LATTICES IN LOCALLY DEFINABLE SUBGROUPS OF $\langle R^n, +\rangle$

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Abstract. Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R. We prove that every connected, abelian, locally definable subgroup of $\langle R^n, +\rangle$ contains a definable generic set.

The goal of this note is to re-formulate some problems which appeared in [4], introduce the notion of a lattice in a locally definable group (a notion which also appeared in that paper, but not under this name) and establish connections between various related concepts. Finally, we return to the main conjecture from [4]:

Every locally definable connected, abelian group, which is generated by a definable set contains a definable generic set.

We prove the conjecture for subgroups of $\langle R^n, +\rangle$, in the context of an o-minimal expansion $\mathcal{M}$ of a real closed field R.

1. Locally definable groups and lattices

We first recall some definitions: Let $\mathcal{M}$ be an arbitrary κ-saturated o-minimal structure (for κ sufficiently large). By a *locally definable* group we mean a group $\langle \mathcal{U}, \cdot\rangle$, whose universe $\mathcal{U} = \bigcup_{n\in\mathbb{N}} X_n$, is a countable union of definable subsets of M^k, for some fixed k, and the group operation is definable when restricted to each $X_m \times X_n$ (equivalently, to each definable subset of $\mathcal{U}\times\mathcal{U}$). We say that a function $f : \mathcal{U} \to M^n$ is *locally definable* if its restriction to each X_i (equivalently, to each definable subset of $\mathcal{U}$) is definable. We let $\dim \mathcal{U}$ be the maximum of $\dim X_n$, $n \in \mathbb{N}$. While some notions treated here make sense under the more general "$\bigvee$-definable group" (no restriction on the number of X_i's), we mostly work in the context of a group which is generated, as a group, by a definable subset and hence it is locally definable. Note that another related concept, that of an *ind-definable group* (see [5]) is identical to our definition when one further assumes that the group is a subset of a fixed M^k.

As was shown in [6], every locally definable group admits a group topology. This topology agrees with the M^k-topology in neighborhoods of generic points, namely, points $g \in \mathcal{U}$ such that $\dim(g/A) = \dim(\mathcal{U})$ (we assume here that all the X_i's above are defined over A). We therefore obtain a definable family of neighborhoods $\{U_t : t \in T\}$ of the identity element, such that $\{gU_t : t \in T, g \in \mathcal{U}\}$ is a basis for the group topology on $\mathcal{U}$. In [2] it was further shown that the topology can be realized by countably many

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definable open charts, each definably homeomorphic to an open subset of M^n, where $n = \dim(\mathcal{U})$.

A subset $X \subseteq \mathcal{U}$ is called *compatible* (see [3]) if for every $Y \subseteq \mathcal{U}$ which is definable, the set $X \cap Y$ is also definable. It easily follows that X itself is also locally definable (namely, given as a countable union of definable subsets of $\mathcal{U}$). As was shown in [3], if $\mathcal{U}$ is locally definable and $\mathcal{H}$ is a normal compatible subgroup of $\mathcal{U}$ then there is a locally definable group $\mathcal{K}$ and a locally definable surjective homomorphism $f : \mathcal{U} \to \mathcal{K}$ whose kernel is $\mathcal{H}$. The converse is true as well, namely if such a homomorphism exists then $\mathcal{H}$ is necessarily compatible.

A locally definable group is called *connected* (see [1]) if it has no compatible subset which is both closed and open, with respect to the group topology. As is shown in [2, Remark 4.3], a locally definable group $\mathcal{U}$ is connected if and only if it is path connected, namely for any two points $x, y \in \mathcal{U}$ there exists a *definable* continuous $\sigma : [0, 1] \to \mathcal{U}$ such that $\sigma(0) = x$ and $\sigma(1) = y$.

A typical example of a locally definable group is obtained by taking a definable subset of a definable group (say, of $\langle R^n, +\rangle$) and letting $\mathcal{U}$ be the subgroup generated by X. When the generating set is definably connected then one obtains a connected locally definable group. We call a locally definable group $\mathcal{U}$ *definably generated* if it is generated, as a group, by some definable subset.

Definition 1.1. *For $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup, we say that* the quotient $\mathcal{U}/\mathcal{H}$ is definable *if there exists a definable group G and a locally definable surjective homomorphism from $\mathcal{U}$ onto G, whose kernel is $\mathcal{H}$.*

Definition 1.2. *A locally definable normal subgroup $\Lambda \subseteq \mathcal{U}$ is called* a lattice in $\mathcal{U}$ *if* $\dim(\Lambda) = 0$ *and $\mathcal{U}/\Lambda$ is definable.*

Notice that any countable group can be realized as a locally definable group, and therefore it is also a lattice in itself.

If $\mathcal{U}$ is the subgroup of R^n generated by the unit n-cube $[-1, 1]^n$ then $\mathbb{Z}^n$ is a lattice in $\mathcal{U}$. The quotient is definably isomorphic to the group H^n, where $H = [0, 1)$, with addition modulo 1.

In [4, Lemma 6.2] we prove the following equivalence:

Lemma 1.3. *Let $\mathcal{U}$ be a locally definable group in an o-minimal structure and Λ a locally definable normal subgroup of dimension* 0*. The following are equivalent.*

(1) Λ *is a lattice in* $\mathcal{U}$*.*
(2) Λ *is compatible, and there exists a definable set $X \subseteq G$ such that* $\Lambda \cdot X = \mathcal{U}$*.*

It is easy to see that every lattice in a locally definable group is countable (the intersection with every definable set is finite). We prove a stronger statement:

Lemma 1.4. *If Λ is a lattice in a locally definable connected group $\mathcal{U}$ then Λ is finitely generated as a group.*

Proof. Let $\phi : \mathcal{U} \to G$ be a locally definable surjective homomorphism onto a definable group G, with $ker\phi = \Lambda$. By compactness there exists a definable set $X \subseteq \mathcal{U}$ such that $\phi(X) = G$, and since ϕ is locally definable, $X \cap \Lambda$ is finite. Similarly, $X^{-1}X \cap \Lambda$ is finite. We now consider the definable equivalence relation on X: $x \sim y$ if and only if $x^{-1}y \in \Lambda$. Because we have definable choice for subsets of $\mathcal{U}$ ([3, Corollary 8.1]), we can choose a subset $X' \subseteq X$, call it X again, such that every coset of Λ has exactly one representative in X'. We assume then that $X = X'$ and that $e \in X$.

Consider the topological closure (with respect to the group topology), $Cl(X) \subseteq \mathcal{U}$.

Claim There exists a finite set $F \subseteq \Lambda$ such that for every $g \in \Lambda$, if $gCl(X) \cap Cl(X) \neq \emptyset$ then $g \in F$.

Proof of Claim. Let $X' \subseteq \mathcal{U}$ be any definable open set containing $Cl(X)$. By saturation, there is a finite $F \subseteq \Lambda$, which we may assume is minimal, such that $X' \subseteq F \cdot X$. Because $gX \cap hX = \emptyset$ for every $g \neq h \in \Lambda$, if $gX \cap X' \neq \emptyset$ then necessarily $g \in F$. Now, if $gCl(X) \cap Cl(X) \neq \emptyset$ then necessarily $gX \cap X' \neq \emptyset$ so $g \in F$. □

We now claim that F generates $\mathcal{U}$, namely every element of $\mathcal{U}$ is a finite word in F and F^{-1}.

Take $\lambda \in \Lambda$. Since $\mathcal{U}$ is path connected, there exists a definable path $\gamma : [0, 1] \to \mathcal{U}$, with $\gamma(0) = e$ and $\gamma(1) = \lambda$. Let $\Gamma \subseteq \mathcal{U}$ be the image of γ. Because Γ is definable it can be covered by finitely many Λ-translates of X. By taking a minimal number of translates, we obtain $\lambda_1, \ldots, \lambda_k \in \Lambda$ (possibly with repetitions), such that $e \in \lambda_1 X$, $\lambda \in \lambda_k X$ and for $i = 1, \ldots, k-1$, we have $Cl(\lambda_i X) \cap Cl(\lambda_{i+1} X) \neq \emptyset$.

By the Claim, it follows that $\lambda_{i+1}^{-1}\lambda_i \in F$, for $i = 1, \ldots, k-1$. But since $e \in X$, we must have $\lambda_1 = e$ and $\lambda_k = \lambda$, so $\lambda_1, \ldots, \lambda_k$ are all in the group generated by F, and in particular, λ belongs to that group. □

We say that $\mathcal{U}$ *admits a lattice* if there is a lattice in $\mathcal{U}$. Note that not every locally definable group admits a lattice. For example, if $r \in R$ is larger than all elements of $\mathbb{N}$ then the subgroup of $\langle R, +\rangle$ given by $\bigcup[-r^n, r^n]$ does not admit any lattice.

As we point out in [4], there are many consequences, for a given group $\mathcal{U}$, to the fact that it admits a lattice. Hence, our main question is:

Question 1 Which locally definable groups in $\mathcal{M}$ admit a lattice?

We start with some basic observations.

Definition 1.5. *A definable subset X of a locally definable group $\mathcal{U}$ is called* left generic *in $\mathcal{U}$ if there exists a bounded set $\Delta \subseteq \mathcal{U}$ (namely, $|\Delta| < \kappa$) such that $\mathcal{U} = \Delta \cdot X$. Equivalently, for every definable $Y \subseteq \mathcal{U}$ there is a finite set $F \subseteq \mathcal{U}$ such that $Y \subseteq F \cdot X$.*

Lemma 1.3 immediately gives:

Lemma 1.6. *If a locally definable group $\mathcal{U}$ admits a lattice then $\mathcal{U}$ contains a definable left generic set.*

Lemma 1.7. *Let $\mathcal{U}$ be a connected locally definable group which contains a left generic definable set X (e.g. if $\mathcal{U}$ admits a lattice). Then $\mathcal{U}$ is definably generated.*

Proof. Let $X \subseteq \mathcal{U}$ be a definable, left generic set, namely there is a bounded set $\Delta \subseteq \mathcal{U}$ such that $\Delta \cdot X = \mathcal{U}$. The group generated by X, call it $\mathcal{H}$, is therefore locally definable, of bounded index in $\mathcal{U}$ (since $\langle \Delta \rangle \cdot \mathcal{H} = \mathcal{U}$, where $\langle \Delta \rangle$ is the group generated by Δ). But then, if $Y \subseteq \mathcal{U}$ is a definable set then $Y \cap \mathcal{H}$ and $Y \cap (\mathcal{U} \setminus \mathcal{H})$ are both bounded unions of definable sets. By saturation, this forces $Y \cap \mathcal{H}$ to be definable, hence $\mathcal{H}$ is compatible. It is easy to see that $\mathcal{H}$ is both closed and open so by connectedness of $\mathcal{U}$ must equal $\mathcal{U}$. □

It is now natural to ask:

Question 2 Does every connected, definably generated group admit a lattice?

2. Lattices in abelian groups

We still work in a sufficiently saturated structure $\mathcal{M}$.

Recall that for a locally definable group $\mathcal{U}$, we say that $\mathcal{U}^{00}$ *exists*, if there is a smallest type-definable normal subgroup of $\mathcal{U}$ of bounded index (note that a type-definable subgroup of $\mathcal{U}$ is necessarily contained in a definable subset of $\mathcal{U}$). We denote that subgroup by $\mathcal{U}^{00}$.

One of the main results in [4] is the following: (the equivalence of the bottom three clauses is given in [4, Theorem 7.6]; the addition of Clause (1) is given by Lemma 1.6):

Theorem 2.1. *Let $\mathcal{U}$ be a connected, abelian locally definable group which is generated by a definably compact set. Then there is k so that the following are equivalent:*

(1) $\mathcal{U}$ admits a lattice.
(2) $\mathcal{U}$ admits a lattice, isomorphic to $\mathbb{Z}^k$.
(3) $\mathcal{U}$ contains a definable generic set.
(4) $\mathcal{U}^{00}$ exists, and $\mathcal{U}/\mathcal{U}^{00}$ is isomorphic to $\mathbb{R}^k \times K$, for some compact Lie group K.

In particular, we see that a connected, abelian, locally definable $\mathcal{U}$ admits a lattice if and only if it contains a definable generic set. Note that by (4), the above k is unique.

In [4] we made the conjecture that the conclusions of the above theorem are always true:

Conjecture A. *Let $\mathcal{U}$ be an abelian, connected, definably generated group. Then $\mathcal{U}$ contains a definable generic set (so in particular admits a lattice).*

The number k, in the above theorem, can be viewed as a measure of how "non-definable" the group $\mathcal{U}$ is. Namely, if $k = 0$ then $\mathcal{U}$ is outright definable, while if $k = \dim \mathcal{U} > 0$, then $\mathcal{U}$ will not contain any definable subgroup. We prove these statements below.

In fact, we can define an invariant for every locally definable group $\mathcal{U}$ (not necessarily satisfying Conjecture A) which counts how "non-definable" $\mathcal{U}$ is.

Definition 2.2. *The* $\bigvee$-dimension *of* $\mathcal{U}$*, denoted by* $\mathrm{vdim}(\mathcal{U})$*, is the maximum* k *such that* $\mathcal{U}$ *contains a compatible subgroup isomorphic to* $\mathbb{Z}^k$*, if such* k *exists, and* ∞*, otherwise.*

We prove in Theorem 2.5 below that Conjecture A is equivalent to the following.

Conjecture B. *Let* $\mathcal{U}$ *be an connected, abelian, locally definable group, which is generated by a definably compact set.*

(1) $\mathrm{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$*. In particular,* $\mathrm{vdim}(\mathcal{U})$ *is finite.*
(2) If $\mathcal{U}$ *is not definable, then* $\mathrm{vdim}(\mathcal{U}) > 0$*.*

In Section 3 we prove Conjecture A for definably generated subgroups of $\langle R^n, +\rangle$, where R is a real closed field and $\mathcal{M}$ is an o-minimal expansion of R.

Unless otherwise stated, $\mathcal{U}$ *denotes a connected, abelian, locally definable group, which is generated by a definably compact set.*

Lemma 2.3. *Assume that* $\mathcal{U}$ *contains a definable generic set. If* $\mathcal{U}$ *is not definable, then* $\mathrm{vdim}(\mathcal{U}) > 0$*.*

Proof. By Theorem 2.1, the group $\mathcal{U}^{00}$ exists and for some $k \in \mathbb{N}$, we have $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for a compact Lie group K. We claim that $k > 0$. Indeed, if $k = 0$ then $\mathcal{U}/\mathcal{U}^{00} = K$ is compact. But then, by [4, Lemma 7.3], the preimage of K would be contained in a definable subset of $\mathcal{U}$, and thus $\mathcal{U}$ would be definable, a contradiction.

If we now apply Theorem 2.1 (4) $\Rightarrow$ (2), we see that $\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$ so $\mathrm{vdim}(\mathcal{U}) \geq k > 0$. $\square$

Proposition 2.4. *Assume that* $\mathcal{U}$ *admits a lattice. Then,*

(i) If Λ *is a* 0*-dimensional, compatible subgroup of* $\mathcal{U}$*, then* $\Lambda \simeq \mathbb{Z}^l + F$*, with* $l \leq \mathrm{vdim}(\mathcal{U})$ *and* F *a finite subgroup of* $\mathcal{U}$*. Moreover,* $\mathrm{vdim}(\mathcal{U}) \leq \dim(\mathcal{U})$*.*

(ii) If Λ *is a lattice in* $\mathcal{U}$*, then* $\Lambda \simeq \mathbb{Z}^l + F$*, with* $l = \mathrm{vdim}(\mathcal{U})$ *and* F *a finite subgroup of* $\mathcal{U}$*. If, in addition,* $\mathcal{U}$ *is torsion-free, then* Λ *is isomorphic to* $\mathbb{Z}^l$*, with* $l = \dim(\mathcal{U})$*.*

Proof. This is extracted from the proof of [4, Theorem 7.6].

(i) By Theorem 2.1, there is a number k with $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, where K is a compact Lie group. We fix this number k. By the same theorem, $\mathcal{U}$ admits a lattice isomorphic to $\mathbb{Z}^k$, so $\mathrm{vdim}(\mathcal{U}) \geq k$. Our proof below implies, in particular, that $\mathrm{vdim}(\mathcal{U}) = k$.

By [4, Theorem 7.7], $\dim \mathcal{U} = k + \dim K$, so $k \leq \dim(\mathcal{U})$. Also, by the same result, if $\mathcal{U}$ is torsion-free, then $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^{\dim(\mathcal{U})}$.

Assume now that $\Lambda \subseteq \mathcal{U}$ is a 0-dimensional compatible subgroup. Consider $\phi : \mathcal{U} \to \mathcal{U}/\Lambda$. We claim that $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing $\mathcal{U}^{00}$. Then, since $\phi \restriction X$ is definable, we must have $\ker(\phi) \cap \mathcal{U}^{00} \subseteq \ker(\phi) \cap X$ finite. However, by [4, Theorem 7.4], the group $\mathcal{U}^{00}$ is torsion-free, so $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$.

Consider the map $\pi_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^k \times K$ and let Γ be the image of $\ker(\phi)$ under π_U. We just showed that Γ is isomorphic to $\Lambda = \ker(\phi)$. We claim that Γ is discrete. Indeed, using X as above we can find another definable set X' whose image $\pi_{\mathcal{U}}(X')$ contains an open neighborhood of 0 and no other elements of Γ, so Γ is discrete.

Now, since K is compact, the projection Γ' of Γ into $\mathbb{R}^k$ has a finite kernel $F \subseteq K$. Furthermore, Γ' is a discrete subgroup of $\langle \mathbb{R}^k, + \rangle$, and hence $\Gamma' \simeq \mathbb{Z}^l$, for some $l \leq k$. Therefore, $\Gamma \simeq \mathbb{Z}^l \times F$, so $\Lambda \simeq \mathbb{Z}^l \times F$. In particular, if $\Lambda \simeq \mathbb{Z}^l$, then $l \leq k$, which implies $\mathrm{vdim}\mathcal{U} \leq k \leq \dim \mathcal{U}$, as required. Moreover, $k = \mathrm{vdim}(\mathcal{U})$.

(ii) Assume now that $\Lambda \simeq \mathbb{Z}^l \times F$ is a lattice in $\mathcal{U}$. Namely, $\mathcal{U}/\Lambda$ is a definable group G. We proceed to show that $l = k$. Let $X \subseteq \mathcal{U}$ be a definable set so that $\phi(X) = G$. Then $X + \ker(\phi) = \mathcal{U}$. Thus, $\pi_{\mathcal{U}}(X) + \Gamma = \mathbb{R}^k \times K$. Let Y, F' and Γ' be the projections of $\pi_{\mathcal{U}}(X)$, F and Γ, respectively, into $\mathbb{R}^k$. We have $Y + \Gamma' = \mathbb{R}^k$. Since X is definable, the set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \ldots, \lambda_l$ be the generators of $\ker(\phi)$ and let $v_1, \ldots, v_l \in \mathbb{R}^k$ be their images in Γ'. If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \ldots, v_l$ then $Y + H + F' = \mathbb{R}^k$, and therefore, since Y is compact and F' finite, we must have $H = \mathbb{R}^k$. This implies that $l = k$. □

Theorem 2.5. *Conjecture A is equivalent to Conjecture B. More precisely,*

(i) If $\mathcal{U}$ admits a definable generic set then $\mathcal{U}$ satisfies clauses (1), (2) of Conjecture B.

(ii) Conjecture B implies Conjecture A.

Proof. (i). By Proposition 2.4 and Lemma 2.3.

(ii). By [4, Claim 7.10], in order to prove Conjecture A we may assume that $\mathcal{U}$ is generated by a definably compact set. Hence, $\mathcal{U}$ satisfies the assumptions of Conjecture B.

Let $\Lambda \simeq \mathbb{Z}^k$ be a compatible subgroup of $\mathcal{U}$ with $k = \mathrm{vdim}(\mathcal{U})$. We will prove that the locally definable group $\mathcal{U}/\Lambda$ is actually definable.

Assume that $\mathcal{U}/\Lambda$ is not definable. By Conjecture B(2) (applied to $\mathcal{U}/\Lambda$), there exists some $a \in \mathcal{U}/\Lambda$ such that $\mathbb{Z}a$ is a compatible subgroup of $\mathcal{U}/\Lambda$, and for every n, $na \neq 0$. Let $b \in \mathcal{U}$ be an element that projects via $\phi : \mathcal{U} \to \mathcal{U}/\Lambda$ to a. Clearly, $\mathbb{Z}b \cap \Lambda = \{0\}$. We claim that $\Lambda + \mathbb{Z}b$ is a compatible subgroup of $\mathcal{U}$, contradicting $k = \mathrm{vdim}(\mathcal{U})$. Let $X \subseteq \mathcal{U}$ be definable. The image of $X \cap (\Lambda + \mathbb{Z}b)$ under ϕ is contained in $\phi(X) \cap \mathbb{Z}a$. Since ϕ is locally definable, $\phi(X)$ is definable. Therefore $\phi(X) \cap \mathbb{Z}a$ is finite, by compatibility of $\mathbb{Z}a$. The preimage of this finite set under π is a union of sets $\Lambda + x$, $x \in B$, for some finite $B \subseteq \mathbb{Z}b$. So $X \cap (\Lambda + \mathbb{Z}b)$ is equal to the finite union of the sets $X \cap (\Lambda + x)$, $x \in B$, each of which is finite, because so is $(X - x) \cap \Lambda$ by compatibility of Λ. Hence $X \cap (\Lambda + \mathbb{Z}b)$ is finite, and thus $\Lambda + \mathbb{Z}b$ is compatible. □

We conclude this section with a statement mentioned in the introduction for the number k from Theorem 2.1. We prove it for $\bigvee$-dimension in general.

Corollary 2.6. *Assume Conjecture A holds. Suppose* $\text{vdim}(\mathcal{U}) = \dim(\mathcal{U}) > 0$. *Then* $\mathcal{U}$ *does not contain any non-trivial definable subgroup.*

Proof. Assume that H is a non-trivial definable subgroup of $\mathcal{U}$. Then $\mathcal{U}/H$ has smaller dimension than $\mathcal{U}$, but, as can easily be seen, the same $\bigvee$-dimension with $\mathcal{U}$. So $\text{vdim}(\mathcal{U}/H) > dim(\mathcal{U}/H)$, which contradicts Conjecture B for $\mathcal{U}/H$. □

3. A proof of Conjecture A for subgroups of R^n

We assume here that $\mathcal{M}$ *is an o-minimal expansion of a real closed field* R

Recall that for $X \subseteq R^n$, we write $X(m)$ for the addition of $X - X$ to itself m times. If $0 \in X$ then $X \subseteq X(m)$.

Definition 3.1. *A subset of* R^n *is called* convex with respect to R (or R-convex) *if for all* $x, y \in X$*, the line segment connecting* x *and* y *is also in* X*.*

The R-convex hull *of* X *is the smallest* R*-convex subset of* R^n *containing* X*. It consists of all finite combinations* $\sum_{i=1}^m t_i x_i$*, where the* x_i*'s are in* X*, all* $t_1 \geq 0$ *and* $\sum t_i = 1$*.*

Lemma 3.2. *If* $X \subseteq R^n$ *is definable then the* R*-convex hull of* X *is also definable.*

Proof. More precisely, we claim that the following set equals the R-convex hull of X:

$$X' = \left\{ \sum_{i=1}^{n+1} t_i x_i : t_1 + \cdots + t_{n+1} = 1, t_i \in [0,1], x_i \in X \right\}.$$

Indeed, by Caratheodory's Theorem, every convex combination of any number of points from X can also be realized as a combination of $n+1$ of these points, hence the R-convex hull of X equals X'. (Note that although Caratheodory's theorem is usually proved over the reals the same proof works over any ordered field. Alternatively, the statement over the real numbers implies, by transfer, the same result over any real closed field). □

We denote by Δ the diagonal $\{\langle x, x\rangle : x \in X\}$.

Proposition 3.3. *Let* $\mathcal{M}$ *is an o-minimal expansion of a real closed field* R*. Assume that* $X \subseteq R^n$ *is a definably connected set containing* 0*. Then there is* m *such that* $X(m)$ *(in the sense of the additive group* $\langle R, +\rangle$*) contains the* R*-convex hull of* X*.*

Proof. Given $f : X \to Z$, the fiber power of X is defined as:

$$X \times_f X = \{\langle x, y\rangle \in X \times X : \ f(x) = f(y)\}.$$

Clearly, the diagonal Δ is contained in $X \times_f X$.

Note that for $\langle x_1, x_2\rangle, \langle y_1, y_2\rangle \in X \times_f X$, there is a continuous definable path in $X \times_f X$, connecting the two points if and only if there are definable continuous curves $\gamma_1, \gamma_2 : [0,1] \to X$ such that $\gamma_i(0) = x_i$, $\gamma_i(1) = y_i$, and for every $t \in [0,1]$ we have $f(\gamma_1(t)) = f(\gamma_2(t))$.

Claim 3.4. *For $X \subseteq R^n$, consider the projection $\pi : R^n \to R$ onto the first coordinate. Assume that $\pi(x_1) = \pi(x_2)$, $\pi(y_1) = \pi(y_2)$ (in particular, $\pi_1(x_1 - x_2) = \pi(y_1 - y_2) = 0$). Assume further that $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$ are in the same connected component of $X \times_\pi X$. Then the elements $x_1 - x_2$ and $y_1 - y_2$ are in the same connected component of the set $(X - X) \cap \{0\} \times R^{n-1}$.*

Proof. Note that the image of $X \times_\pi X$ under the binary map $\langle x, y\rangle \mapsto x - y$ is contained in the set $\{0\} \times R^{n-1}$. Consider the restriction of this map to the connected component of $X \times_\pi X$ which contains $\langle x_1, x_2\rangle$ and $\langle y_1, y_2\rangle$. The image is connected and clearly contains $x_2 - x_1$ and $y_2 - y_1$. □

Claim 3.5. *Assume that $x, y \in X$, $\pi(x) = \pi(y)$ and that there is a curve*

$$\gamma = (\gamma_1, \dots, \gamma_n) : [0, 1] \to X$$

connecting x and y inside X (note that $\gamma_1(0) = \gamma_1(1)$). Let Γ be the image of γ.

(1) Assume that γ_1 is constant on $[0, 1]$. Then for all x_1, x_2 in the image of γ, $\langle x_1, x_2\rangle$ and some $\langle z, z\rangle \in \Delta$ are in the same connected component of $X \times_\pi X$.

(2) If for some $a \in (0, 1)$, γ_1 is increasing on $(0, a)$ and decreasing on $(a, 1)$ then $y - x$ and 0 are in the same connected component of $(X - X) \cap \{0\} \times R^{n-1}$.

(3) If for some $a_1 < a_2$ in $(0, 1)$, γ_1 is increasing on $(0, a_1)$, constant on (a_1, a_2) and decreasing on $(a_2, 1)$ then $y - x$ and 0 are in the same connected component of $(X - X) \cap \{0\} \times R^{n-1}$.

Proof. (1) Assume that $x_1 = \gamma(a_1)$ and $x_2 = \gamma(a_2)$, for $a_1 < a_2$. Fix some $a \in (a_1, a_2)$ and let $z = \gamma(a)$. Now re-parameterize the piece of Γ between $\gamma(a)$ and $\gamma(a_2)$, by an injective $\alpha : [a_1, a] \to \Gamma$ so that $\alpha(a) = \gamma(a) = z$ and $\alpha(a_1) = \gamma(a_2) = x_2$ (we can do it using a linear decreasing bijection between the intervals $[a_1, a]$ and $[a, a_2]$). Note that by assumption for all $t \in [a_1, a]$, the function $\pi(\gamma(t)) = \pi(\alpha(t))$. The curves $\gamma|[a_1, a]$ and α witness the fact that $\langle x_1, x_2\rangle$ and $\langle z, z\rangle$ are in the same component of $X \times_\pi X$.

(2) Let $[b_1, b_2]$ be the image of γ under π. By assumptions, $\pi(\gamma_1(0)) = \pi(\gamma_1(1)) = b_1$, $\pi(\gamma_1(a)) = b_2$ and the restrictions of π to the pieces $\gamma([0, a])$ and $\gamma([a, 1])$ are both injective. Let α_1, α_2 be their inverse maps, respectively (so these are maps from $[b_1, b_2]$ into Γ). We have $\alpha_1(b_1) = x$, $\alpha_2(b_1) = y$, $\alpha_1(b_2) = \alpha_2(b_2) = \gamma(a)$. Moreover, for every $t \in [b_1, b_2]$ we have $\pi(\alpha_1(t)) = \pi(\alpha_2(t)) = t$. It follows that $\langle x, y\rangle$ and $\langle\gamma(a), \gamma(a)\rangle$ are in the same component of $X \times_\pi X$, so by Claim 3.4, $y - x$ and 0 are in the same component of $(X - X) \cap \{0\} \times R^{n-1}$.

(3) As in (2), let $[b_1, b_2]$ be the image of γ under π. It is easy to see that $\gamma_1(t) = b_2$ for all $t \in [a_1, a_2]$. Similarly to the proof of (2), $\langle x, y\rangle$ and $\langle\gamma(a_1), \gamma(a_2)\rangle$ are in the same component of $X \times_\pi X$. Using (1), we see that $\langle\gamma(a_1), \gamma(a_2)\rangle$ is in the same component as $\langle z, z\rangle$ for some $z \in \gamma([a_1, a_2])$. Applying Claim 3.4, we conclude that $x - y$ and 0 are in the same component of $(X - X) \cap \{0\} \times R^{n-1}$. □

We now return to the proof of the main proposition. So, X is a definably connected subset of R^n containing 0, and we want to show that for some m, the convex hull of X is contained in $X(m)$.

We will use induction on n. If $n = 1$ then X is already convex. So, we assume that the result is true for $X \subseteq R^n$ and prove it for $X \subseteq R^{n+1}$. We take $x, y \in X$ and first want to show that for some m the line segment $[x, y]$ (i.e the line connecting x and y in R^{n+1}) is contained in $X(m)$.

Using a linear automorphism of R^n, we may assume that $\pi(x) = \pi(y) = 0$. Since X is definably connected, there exists a definable curve $\gamma : [0, 1] \to X$ connecting x and y. Let $\Gamma \subseteq X$ be the image of γ and again let $\gamma_1 = \pi \circ \gamma$.

Notation: For $f : [0, 1] \to R$ continuous, let $k = k(f)$ be the minimal natural number so that there are $0 = a_0 < a_1 < \cdots < a_k = 1$ and f is either constant or strictly monotone on $[a_i, a_{i+1}]$.

We consider the map $\gamma_1 : [0, 1] \to R$ and prove the result by sub-induction on $k(\gamma_1)$.

Assume first that $k(\gamma_1) = 1$, namely that γ_1 is constant on $[0, 1]$. In this case, Γ is contained in $\{0\} \times R^n$, so we can work in R^n and use the inductive hypothesis to conclude that the line segment $[x, y]$ is contained in $\Gamma(m)$ for some m. Clearly, $\Gamma(m) \subseteq X(m)$ so we are done.

Assume then that $k(\gamma_1) > 1$, so γ_1 is not constant. Without loss of generality, γ_1 takes some positive value on $(0, 1)$, so let $a \in (0, 1)$ be a point where γ_1 takes its maximum value in $[0, 1]$.

Case 1 Assume first that γ_1 is not locally constant at a.

Then there are $a_1 < a < a_2$ such that γ_1 is increasing on (a_1, a), decreasing on (a, a_2), $\gamma_1(a_1) = \gamma_1(a_2)$, and furthermore, either a_1 or a_2 are local minimum for γ_1. Indeed, we take $a_1' < a$ to be the minimum of all points t such that γ_1 is increasing on (t, a), take $a_2' > a$ be the maximum of all $t > a$ such that γ_1 is decreasing on (a, t). (In this case, a_1' and a_2' are local minima for γ_1). We then compare $\gamma_1(a_1')$ and $\gamma_1(a_2')$. If $\gamma_1(a_1') > \gamma_1(a_2')$ then we take $a_1 := a_1'$ and take a_2 to be the unique element of the interval (a, a_2') such that $\gamma_1(a_2) = \gamma_1(a_1)$. Otherwise, we do the opposite.

Let $x_1 = \gamma(a_1)$ and $x_2 = \gamma(a_2)$. Consider now the curve Γ' which is the image of $[a_1, a_2]$ under γ. By Claim 3.5 (2), $x_2 - x_1$ and 0 are in the same connected component of $(\Gamma' - \Gamma') \cap \{0\} \times R^n$. But then, we can view this component as living in R^n, so by inductive hypothesis there exists m such that the line segment connecting 0 and $x_2 - x_1$ is contained in $(\Gamma' - \Gamma')(m)$. By adding x_1 to both sides, we see that the line segment connecting x_1 and x_2 is contained in $(X - X)(m + 1)$. Hence, after replacing X with $X(m)$, we can also replace the original curve Γ with a new curve Γ'', in which the piece $\gamma([a_1, a_2])$ was replaced by a linear segment all of whose points project to the same point $\pi(x_1)$. Let $\gamma'' : [0, 1] \to X$ be the map whose image is Γ'' (so $\gamma'' = \gamma$ everywhere, except on $[a_1, a_2]$, in which the image is linear and γ_1'' is constant). Because a_1 or a_2 is a local minimum of γ_1, it is easy to see that $k(\gamma_1'') = k(\gamma_1) - 1$. By sub-inductive hypothesis, the line connecting x and y is contained in some $X(m')$.

Case 2 Assume that γ_1 is locally constant at a.

So, there are $a_1' \leq a \leq a_2'$ such that γ_1 is constant on $[a_1', a_2']$ and this is a maximal such interval. As in Case 1, we can find $a_1 < a_1'$ and $a_2 > a_2'$ such that γ_1 is increasing on $[a_1, a_1']$, decreasing on $[a_2', a_2]$, $\gamma_1(a_1) = \gamma_1(a_2)$ and furthermore, either a_1 or a_2 is a local minimum of γ_1.

Let Γ' be the piece of Γ connecting $\gamma(a_1)$ and $\gamma(a_2)$. Then, by Claim 3.5(3), the points $\gamma(a_2) - \gamma(a_1)$ and 0 are in the same component of $(\Gamma' - \Gamma') \cap \{0\} \times R^n$. Again, by inductive hypothesis, the line segment connecting 0 and $\gamma(a_2) - \gamma(a_1)$ is contained in $(\Gamma' - \Gamma')(m)$ for some m, so the line segment connecting $\gamma(a_1)$ and $\gamma(a_2)$ is contained in $X(m+1)$. As in Case (1), we can replace Γ by Γ'', in which the piece $\gamma([a_1, a_2])$ is replaced by the line segment connecting $\gamma(a_1)$ and $\gamma(a_2)$. Again, the map $\gamma'' : [0,1] \to X$ whose image is Γ'' now satisfies $k(\gamma_1'') = k(\gamma_1) - 2$ (because we replaced three pieces by one). By sub-inductive hypothesis, the line connecting x and y is in some $X(m')$.

We therefore showed that for every $x, y \in X$, there exists m such that the line segment $[x, y]$ is contained in $X(m)$. To see that we can find a uniform m for all $x, y \in X$, we use logical compactness (writing a type $p(x, y)$, which says that the line segment $[x, y]$ is not contained in any $X(m)$). This ends the proof of the proposition $\square$

Question it is interesting to ask what is the required m. the argument suggests that it depends on the possible number of "twistings" of the curve connecting two points in X. But maybe this is just an effect of the proof and one can find uniform such m which depends only on the ambient R^n)?

Theorem 3.6. *Let $\mathcal{M}$ be an o-minimal expansion of a real closed field R and let X be a definably connected subset of R^n containing 0, and $\mathcal{U}$ the subgroup of $\langle R^n, + \rangle$ generated by X. Then $\mathcal{U}$ contains a definable generic set.*

Proof. As was shown in [4, Claim 7.10], it is sufficient to prove the result for bounded X so we assume that X is bounded. We prove the result by induction on n.

For $n = 1$, the set X is an interval containing 0, so $X - X$ is an interval whose endpoints are $\pm a$ for some $a > 0$. The group generated by X is them equal to $(X - X) + \mathbb{Z}a$, so $X - X$ is generic in it.

We assume the result for $k < n$ and prove it for $X \subseteq R^n$.

By Proposition 3.3, there exists an m such that $X(m)$ contains the R-convex hull X' of X. By Lemma 3.2, this R-convex hull is definable. Clearly, $X \subseteq X'$ so we may assume from now on that X itself is convex with respect to R. Since $\mathcal{U}$ is closed in R^n, we can replace X by its topological closure, still R-convex, and assume that X is closed.

Since X is bounded and convex, we can find an $n-1$ dimensional subspace $H \subseteq R^n$ and points $x_1, x_2 \in X$, such that X lies entirely between the planes $x_1 + H$ and $x_2 + H$. Let ℓ be the interior of the line segment connecting x_1 and x_2. By convexity of X, $\ell \subseteq X$. Without loss of generality, $0 \in \ell$ and, moreover, 0 divides the segment ℓ into two equal parts (if not, we replace X by $X - y \subseteq X - X$ for some $y \in X$. The set $X - y$ is convex and still generates $\mathcal{U}$).

Let $v = x_2 - x_1$, and let $\pi_H : R^n \to H$ be the projection onto H along the line ℓ (namely, in the direction of v). Because X is trapped between $x_1 + H$ and $x_2 + H$, for every point $x \in X$, we have $|x - \pi_H(x)| < 1/2|v|$. It follows that $X \subseteq \pi_H(X) + \ell$.

We also claim that $\pi_H(X)+\ell \subseteq (X-X)+(X-X)$. Indeed, every element in $\pi_H(X) + \ell$ is of the form $y = \pi_H(x) + tv$, for $|t| < 1/2$. We also have $x - \pi_H(x) = t'v$, for $|t'| < 1/2$, hence $y = x + (t - t')v \subseteq X + \ell - \ell \subseteq X(2)$.

To sum up, we found a subset $X_1 = \pi_H(X)$ of H and a line segment ℓ such that $X \subseteq X_1 + \ell \subseteq X(2)$. This implies that $\mathcal{U}$ is generated by $X_1 + \ell$.

Let $\mathcal{U}_1$ be the subgroup of H generated by X_1 and let $\mathcal{I}$ be the 1-dimensional subgroup of the group Rv generated by ℓ. Clearly, $\mathcal{U} = \mathcal{U}_1 + \mathcal{I}$. By induction, $\mathcal{U}_1$ contains a definable generic subset Y and $\mathcal{I}$ contains a definable generic set I. The set $Y + I$ is therefore generic in $\mathcal{U}$, thus finishing the proof. □

Note that the above argument implies that $\mathcal{U}$ is generated by a convex set and therefore is itself convex in R^n. This immediately implies that $\mathcal{U}$ is divisible. In [4], we prove more generally that Conjecture A implies that every connected divisible, definably generated abelian group is divisible.

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