for a *small* set X (Definition 4.1), $X \cap \mathcal{A}$ is finite (Corollary 4.12).

4.1. **Cone decomposition theorem.**

In this subsection we recall all necessary background from [14]. The following definition is taken essentially from [12].

Definition 4.1. Let $X \subseteq \mathbb{R}^n$ be a definable set. We call X *large* if there is some m and an $\mathcal{L}$-definable function $f : \mathbb{R}^{nm} \to \mathbb{R}$ such that $f(X^m)$ contains an open interval in $\mathbb{R}$. We call X *small* if it is not large.

The notion of a cone is based on that of a supercone, which in its turn generalizes the notion of being co-small in an interval. Both supercones and cones are unions of special families of sets, which not only are definable, but they are so in a very uniform way. Although this uniformity is not fully exploited in this paper, we include it here to match the definitions from [14].

Definition 4.2 ([14])**.** A *supercone* $J \subseteq \mathbb{R}^k$, $k \geq 0$, and its *shell* $sh(J)$ are defined recursively as follows:

- $\mathbb{R}^0 = \{0\}$ is a supercone, and $sh(\mathbb{R}^0) = \mathbb{R}^0$.
- A definable set $J \subseteq \mathbb{R}^{n+1}$ is a supercone if $\pi(J) \subseteq \mathbb{R}^n$ is a supercone and there are $\mathcal{L}$-definable continuous $h_1, h_2 : sh(\pi(J)) \to \mathbb{R} \cup \{\pm\infty\}$ with $h_1 < h_2$, such that for every $a \in \pi(J)$, J_a is contained in $(h_1(a), h_2(a))$ and it is co-small in it. We let $sh(J) = (h_1, h_2)_{sh(\pi(J))}$.

Note that, $sh(J)$ is an open cell in $\mathbb{R}^k$ and $cl(sh(J)) = cl(J)$.

Recall that in our notation we identify a family $\mathcal{J} = \{J_g\}_{g\in S}$ with $\bigcup_{g\in S}\{g\}\times J_g$. In particular, $cl(\mathcal{J})$ and $\pi_n(\mathcal{J})$ denote the closure and a projection of that set, respectively.

Definition 4.3 (Uniform families of supercones [14])**.** Let $\mathcal{J} = \bigcup_{g\in S}\{g\}\times J_g \subseteq \mathbb{R}^{m+k}$ be a definable family of supercones. We call $\mathcal{J}$ *uniform* if there is a cell $V \subseteq \mathbb{R}^{m+k}$ containing $\mathcal{J}$, such that for every $g \in S$ and $0 < j \le k$,

$$cl(\pi_{m+j}(\mathcal{J})_g) = cl(\pi_{m+j}(V)_g).$$

We call such a V a *shell* for $\mathcal{J}$.

Remark 4.4. A shell for a uniform family of supercones $\mathcal{J}$ need not be unique. Also, one can identify a supercone $J \subseteq \mathbb{R}^k$ with a uniform family of supercones $\mathcal{J} \subseteq M^{m+k}$ with $\pi_m(\mathcal{J})$ a singleton; in that case, a shell for $\mathcal{J}$ is unique and equals that of J.

Definition 4.5 (Cones [14] and H-cones[1])**.** A set $C \subseteq \mathbb{R}^n$ is a *k-cone*, $k \ge 0$, if there are a definable small $S \subseteq \mathbb{R}^m$, a uniform family $\mathcal{J} = \{J_g\}_{g\in S}$ of supercones in $\mathbb{R}^k$, and an $\mathcal{L}$-definable continuous function $h : V \subseteq \mathbb{R}^{m+k} \to \mathbb{R}^n$, where V is a shell for $\mathcal{J}$, such that

(1) $C = h(\mathcal{J})$, and
(2) for every $g \in S$, $h(g,-) : V_g \subseteq \mathbb{R}^k \to \mathbb{R}^n$ is injective.

We call C a *k-H-cone* if, in addition, $S \subseteq P^m$ and $h : \mathcal{J} \to \mathbb{R}^n$ is injective. An *(H-)cone* is a k-(H-)cone for some k.

The cone decomposition theorem [14, Theorem 5.1] is a statement about definable sets and functions. Here we are only interested in a decomposition of sets into H-cones. Before stating the *H-cone decomposition theorem*, we need the following fact.

Fact 4.6. *Let $S \subseteq \mathbb{R}^n$ be an A-definable small set. Then S is a finite union of sets of the form $f(X)$, where*

- *$f : Z \subseteq \mathbb{R}^m \to \mathbb{R}^n$ is an $\mathcal{L}_A$-definable continuous map,*
- *$X \subseteq P^m \cap Z$ is A-definable, and*
- *$f : X \to \mathbb{R}^l$ is injective.*

Proof. By [14, Lemma 3.11], there is an $\mathcal{L}_A$-definable map $h : \mathbb{R}^m \to \mathbb{R}^n$ such that $X \subseteq h(P^m)$. The result follows from [15, Theorem 2.2]. □

Fact 4.7 (H-cone decomposition theorem)**.** *Let $X \subseteq \mathbb{R}^n$ be an A-definable set. Then X is a finite union of A-definable H-cones.*

Proof. By [14, Theorem 5.12] and [15, Theorem 2.2], X is a finite union of A-definable cones $h(\mathcal{J})$ with $h : \mathcal{J} \to \mathbb{R}^n$ injective (such $h(\mathcal{J})$ is called *strong cone* in the above references). By Fact 4.6, it is not hard to see that $h(\mathcal{J})$ is a finite union of A-definable H-cones. □

We next recall the notion of 'large dimension' from [14].

Definition 4.8 (Large dimension [14])**.** Let $X \subseteq \mathbb{R}^n$ be definable. If $X \ne \emptyset$, the *large dimension* of X is the maximum $k \in \mathbb{N}$ such that X contains a k-cone. The large dimension of the empty set is defined to be $-\infty$. We denote the large dimension of X by $\operatorname{ldim}(X)$.

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[1]The letter 'H' derives from 'Hamel basis' - see [9] for the motivating example $\langle \mathbb{R}, <, +, H\rangle$.

Some basic properties of the large dimension that will be used in the sequel are the following (see [14, Lemma 6.11]): for every two definable sets $X, Y \subseteq \mathbb{R}^n$,

- if $X \subseteq Y$, then $\text{ldim}X \leq \text{ldim}Y$.
- if X is $\mathcal{L}$-definable, then $\text{ldim}X = \dim X$.
- X is small if and only if $\text{ldim}X = 0$.

4.2. **Point counting.** We now proceed to the proof of Theorem 1.3 (B). We need several preparatory lemmas. First, a very useful fact.

Fact 4.9. *For every $A \subseteq \mathbb{R}$ with $A \backslash P$ dcl-independent over P, we have $\text{dcl}_{\mathcal{L}(P)}(A) = \text{dcl}(A)$.*

Proof. Take $x \in \text{dcl}_{\mathcal{L}(P)}(A)$. That is, the set $\{x\}$ is A-definable in $\langle \mathcal{R}, P\rangle$. By [14, Assumption III], since $A \setminus P$ is dcl-independent over P, we have that $cl(\{x\})$ is $\mathcal{L}_A$-definable. But $cl(\{x\}) = \{x\}$. So $x \in \text{dcl}(A)$. □

The following lemma is crucial and relies on the fact that P is dcl-independent.

Lemma 4.10. *Let $h : Z \subseteq P^m \times \mathbb{R}^k \to \mathbb{R}^n$ be a definable injective map. Let $B \subseteq \mathbb{R}$ be a finite set. Then there is a finite set $S_0 \subseteq P^m$ such that*

$$h\left(\bigcup_{g \in P^m \setminus S_0} \{g\} \times Z_g\right) \cap \text{dcl}(B)^n = \emptyset.$$

Proof. Suppose h is A-definable, with A finite. Let $A_0 \subseteq A \cup B$ and $P_0 \subseteq P$ be finite so that $A \cup B \subseteq \text{dcl}(A_0 P_0)$ and A_0 is dcl-independent over P. Suppose $q = h(g, t)$, where $g \in P^m$, $t \in Z_g$ and $q \in \text{dcl}(B)$. By injectivity of h, all coordinates of g are in

$$\text{dcl}_{\mathcal{L}(P)}(Aq) \subseteq \text{dcl}_{\mathcal{L}(P)}(AB) \subseteq \text{dcl}_{\mathcal{L}(P)}(A_0 P_0) = \text{dcl}(A_0 P_0).$$

Since P is dcl-independent, there can be at most $|A_0|$ many such g's, and hence so can q's. □

Two particular cases of the above lemma are the following (recall, $\mathcal{A} \subseteq \text{dcl}(\emptyset)$).

Corollary 4.11. *Let $C = h\left(\bigcup_{g \in S} \{g\} \times J_g\right)$ be an H-cone. Then there is a finite set $S_0 \subseteq S$ such that $h\left(\bigcup_{g \in S \setminus S_0} \{g\} \times J_g\right)$ contains no algebraic points.*

Corollary 4.12. *Every small set contains only finitely many algebraic points.*

Proof. By Lemma 4.10, for $k = 0$, and Fact 4.6. □

The key lemma in the inductive step of the proof of Theorem 4.15 is the following.

Lemma 4.13. *Let $J \subseteq \mathbb{R}^k$ be a supercone with shell Z, and $B \subseteq \mathbb{R}$ finite. Then there is an $\mathcal{L}$-definable set $F \subseteq Z$ with $\dim(F) < k$, such that*

$$Z \cap \text{dcl}(B)^k \subseteq J \cup F.$$

Proof. By induction on k. For $k = 0$, the statement is trivial. For $k > 0$, assume $J = \bigcup_{g \in \Gamma} \{g\} \times J_g$, where $\Gamma \subseteq \mathbb{R}^{k-1}$ is a supercone. By inductive hypothesis, there is $F_1 \subseteq \pi(Z)$, such that

$$\pi(Z) \cap \text{dcl}(B)^{k-1} \subseteq \Gamma \cup F_1.$$

Since $\dim(F_1 \times \mathbb{R}) < k$, it suffices to write $\left(\bigcup_{g\in\Gamma}\{g\} \times Z_g\right) \cap \mathrm{dcl}(B)^k$ as a subset of $J \cup F_2$, for some $F_2 \subseteq Z$ with $\dim(F_2) < k$. Let

$$X = \bigcup_{g\in\Gamma} \{g\} \times (Z_g \setminus J_g).$$

So we need to prove that $X \cap \mathrm{dcl}(B)^k$ is contained in an $\mathcal{L}$-definable set $F_2 \subseteq Z$ with $\dim(F_2) < k$. By [15, Theorem 2.2] and [14, Corollary 5.11], X is a finite union of sets $X_1, \dots, X_l$, each of the form

$$X_i = f\left(\bigcup_{g\in S} \{g\} \times U_g\right),$$

where

- $f : V \subseteq \mathbb{R}^{m+k-1} \to \mathbb{R}^k$ is an $\mathcal{L}$-definable continuous map,
- $U \subseteq (S \times \Gamma) \cap V$ is a definable set, and
- $f_{\restriction U}$ is injective.

Using Fact 4.6, we may further assume that $S \subseteq P^m$. By Lemma 4.10, for $h = f$, there is a finite set $S_0 \subseteq P^m$ such that

$$f\left(\bigcup_{g\in S\setminus S_0} \{g\} \times U_g\right) \cap \mathrm{dcl}(B)^k = \emptyset.$$

For each $i = 1, \dots, l$, and X_i as above, set

$$D_i = f\left(\bigcup_{g\in S_0} \{g\} \times U_g,\right).$$

Then $F_2 = \bigcup_{i=1}^l D_i$ satisfies the required properties. □

Corollary 4.14. *Let $C = h(J) \subseteq \mathbb{R}^n$, where $J \subseteq \mathbb{R}^k$ is a supercone with shell Z, and $h : Z \to \mathbb{R}^n$ an $\mathcal{L}$-definable and injective map. Then there is a definable set $F \subseteq Z$ with $\dim(F) < k$, such that all algebraic points of $h(Z)$ are contained in $h(J \cup F)$.*

Proof. Suppose h is $\mathcal{L}_B$-definable, and take F be as in Lemma 4.13. Let $x = h(y) \in h(Z)$ be an algebraic point. In particular, $x \in \mathrm{dcl}(\emptyset)$. Since h is $\mathcal{L}$-definable and injective, $y \in \mathrm{dcl}(B) \subseteq J \cup F$. □

Theorem 4.15. *For every definable set X, $X \setminus X_t^{alg_{\mathbb{Q}}}$ has few algebraic points.*

Proof. Let $X \subseteq \mathbb{R}^n$ be a definable set. We work by induction on the large dimension of X. If $\mathrm{ldim}(X) = 0$, then X is small and the statement follows from Corollary 4.12. Assume $\mathrm{ldim}(X) = k > 0$. By Facts 4.7 and 2.14(3), we may assume that X is a k-H-cone, say $h(\mathcal{J})$ with $\mathcal{J} \subseteq \mathbb{R}^{m+k}$. By Corollary 4.11, we may further assume that $\pi_m(\mathcal{J})$ is a singleton, and hence, that $X = h(J) \subseteq \mathbb{R}^n$, where $J \subseteq \mathbb{R}^k$ is a supercone. Let Z be the shell of J, and $F \subseteq Z \setminus J$ as in Corollary 4.14. We have that $X \subseteq h(Z \setminus F) \cup h(F)$. By Fact 2.14(3), it suffices to show the statement for each of $X \cap h(Z \setminus F)$ and $X \cap h(F)$.

$X \cap h(F)$. We have

$$\mathrm{ldim}(X \cap h(F)) \leq \mathrm{ldim}\, h(F) = \dim h(F) < k,$$

and hence we conclude by inductive hypothesis.

$X \cap h(Z \setminus F)$. Observe that

$$h(Z \setminus F)^{alg_{\mathbb{Q}}} \subseteq (X \cap h(Z \setminus F))_t^{alg_{\mathbb{Q}}}.$$

Indeed, let $T \subseteq h(Z \setminus F)$ be a $\mathbb{Q}$-set. We need to show that $T \subseteq cl(X \cap T)$. By the conclusion of Corollary 4.13, $T \cap \mathcal{A}^n \subseteq T \cap X$. Since the set of algebraic points $\mathcal{A}$ is dense in Y, we obtain that

$$T \subseteq cl(T \cap \mathcal{A}^n) \subseteq cl(T \cap X),$$

as required. Hence, by Fact 2.3, the sets

$$(X \cap h(Z \setminus F)) \setminus (X \cap h(Z \setminus F))_t^{alg_{\mathbb{Q}}} \subseteq h(Z \setminus F) \setminus h(Z \setminus F)^{alg_{\mathbb{Q}}}$$

has few algebraic points. □

We now turn to the proof of Theorem 1.4. Note that Theorem 4.15 implies that if a definable set X contains many algebraic points, then it is dense in an infinite semialgebraic set. However, the last conclusion by itself does not guarantee that X contains an infinite set definable in $\langle \overline{\mathbb{R}}, P\rangle$. For example, let $\mathcal{R} = \langle \overline{\mathbb{R}}, \exp\rangle$ and $X = e^P$. Then X is definable (in $\langle \mathcal{R}, P\rangle$), and dense in $\mathbb{R}$. Suppose, towards a contradiction, that it contains an infinite set Y definable in $\langle \overline{\mathbb{R}}, P\rangle$. Then Y must be small in the sense of $\langle \overline{\mathbb{R}}, P\rangle$. Indeed, e^P is small in the sense of $\widetilde{\mathcal{R}}$, and smallness is preserved under reducts, by [14, Corollary 3.12]. Now, since Y is small in the sense of $\langle \overline{\mathbb{R}}, P\rangle$, by [13], there is a semialgebraic $h : \mathbb{R}^n \to \mathbb{R}$ and $S \subseteq P^n$, such that $h_{\restriction S}$ is injective and $h(S) = Y \subseteq e^P$. We leave it to the reader to verify that this statement contradicts the dcl-independence of P.

We need two preliminary lemmas.

Lemma 4.16. *Let $J \subseteq \mathbb{R}^k$ be a supercone. Then there is $b \in \mathcal{A}^k$, such that*

$$(b + P^k) \cap sh(J) \subseteq J.$$

In particular, J contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$.

Proof. Denote $Z = sh(J)$. We work by induction on k. For $k = 0$, $J = P^0 = \mathbb{R}^0 = \{0\}$, and the statement holds. Now let $k > 1$. By inductive hypothesis, there is $b_1 \in \mathcal{A}^{k-1}$, such that

$$(b_1 + P^{k-1}) \cap \pi(Z) \subseteq \pi(J).$$

Let $S = (b_1 + P^{k-1}) \cap \pi(Z)$. For every $t \in S$, the set $(Z_t \setminus J_t) - P$ is small, and hence $\bigcup_{t\in S}(Z_t \setminus J_t) - P$ is also small. By Lemma 4.12, the last set contains only finitely many algebraic points. So there is

$$b_2 \in \mathcal{A} \setminus \bigcup_{t\in S}((Z_t \setminus J_t) - P).$$

But then for every $p \in P$ and $t \in S$, if $b_2 + p \in Z_t$, then $b_2 + p \in J_t$. That is, $(b_2 + P) \cap Z_t \subseteq J_t$. Therefore, for $b = (b_1, b_2) \in \mathcal{A}^k$, we have that

$$(b + P) \cap Z \subseteq J.$$

For the "in particular" clause, let $B \subseteq sh(J)$ be any $\emptyset$-definable open box, and b as above. Then $(b + P^k) \cap B \subseteq J$ is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$. It is also infinite, by density of P in $\mathbb{R}$. □

Question 4.17. *Let $J \subseteq \mathbb{R}^k$ be a supercone. Does J contain a set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P\rangle$ and has large dimension k?*

Lemma 4.18. *Let $X \subseteq \mathbb{R}^n$ be a definable set and $T \subseteq \mathbb{R}^n$ a $\mathbb{Q}$-set, such that $\mathcal{A}^n \cap T \subseteq X$. Then* $\text{ldim}(X \cap T) = \dim T$.

Proof. Clearly, $\text{ldim}(X \cap T) \leq \text{ldim}T = \dim T$. Let $k = \dim T$. The set $X \cap T$ is a finite union of H-cones. By Corollary 4.11, there are finitely many cones $h_i(J_i)$ contained in $X \cap T$ and containing all algebraic points of $X \cap T$. Since $\mathcal{A}^n \cap T \subseteq X$, $\mathcal{A}^n \cap T$ is contained in the union of those cones. So

$$T \subseteq cl(\mathcal{A}^n \cap T) \subseteq \bigcup_i cl(h_i(J_i)),$$

implying that for some i, $\dim cl(h_i(J_i)) \geq k$. Therefore, some J_i is a supercone in $\mathbb{R}^k$, implying that $\text{ldim}(X \cap T) \geq k$. □

Theorem 4.19. *Let $X \subseteq \mathbb{R}^n$. If X contains many algebraic points, then it contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$.*

Proof. The beginning of the proof is similar to that of Theorem 4.15, and thus we are brief. We work by induction on $\text{ldim}(X) = 0$. If $\text{ldim}X = 0$, then X is small and the statement holds trivially by Corollary 4.12. For $\text{ldim}X = k > 0$, we may assume that $X = h(J)$ is a k-cone, with $J \subseteq \mathbb{R}^k$. Let Z be the shell of J, and $F \subseteq Z \setminus J$ as in Corollary 4.14. So one of $X \cap h(F)$ and $X \cap h(Z \setminus F)$ must contain many algebraic points. If the former one does, then we can conclude by inductive hypothesis. If the latter one does, then by Fact 2.3, there is a $\mathbb{Q}$-cell $T \subseteq h(Z \setminus F)$. By the conclusion of Corollary 4.12, $\mathcal{A}^n \cap T \subseteq X$. By Lemma 4.18, $\text{ldim}X \cap T = \dim T$. Also,

$$T \subseteq \text{cl}(\mathcal{A}^n \cap T) \subseteq cl(X \cap T),$$

and hence if follows easily that

$$\dim cl(X \cap T) = \text{ldim}X \cap T.$$

Now, if T is open, then $\text{ldim}X \cap T = n$, and hence $X \cap T$ contains a supercone in $\mathbb{R}^n$ (by [14, Theorem 5.7(1)]). By Lemma 4.16, $X \cap T$ contains an infinite set which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$. Suppose $T = \Gamma(f)$ and let $\pi : \mathbb{R}^n \to \mathbb{R}^k$ be a coordinate projection that is injective on T. Then $\text{ldim}\pi(X \cap T) = k$ and hence $\pi(X \cap T)$ contains a supercone in $\mathbb{R}^k$, and thus, by Lemma 4.16, an infinite set S which is $\emptyset$-definable in $\langle \overline{\mathbb{R}}, P \rangle$. Then $\Gamma(f_{\restriction S})$ is contained in X and is as desired. □

We conclude with a remark that goes also beyond the scope of this section.

Remark 4.20. Let $X \subseteq \mathbb{R}^n$ and $P \subseteq \mathbb{R}$ be as in Theorem 1.3. Define

$$X_P^{alg} = \bigcup \{Y \subseteq X : Y \text{ infinite } \emptyset\text{-definable in } \langle \overline{\mathbb{R}}, P \rangle\}.$$

It is natural to ask whether $X \setminus X_P^{alg}$ has few algebraic points. An affirmative answer to this question would strengthen Theorem 1.3, and its contrapositive would imply Theorem 1.4. For the case of dense pairs, it is actually not too hard to adjust the proofs of Lemma 3.2 and Theorem 3.3 and obtain an affirmative answer. For the case of dense independent sets, the question is open, and it is possible that an affirmative answer to Question 4.18 could be relevant.

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