SMALL SETS IN DENSE PAIRS

PANTELIS E. ELEFTHERIOU

We consider expansions $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$ of an o-minimal structure $\mathcal{M}$ by a set $P \subseteq M$, such that the geometric behavior on the class of all definable sets is tame. An important category of such structures is when every definable open set is already definable in $\mathcal{M}$ ([1, 2, 4, 5, 7]). Three main examples of this category are:

(1) Dense pairs
(2) Expansions of $\mathcal{M}$ by a dense independent set
(3) Expansions of real closed field $\mathcal{M}$ by a dense divisible subgroup P of $\langle M^{>0}, \cdot\rangle$ with the Mann property.

In [7], all these examples were put under a common perspective, and a cone decomposition theorem was proved for their definable sets. That theorem aimed to provide an understanding of all definable sets in terms of sets definable in $\mathcal{M}$ and 'P-bound' sets. Corollary 5 below further reduces the study of P-bound sets to that of subsets of some P^l definable in $\widetilde{\mathcal{M}}$.

Notation. We fix an o-minimal expansion $\mathcal{M} = \langle M, <, +, 0, \dots\rangle$ of an ordered group with a distinguished positive element 1. We denote by $\mathcal{L}$ its language, and by dcl the usual definable closure operator in $\mathcal{M}$. An '$\mathcal{L}_A$-definable' set is a set definable in $\mathcal{M}$ with parameters from A. We also fix some $P \subseteq M$ and denote $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P\rangle$. An '($A$-)definable set' is a set definable in $\widetilde{\mathcal{M}}$ with parameters from A. We drop the above indices 'A', if we do not want to specify the parameters. Finally, we let D denote a subset of M.

Definition 1 ([4])**.** A set $X \subseteq M^n$ is called *P-bound over A* if there is an $\mathcal{L}_A$-definable function $h : M^m \to M^n$ such that $X \subseteq h(P^m)$.

In the aforementioned examples, P-boundness amounts to a precise topological notion of smallness ([7, Definition 2.1]), as well as to the classical notion of P-internality from geometric stability theory ([7, Corollary 3.12]). In [7], we asked:

Question 2. *Is every P-bound set in definable bijection with a subset of P^n, for some n?*

The main difficulty in answering the above question is that in $\widetilde{\mathcal{M}}$, most 'choice properties' generally fail. For example, it is known that a dense pair does not eliminate imaginaries and does not admit definable Skolem functions ([2, Section 5]. If P is a dense independent set, then $\widetilde{\mathcal{M}}$ eliminates imaginaries but does not admit definable Skolem functions ([3]). We observe here (Corollary 5 below) that all that is needed in order to answer the above question is that the induced structure on P by $\mathcal{M}$ eliminates imaginaries.

Definition 3. Let $D, P \subseteq M$. The *D-induced structure on P by $\mathcal{M}$*, denoted by $P_{ind(D)}$, is a structure whose language is

$$\mathcal{L}_{ind(D)} = \{R_{\phi(x)}(x) : \phi(x) \in \mathcal{L}_D\}$$

and, for every tuple $a \subseteq P$,

$$P_{ind(D)} \models R_\phi(a) \;\Leftrightarrow\; \mathcal{M} \models \phi(a).$$

We prove (Proposition 6) that in our examples, $P_{ind(D)}$ eliminates imaginaries, for any $D \subseteq M$ which is dcl-independent over P. We work in a general setting. Consider the following properties for $\widetilde{\mathcal{M}}$ and D:

(OP) (Open definable sets are $\mathcal{L}$-definable.) For every set A such that $A \setminus P$ is dcl-independent over P, and for every A-definable set $V \subset M^n$, its topological closure $\overline{V} \subseteq M^n$ is $\mathcal{L}_A$-definable.

$(\mathrm{dcl})_D$ Let $B, C \subseteq P$ and

$$A = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD) \cap P.$$

Then

$$\mathrm{dcl}(AD) = \mathrm{dcl}(BD) \cap \mathrm{dcl}(CD).$$

$(\mathrm{ind})_D$ Every A-definable set in $P_{ind(D)}$ is the trace of an $\mathcal{L}_{AD}$-definable set.

Properties (OP) and $(\mathrm{ind})_D$ already appear in the literature and are known for our three examples ([7]). Property $(\mathrm{dcl})_D$ is introduced here. We prove the following results ([6]).

Theorem 4. *Suppose* (OP)*,* $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D*. Then* $P_{ind(D)}$ *eliminates imaginaries.*

Corollary 5. *Suppose* (OP)*,* $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and* D*, and let* $X \subseteq M^n$ *be a* D*-definable set. If* X *is* P*-bound over* D*, then there is a* D*-definable injective map* $\tau : X \to P^l$*.*

Proof. Let $h : M^m \to M^n$ be an $\mathcal{L}_A$-definable map such that $X \subseteq h(P^m)$, and consider the following equivalence relation E on M^m:

$$xEy \Leftrightarrow h(x) = h(y).$$

Note that $E \cap (P^m \times P^m)$ is an equivalence relation on P^m, which is $\emptyset$-definable in $P_{ind(D)}$. Since $P_{ind(D)}$ eliminates imaginaries, there is a $\emptyset$-definable in $P_{ind(D)}$ map $f : P^m \to P^l$, for some l, such that for every $x, y \in P^m$,

$$f(x) = f(y) \Leftrightarrow xEy.$$

Define $\tau : X \to P^l$, given by $\tau(h(x)) = f(x)$. Then τ is well-defined, injective and D-definable (in $\widetilde{\mathcal{M}}$). □

We verify $(\mathrm{dcl})_D$ in our three main examples.

Proposition 6. *Let* $\widetilde{\mathcal{M}} = \langle \mathcal{M}, P \rangle$ *be a dense pair, or an expansion of* $\mathcal{M}$ *by a dense independent set or by a dense divisible multiplicative group with the Mann Property. Let* $D \subseteq M$ *be* dcl*-independent over* P*. Then* $(\mathrm{dcl})_D$ *holds. Hence* $P_{ind(D)}$ *eliminates imaginaries.*

The assumption that D is dcl-independent over P is necessary. Namely, without it, $P_{ind(D)}$ need not eliminate imaginaries. However, even without it, we still obtain the following corollary, which in particular applies to our examples.

Corollary 7. *Suppose* (OP)*,* $(\mathrm{dcl})_D$ *and* $(\mathrm{ind})_D$ *hold for* $\widetilde{\mathcal{M}}$ *and every* $D \subseteq M$ *which is* dcl*-independent over* P*. Let* $X \subseteq M^n$ *be an* A*-definable set. If* X *is* P*-bound over* A*, then there is an* $A \cup P$*-definable injective map* $\tau : X \to P^l$*.*

Allowing parameters from P is standard practice when studying definability in this context; see for example also [7, Lemma 2.5, Corollary 3.24].

Finally, we show that Theorems A and B are optimal also in the following way. Let D be dcl-independent over P. Suppose (OP) and $(\text{ind})_D$ hold for $\widetilde{\mathcal{M}}$ and D. Then:

$$P_{ind(D)} \text{ eliminates imaginaries } \Leftrightarrow (\text{dcl})_D.$$

If we do not assume (OP), the above two properties need not hold. We do not know whether they hold, if we assume (OP) and $(\text{ind})_D$. Finally, (OP) does not imply $(\text{ind})_D$, but we do not know whether $(\text{ind})_D$ is necessary for P_{ind} to eliminate imaginaries.

References

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