ON WEAKLY O-MINIMAL NON-VALUATIONAL STRUCTURES

PANTELIS ELEFTHERIOU, ASSAF HASSON, AND GIL KEREN

Abstract. For a weakly o-minimal expansion $\mathcal{M} = \langle M, <, +, \ldots\rangle$ of an ordered group, we introduce the notion 'no external limits' and prove that $\mathcal{M}$ is o-minimal if and only if it has no external limits and admits definable Skolem functions. We then show that all known examples of weakly o-minimal non-valuational expansions of ordered groups have no external limits and thus obtain a large collection of such structures that do not have definable Skolem functions, extending a result from Shaw [8].

1. Extended abstract

A structure $\mathcal{M} = \langle M, <, \ldots\rangle$ is called *o-minimal* if every definable subset of M is a finite union of points and intervals [3, 7]. $\mathcal{M}$ is called *weakly o-minimal* if every definable subset of M is a finite union of convex sets [2, 6]. Examples of weakly o-minimal structures are:

(a) $\mathcal{R} = \langle R, Fin(R)\rangle$, a non-archimedean real closed field R expanded by its natural valuation ring $Fin(R)$.
(b) $\mathcal{R} = \langle \mathbb{R}_{alg}, (0,\pi)\rangle$, the field of real algebraic numbers expanded by the convex set $(0,\pi)$.

These are the archetypical examples of two categories of weakly o-minimal structures that can be distinguished by their 'definable cuts'. A pair (C, D) of non-empty subsets of M is called a *cut in* $\mathcal{M}$ if $C < D$ and $C \cup D = M$. It is called a *definable cut* if C (and D) are definable. If $\mathcal{M} = \langle M, <, +, \ldots\rangle$ expands an ordered group, then $\mathcal{M}$ is called *non-valuational* if for every definable cut (C, D) in $\mathcal{M}$, the infimum $\inf\{y - x : x \in C, y \in D\}$ exists in M (and must equal 0). Otherwise, it is called *valuational*. Example (a) above is valuational and (b) is non-valuational. We denote by $\overline{M}$ the set of all definable cuts (C, D) in M such that C has no maximum element. Then $\overline{M}$ has a natural order, where $(C, D) < (C', D')$ if and only if $C < C'$, which extends the order $<$ of M, where $a \in M$ is identified with $\big((-\infty, a), [a, +\infty)\big)$.

In [6], a weak cell decomposition theorem was proved for every weakly o-minimal structure $\mathcal{M}$. In [9], a strong cell decomposition theorem was shown in case $\mathcal{M}$ is non-valuational, exhibiting its resemblance to o-minimal structures. In the current note, we introduce the notion of having no external limits (Definition 1.1 below) and use it to prove that a large collection of weakly o-minimal non-valuational structures do not admit definable Skolem functions.

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Canonical examples of weakly o-minimal non-valuational structures are obtained by considering dense pairs of o-minimal structures, as introduced in [4]: let $\mathcal{N}$ be an o-minimal structure and $\mathcal{M}_1 \prec \mathcal{N}$ a dense elementary substructure. The theory of dense pairs is the theory of the structure $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ obtained by expanding $\mathcal{N}$ with a unary predicate for the universe M of $\mathcal{M}_1$. By a result of [1] the structure $\mathcal{M}$ induced from $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ on M is weakly o-minimal. Moreover, any definable set in $\mathcal{M}$ is of the form $M^n \cap S$ where $S \subseteq N^n$ is $\mathcal{N}$-definable, [4, Theorem 2]. Thus, any definable cut in $\mathcal{M}$ is of the form $\big((-\infty, a) \cap M, (a, +\infty)\big)$ for some $a \in N$. Since M is dense in N, we obtain that $\mathcal{M}$ is non-valuational.

We now give the main definition and results of this note.

Definition 1.1. A weakly o-minimal structure $\mathcal{M}$ *has no external limits* if for every definable $f : (a, b) \to M$ where $a, b \in M \cup \{\pm\infty\}$ and $\lim_{t\to a^+} f(t)$ or $\lim_{t\to a^+} f(t)$ exist in $\overline{M}$, then that limit exists in M. Otherwise, we say that $\mathcal{M}$ *has external limits.*

Theorem 1.2. *Let $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ be a dense pair and $\mathcal{M}$ the induced structure on the universe M of $\mathcal{M}_1$. Then every ordered reduct of $\mathcal{M}$ has no external limits.*

By [1], an expansion of an o-minimal structure by any number of convex sets is weakly o-minimal.

Theorem 1.3. *Let $\mathcal{M}_1 = \langle M, <, +, \ldots \rangle$ be an o-minimal expansion of an ordered group. Let $\mathcal{M} = \langle \mathcal{M}_1, \{C_i\}_{i\in I} \rangle$ be an expansion of $\mathcal{M}_1$ by a number of convex sets. Assume that $\mathcal{M}$ is non-valuational. Then there is a dense pair $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ of o-minimal structures such that $\mathcal{M}$ is the induced structure on M.*

It follows that the structure $\mathcal{M}$ from the last theorem has no external limits.

Although it is easy to construct a weakly o-minimal non-valuational structure which has external limits, we conjecture the following:

Conjecture 1.4. Let $\mathcal{M} = \langle M, <, +, \cdots \rangle$ be a weakly o-minimal non-valuational expansion of an ordered group. Then $\mathcal{M}$ has no external limits.

Theorem 1.5. *Let $\mathcal{M} = \langle M, <, +, \ldots \rangle$ be a weakly o-minimal expansion of an ordered group. Then $\mathcal{M}$ is o-minimal if and only if it has no external limits and admits definable Skolem functions.*

Corollary 1.6. *The weakly o-minimal structures from Theorems 1.2 and 1.3 do not admit definable Skolem functions.*

The reader may wonder if all weakly o-minimal non-valuational expansions of ordered groups can be obtained as (reducts of) the induced structure from a dense pair $\langle \mathcal{N}, \mathcal{M}_1 \rangle$ on the universe M of $\mathcal{M}_1$. This is not the case, as it can be shown with the structure $\big\langle \mathbb{Q}, <, +, \big\{\{(x, y) : x < \alpha y\}\big\}_{\alpha\in\mathbb{Q}(\pi)}\big\rangle$.

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Zukunftskolleg and Department of Mathematics and Statistics, University of Konstanz, Box 216, 78457 Konstanz, Germany
E-mail address: panteleimon.eleftheriou@uni-konstanz.de

Department of mathematics, Ben Gurion University of the Negev, Be'er Sheva, Israel
E-mail address: hassonas@math.bgu.ac.il

Department of mathematics, Ben Gurion University of the Negev, Be'er Sheva, Israel
E-mail address: cruvadom@gmail.com