

DEFINABLE QUOTIENTS OF LOCALLY DEFINABLE GROUPS

PANTELIS E. ELEFThERIOU AND YA'ACOV PETERZIL

ABSTRACT. We study locally definable abelian groups \mathcal{U} in various settings and examine conditions under which the quotient of \mathcal{U} by a discrete subgroup might be definable. This turns out to be related to the existence of the type-definable subgroup \mathcal{U}^{00} and to the divisibility of \mathcal{U} .

1. INTRODUCTION

This is the first of two papers (originally written as one) around groups definable in o-minimal expansions of ordered groups. The ultimate goal of this project is to reduce the analysis of such groups to semi-linear groups and to groups definable in o-minimal expansions of real closed fields. This reduction is carried out in the second paper ([8]). In the current paper, we prove a crucial lemma in that perspective, Theorem 3.10 below. This theorem is proved by analyzing \forall -definable abelian groups in various settings and investigating when such groups have definable quotients of the same dimension. The analysis follows closely known work on definably compact groups. We make strong use of their minimal type-definable subgroups of bounded index, and of the solution to so-called Pillay's conjecture in various settings.

In the rest of this introduction we recall the main definitions and state the results of this paper.

Until Section 3, and unless stated otherwise, \mathcal{M} denotes a sufficiently saturated, not necessarily o-minimal, structure.

If \mathcal{M} is κ -saturated, by *bounded* cardinality we mean cardinality smaller than κ . Since “bounded” has a different meaning in the context of an ordered structure we use “small” to refer to subsets of M^n of bounded cardinality. Every small definable set is therefore finite.

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1.1. \forall -definable and locally definable sets. A \forall -definable group is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{i \in I} X_i$ of definable subsets of M^n for some fixed n (where $|I|$ is bounded) and for every $i, j \in I$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). Following [6], we say that $\langle \mathcal{U}, \cdot \rangle$ is *locally definable* if $|I|$ is countable. We are mostly interested here in *definably generated* groups, namely \forall -definable groups which are generated as a group by a definable subset. These groups are locally definable. An important example of such groups is the universal cover of a definable group (see [7]). In [12, Section 7] a more general notion is introduced, of an Ind-definable group, where the X_i 's are not assumed to be subsets of the same sort and there are definable maps which connect them to each other.

A map $\phi : \mathcal{U} \rightarrow \mathcal{H}$ between \forall -definable (locally definable) groups is called \forall -definable (*locally definable*) if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, $\text{graph}(\phi) \cap (X \times Y)$ is a definable set. Equivalently, the restriction of ϕ to any definable set is definable.

Remark 1.1. If in the above definition, instead of M^n we allow all X_i 's to be subsets of a fixed sort S then the analogous definition of groups and maps works in \mathcal{M}^{eq} . This will allow us to discuss locally definable maps from a locally definable group \mathcal{U} onto an interpretable group \mathcal{V} .

1.2. Compatible subgroups.

Definition 1.2. (See [6]) For a \forall -definable group \mathcal{U} , we say that $\mathcal{V} \subseteq \mathcal{U}$ is a *compatible subset* of \mathcal{U} if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case \mathcal{V} itself is a bounded union of definable sets).

Clearly, the only compatible \forall -definable subsets of a definable group are the definable ones. Note that if $\phi : \mathcal{U} \rightarrow \mathcal{V}$ is a \forall -definable homomorphism between \forall -definable groups then $\ker(\phi)$ is a compatible \forall -definable normal subgroup of \mathcal{U} . Compatible subgroups are used in order to obtain \forall -definable quotients, but for that we need to restrict ourselves to locally definable groups. Together with [6, Theorem 4.2], we have:

Fact 1.3. *If \mathcal{U} is a locally definable group and $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup then \mathcal{H} is a compatible subgroup of \mathcal{U} if and only if there exists a locally definable surjective homomorphism of locally definable groups $\phi : \mathcal{U} \rightarrow \mathcal{V}$ whose kernel is \mathcal{H} .*

1.3. Connectedness. If \mathcal{M} is an o-minimal structure and $\mathcal{U} \subseteq M^n$ is a \forall -definable group then, by [2, Theorem 4.8], it can be endowed with a manifold-like topology τ , making it into a topological group. Namely, there exists a bounded collection $\{U_i : i \in I\}$ of definable subsets of \mathcal{U} , whose union equals \mathcal{U} , such that each U_i is in definable bijection with an open subset of M^k ($k = \dim \mathcal{U}$), and the transition maps are continuous. The group operation and group inverse are continuous with respect to this induced

topology. Moreover, the U_i 's are definable over the same parameters which define \mathcal{U} . The topology τ is determined by the ambient topology of M^n in the sense that at every generic point of \mathcal{U} the two topologies coincide. From now on, whenever we refer to a topology on G , it is τ we are considering.

Definition 1.4. (See [1]) In an o-minimal structure, a \forall -definable group \mathcal{U} is called *connected* if there exists no \forall -definable compatible subset $\emptyset \subsetneq \mathcal{V} \subsetneq \mathcal{U}$ which is both closed and open with respect to the group topology.

1.4. Definable quotients.

Definition 1.5. Given a \forall -definable group \mathcal{U} and $\Lambda_0 \subseteq \mathcal{U}$ a normal subgroup, we say that \mathcal{U}/Λ_0 is *definable* if there exists a definable group \overline{K} and a surjective \forall -definable homomorphism $\mu : \mathcal{U} \rightarrow \overline{K}$ whose kernel is Λ_0 .

One can define the notion of an *interpretable quotient* by replacing “ \overline{K} definable” by “ \overline{K} interpretable” in the above definition. Note, however, that in case \mathcal{M} is an o-minimal structure and \mathcal{U} is locally definable, such as in Section 3 below, by [6, Corollary 8.1], the group \mathcal{U} has strong definable choice for definable families of subsets of \mathcal{U} . Namely, for every definable family of subsets of \mathcal{U} , $\{X_t : t \in T\}$, there is a definable function $f : T \rightarrow \bigcup X_t$ such that for every $t \in T$, $f(t) \in X_t$ and if $X_{t_1} = X_{t_2}$ then $f(t_1) = f(t_2)$. In particular, every interpretable quotient of \mathcal{U} would be definably isomorphic to a definable group.

1.5. Results. Our results concern the existence of the type-definable group \mathcal{U}^{00} , for a \forall -definable abelian group \mathcal{U} . Recall ([12, Section 7]) that for a definable, or \forall -definable group \mathcal{U} , we write \mathcal{U}^{00} for the smallest, if such exists, type-definable subgroup of \mathcal{U} of bounded index. In particular we require that \mathcal{U}^{00} is contained in a definable subset of \mathcal{U} . From now on we use the expression “ \mathcal{U}^{00} exists” to mean that “there exists a smallest type-definable subgroup of \mathcal{U} of bounded index, which we denote by \mathcal{U}^{00} ”. Note that a type definable subgroup \mathcal{H} of \mathcal{U} has bounded index if and only if there are no new cosets of \mathcal{H} in \mathcal{U} in elementary extensions of \mathcal{M} .

When \mathcal{U} is a definable group in a NIP structure, then \mathcal{U}^{00} exists (see Shelah’s theorem in [18]). When \mathcal{U} is a \forall -definable group in a NIP structure or even in an o-minimal one, then \mathcal{U}^{00} may not always exist. However, if we assume that *some* type-definable subgroup of bounded index exists, then there is a smallest one (see [12, Proposition 7.4]). Recall that a definable $X \subseteq \mathcal{U}$ is called *left generic* if boundedly many translates of X cover \mathcal{U} . In Section 2, we prove the following theorem for \forall -definable groups:

Theorem 2.6. *Let \mathcal{U} be an abelian \forall -definable group in a NIP structure. If the definable non-generic sets in \mathcal{U} form an ideal and \mathcal{U} contains at least one definable generic set, then \mathcal{U}^{00} exists.*

We also prove (Corollary 2.12) that when we work in o-minimal expansions of ordered groups, for a \forall -definable abelian group which contains a

definable generic set and is generated by a definably compact set, the non-generic definable subsets do form an ideal (this is a generalization of the same result from [16] for definably compact group, which itself relies heavily on work in [5]).

In Section 3, we use these results to establish the equivalence of the following conditions.

Theorem 3.9. *Let \mathcal{U} be a connected abelian \forall -definable group in an o-minimal expansion of an ordered group, with \mathcal{U} definably generated. Then there is $k \in \mathbb{N}$ such that the following are equivalent:*

- (i) \mathcal{U} contains a definable generic set.
- (ii) \mathcal{U}^{00} exists.
- (iii) \mathcal{U}^{00} exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, where \mathbb{T} is the circle group and $r \in \mathbb{N}$.
- (iv) There is a definable group G , with $\dim G = \dim \mathcal{U}$, and a \forall -definable surjective homomorphism $\phi : \mathcal{U} \rightarrow G$.

If \mathcal{U} is generated by a definably compact set, then (ii) is strengthened by the condition that $k + r = \dim \mathcal{U}$.

We conjecture, in fact, that the conditions of Theorem 3.9 are always true.

Conjecture A. *Let \mathcal{U} be a connected abelian \forall -definable group in an o-minimal structure, which is definably generated. Then*

- (i) \mathcal{U} contains a definable generic set.
- (ii) \mathcal{U} is divisible.

We do not know if Conjecture A is true, even when \mathcal{U} is a subgroup of a definable group. We do show that it is sufficient to prove (i) under restricted conditions, in order to deduce the full conjecture. In a recent paper (see [9]) we prove that Conjecture A holds for definably generated subgroups of $\langle \mathbb{R}^n, + \rangle$, in an o-minimal expansion of a real closed field R .

Finally, we derive the theorem that is used in [8].

Theorem 3.10. *Let \mathcal{U} be a connected abelian \forall -definable group in an o-minimal expansion of an ordered group, with \mathcal{U} definably generated. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leq \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that \mathcal{U}/Λ' is a definable group.

If \mathcal{U} is generated by a definably compact set, then \mathcal{U}/Λ' is moreover definably compact.

1.6. Notation. Given a group $\langle G, \cdot \rangle$ and a set $X \subseteq G$, we denote, for every $n \in \mathbb{N}$,

$$X(n) = \overbrace{XX^{-1} \cdots XX^{-1}}^{n\text{-times}}$$

We assume familiarity with the notion of definable compactness. Whenever we write that a set is definably compact, or definably connected, we assume in particular that it is definable.

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2. \forall -DEFINABLE GROUPS AND TYPE-DEFINABLE SUBGROUPS OF BOUNDED INDEX

In this section, unless stated otherwise, \mathcal{M} denotes a sufficiently saturated, not necessarily o -minimal, structure.

2.1. Definable quotients of \forall -definable groups. We begin with a criterion for definability (and more generally interpretability) of quotients.

Lemma 2.1. *Let $\langle \mathcal{U}, \cdot \rangle$ be a \forall -definable group and Λ_0 a small normal subgroup of \mathcal{U} . Then the following are equivalent:*

- (1) *The quotient \mathcal{U}/Λ_0 is interpretable in \mathcal{M} .*
- (2) *There is a definable $X \subseteq \mathcal{U}$ such that (a) $X \cdot \Lambda_0 = \mathcal{U}$ and (b) for every definable $Y \subseteq \mathcal{U}$, $Y \cap \Lambda_0$ is finite.*
- (3) *There is a definable $X \subseteq \mathcal{U}$ such that (a) $X \cdot \Lambda_0 = \mathcal{U}$ and (b) $X \cap \Lambda_0$ is finite.*

Proof. (1 \Rightarrow 2). We assume that there is a \forall -definable surjective $\mu : \mathcal{U} \rightarrow \overline{K}$ with kernel Λ_0 , and \overline{K} interpretable. By saturation, there is a definable subset $X \subseteq \mathcal{U}$ such that $\mu(X) = \overline{K}$ and hence $X \cdot \Lambda_0 = \mathcal{U}$. Given any definable $Y \subseteq \mathcal{U}$, the restriction of μ to Y is definable and thus the small set $\ker(\mu|_Y) = Y \cap \Lambda_0$ is definable and, hence, finite.

(2 \Rightarrow 3). This is obvious.

(3 \Rightarrow 1) We claim first that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap \Lambda_0$ is finite. Indeed, since $Y \subseteq X \cdot \Lambda_0$ and Λ_0 is small, by saturation there exists a finite $F \subseteq \Lambda_0$ such that $Y \subseteq X \cdot F$. We assume that $X \cap \Lambda_0$ is finite, and since F is a finite subset of Λ_0 it follows that $(X \cdot F) \cap \Lambda_0$ is finite which clearly implies $Y \cap \Lambda_0$ finite.

Fix a finite $F_1 = XX^{-1} \cap \Lambda_0$ and $F_2 = XXX^{-1} \cap \Lambda_0$.

We now define on X an equivalence relation $x \sim y$ if and only if $xy^{-1} \in \Lambda_0$ if and only if $xy^{-1} \in F_1$. This is a definable relation since F_1 is finite. We can also define a group operation on the equivalence classes: $[x] \cdot [y] = [z]$ if and only if $xyz^{-1} \in \Lambda_0$ if and only if $xyz^{-1} \in F_2$. The interpretable group we get, call it \overline{K} , is clearly isomorphic to \mathcal{U}/Λ_0 , and we have a \forall -definable homomorphism from \mathcal{U} onto \overline{K} , whose kernel is Λ_0 . \square

We will return to definable quotients of \forall -definable groups in Section 3. We now focus on the existence of \mathcal{U}^{00} for a \forall -definable group \mathcal{U} .

2.2. Subgroups of bounded index of \forall -definable groups. Let \mathcal{U} be a \forall -definable group in an o-minimal structure. It is not always true that \mathcal{U} has some type-definable subgroup of bounded index. For example, consider a sufficiently saturated ordered divisible abelian group $\langle G, <, + \rangle$ and in it take an infinite increasing sequence of elements $0 < a_1 < a_2 < \dots$ such that, for every $n \in \mathbb{N}$, we have $na_i < a_{i+1}$. The subgroup $\bigcup_i (-a_i, a_i)$ of G is a \forall -definable group which does not have any type-definable subgroup of bounded index. However, as is shown in [12] (see Proposition 6.1 and Proposition 7.4), if \mathcal{U} does have some type-definable subgroup of bounded index then it has a smallest one; namely \mathcal{U}^{00} exists.

Our goal here is to show, under various assumptions on \mathcal{U} , that the ideal of non-generic definable sets gives rise to type-definable subgroups of bounded index.

As is shown in [16], using Dolich's results in [5], if G is a definably compact, abelian group in an o-minimal expansion of a real closed field then the non-generic definable sets form an ideal. Later, it was pointed out in [10] and [14, Section 8] that the same proof works in expansions of groups. We start by re-proving an analogue of the result for \forall -definable groups (see Lemma 2.11 below). We first define the corresponding notion of genericity and prove some basic facts about it.

Definition 2.2. Let \mathcal{U} be a \forall -definable group. A definable $X \subseteq \mathcal{U}$ is called *left-generic* if there is a small subset $A \subseteq \mathcal{U}$ such that $\mathcal{U} = \bigcup_{g \in A} gX$. We similarly define *right-generic*. The set X is called *generic* if it is both left-generic and right-generic.

It is easy to see that a definable $X \subseteq \mathcal{U}$ is generic if and only if for every definable $Y \subseteq \mathcal{U}$, there are finitely many translates of X which cover Y .

Fact 2.3. (1) *If \mathcal{U} is a \forall -definable group, then every \forall -definable subgroup of bounded index is a compatible subgroup. In particular, if $X \subseteq \mathcal{U}$ is a definable left-generic set, then the subgroup generated by X is a compatible subgroup.*

(2) *Assume that \mathcal{U} is a \forall -definable group in an o-minimal structure. If \mathcal{U} is connected and $X \subseteq \mathcal{U}$ is a left-generic set, then X generates \mathcal{U} .*

Proof. (1) Assume that \mathcal{V} is a \forall -definable subgroup of bounded index. We need to see that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap \mathcal{V}$ is definable. Since \mathcal{V} has bounded index in \mathcal{U} its complement in \mathcal{U} is also a bounded union of definable sets, hence a \forall -definable set. But then $Y \cap \mathcal{V}$ and $Y \setminus \mathcal{V}$ are both \forall -definable sets, so by compactness $Y \cap \mathcal{V}$ must be definable.

(2) Assume now that \mathcal{U} is a \forall -definable connected group in an o-minimal structure and $X \subseteq \mathcal{U}$ is a left-generic set. By (1), the group \mathcal{V} generated by X is compatible, of bounded index. But then $\dim \mathcal{V} = \dim \mathcal{U}$, so by [1, Proposition 1], $\mathcal{V} = \mathcal{U}$. \square

Fact 2.4. *Let $\langle \mathcal{U}, + \rangle$ be an abelian, definably generated group. If $X \subseteq \mathcal{U}$ is a definable set then X is generic if and only if there exists a finitely generated (in particular countable) group $\Gamma \leq \mathcal{U}$ such that $\mathcal{U} = X + \Gamma$.*

Proof. Clearly, if Γ exists then X is generic. For the converse, assume that \mathcal{U} is generated by the definable set $Y \subseteq \mathcal{U}$, with $0 \in Y$. Because X is generic in \mathcal{U} , there is a finite set $F \subseteq \mathcal{U}$ such that the sets $-Y$, Y and $X + X$ are all contained in $X + F$.

Let $Y(n)$ be as in the notation from Section 1.6. If we now let Γ be the group generated by F , then $\mathcal{U} = \bigcup_n Y(n) = X + \Gamma$. \square

We next show that under some suitable conditions we can guarantee the existence of \mathcal{U}^{00} . We do it first in the general context of NIP theories. We recall a definition [16]:

Definition 2.5. Given a \vee -definable group \mathcal{U} and a definable set $X \subseteq \mathcal{U}$,

$$Stab_{ng}(X) = \{g \in \mathcal{U} : gX\Delta X \text{ is non-generic in } \mathcal{U}\}.$$

Theorem 2.6. *Let \mathcal{U} be an abelian \vee -definable group in a NIP structure \mathcal{M} . Assume that the non-generic definable subsets of \mathcal{U} form an ideal and that \mathcal{U} contains some definable generic set. Then for any definable generic set X , the set $Stab_{ng}(X)$ is a type-definable group and has bounded index in \mathcal{U} . In particular, by [12, Proposition 7.4], \mathcal{U}^{00} exists.*

Proof. The fact the definable non-generic sets form an ideal implies that for every definable set X , the set $Stab_{ng}(X)$ is a subgroup. Note however that if X is a non-generic set then $Stab_{ng}(X) = \mathcal{U}$ and therefore will not in general be type-definable (unless \mathcal{U} itself was definable).

We assume now that $X \subseteq \mathcal{U}$ is a definable generic set and show that $Stab_{ng}(X)$ is type-definable. First note that for every $g \in \mathcal{U}$, if $gX\Delta X$ is non-generic, then in particular $gX \cap X \neq \emptyset$ and therefore $g \in XX^{-1}$. It follows that $Stab_{ng}(X)$ is contained in XX^{-1} .

Next, note that a subset of \mathcal{U} is generic if and only if finitely many translates of it cover X (since X itself is generic). Now, for every n , we consider the statement in g : “ n many translates of $gX\Delta X$ do not cover X ”. Here again we note that for $h(gX\Delta X) \cap X$ to be non-empty we must have $h \in XX^{-1} \cup X(gX)^{-1}$. Hence, it is sufficient to write the first-order formula saying that for every $h_1, \dots, h_n \in XX^{-1} \cup X(gX)^{-1}$, $X \not\subseteq \bigcup_{i=1}^n h_i(gX\Delta X)$. The union of all these formulas for every n , together with the formula for XX^{-1} is the type which defines $Stab_{ng}(X)$.

It remains to see that $Stab_{ng}(X)$ has bounded index in \mathcal{U} . This is a similar argument to the proof of [12, Corollary 3.4] but in that paper the amenability of definable groups and, as a result, the fact that every generic set has positive measure, played an important role. Since a generic subset of a \vee -definable group may require infinitely many translates to cover the group, we cannot a-priori conclude that it has positive measure, even if the group is amenable. Assume then towards contradiction that $Stab_{ng}(X)$ had

unbounded index and fix a small elementary substructure \mathcal{M}_0 over which all data is definable. Then we can find a sequence $g_1, \dots, g_n, \dots \in \mathcal{U}$ of indiscernibles over \mathcal{M}_0 , which are all in different cosets of $Stab_{ng}(X)$. In particular, it means that $g_i X \Delta g_j X$ is generic, for $i \neq j$.

Consider now the sequence $X_i = g_{2i} X \Delta g_{2i+1} X$, $i \in \mathbb{N}$. By NIP, there is a k , such that the sequence $\{X_i : i \in \mathbb{N}\}$ is k -inconsistent.

Consider now the type $tp(g_i/M_0)$ and find some M_0 -definable set W containing g_i . Because of indiscernibility, all g_i 's are in W . It follows that all the $g_i X$, and therefore also all X_i , are contained in WX . Because each X_i is generic, finitely many translates of X_i cover WX . By indiscernibility, there is some ℓ such that for every i there are ℓ -many translates of X_i which cover WX .

We then have countably many sets $X_i \subseteq WX$, such that on one hand the intersection of every k of them is empty and on the other hand there is some ℓ such that for each i , ℓ -many translates of X_i cover WX . To obtain a contradiction it is sufficient to prove the following lemma (it is here that we need to find an alternative argument to the measure theoretic one):

Lemma 2.7. *Let G be an arbitrary abelian group, $A \subseteq G$ an arbitrary subset. For every k and ℓ there is a fixed number $N = N(k, \ell)$ such that there are at most N subsets of A with the property that each covers A with ℓ -many translates and every k of them have empty intersection.*

Proof. We are going to use the following fact about abelian groups, taken from [13] (see problems 7 and 16 on p. 82):

Fact 2.8. *For every abelian group G , and for every set $A \subseteq G$ and m , it is not possible to find $A_1, \dots, A_{m+1} \subseteq A$ pairwise disjoint such that each A_i covers A by m -many translates.*

Returning to the proof of the lemma, we are going to show that $N = k\ell$ works. Assume for contradiction that there are $k\ell + 1$ subsets $X_1, \dots, X_{k\ell+1}$ of A , each covering A by ℓ -many translates, with an empty intersection of every k of them. We work with the group $G' = G \times C_k$, where $C_k = \{0, \dots, k-1\}$ is the cyclic group. For $i = 1, \dots, k\ell + 1$, we define $Y_i \subseteq G'$ as follows: For $x \in G$ and $n \in \mathbb{N}$, we have $(x, n) \in Y_i$ if and only if $x \in X_i$ and n is the maximum number such that for some distinct $i_1, \dots, i_n < i$, we have $x \in X_{i_1} \cap \dots \cap X_{i_n} \cap X_i$. Notice that even though i might be larger than k , because of our assumption that every k sets among the X_i 's intersect trivially, the maximum n we pick is indeed at most $k-1$. Note also that the projection of each Y_i on the first coordinate is X_i .

We claim that the Y_i 's are pairwise disjoint. Indeed, if $x \in X_i \cap X_j$ and $i < j$ then by the definition of the sets, if $(x, n) \in Y_i$ and $(x, n') \in Y_j$ then $n < n'$, so $Y_i \cap Y_j = \emptyset$.

Now, let $A' = A \times C_k$. We claim that each Y_i covers A' by $k\ell$ -many translates. Indeed, if $A \subseteq \bigcup_{j=1}^{\ell} g_{ij} \cdot X_i$ then

$$A' \subseteq \bigcup_{p \in C_k} \bigcup_{j=1}^{\ell} (g_{ij}, p) \cdot Y_i.$$

We therefore found $N + 1$ pairwise disjoint subsets of A' , each covering A' in N translates, contradicting Fact 2.8. \square

Thus, as pointed out above we reached a contradiction, so $\text{stab}_{ng}(X)$ does have bounded index in \mathcal{U} . This ends the proof of Theorem 2.6. \square

Remark 2.9. The last theorem implies that for a \forall -definable abelian group $\langle \mathcal{U}, + \rangle$ in a NIP structure, if the non-generic definable sets form an ideal, then \mathcal{U}^{00} exists if and only if \mathcal{U} contains a definable generic set (we have just proved the right-to-left direction. The converse is immediate since every definable set containing \mathcal{U}^{00} is generic).

We are now ready to show (Corollary 2.12 below) that when we work in o-minimal expansions of ordered groups, for a \forall -definable abelian group which contains a definable generic set and is generated by a definably compact set, the assumptions of Theorem 2.6 are satisfied. We begin by proving that we can obtain Dolich's result in this setting.

Fact 2.10. *Let \mathcal{M} be an o-minimal expansion of an ordered group and let $\mathcal{M}_0 \preceq \mathcal{M}$ be a small elementary submodel. If \mathcal{U} is a \forall -definable group over \mathcal{M}_0 and $X_t \subseteq \mathcal{U}$ is a t -definable, definably compact set such that $X_t \cap \mathcal{M}_0 = \emptyset$, then there are t_1, \dots, t_k , all of the same type as t over \mathcal{M}_0 such that $X_{t_1} \cap \dots \cap X_{t_k} = \emptyset$.*

Proof. We need to translate the problem from the group topology to the M^n -topology. As we already noted it is shown in [2] that \mathcal{U} can be covered by a fixed collection of \mathcal{M}_0 -definable open sets $\bigcup_i V_i$ such that each V_i is definably homeomorphic to an open subset of M^n . By logical compactness, X_t is contained in finitely many V_i 's, say V_1, \dots, V_m . Now, by definable compactness, we can replace each of the V_i 's by an open set W_i such that $\text{Cl}(W_i) \subseteq V_i$ and X_t is still contained in W_1, \dots, W_m . Each $X(i) = X_t \cap \text{Cl}(W_i)$ is definably compact and we finish the proof as in [10, Lemma 3.10]. \square

For a \forall -definable group \mathcal{U} , we call a definable $X \subseteq \mathcal{U}$ *relatively definably compact* if the closure of X in \mathcal{U} is definably compact. Clearly, X is relatively definably compact if and only if it is contained in some definably compact subset of \mathcal{U} .

Lemma 2.11. *Let \mathcal{M} be an o-minimal expansion of an ordered group. Assume that \mathcal{U} is a \forall -definable abelian group, and $X, Y \subseteq \mathcal{U}$ are definable, with X relatively definably compact. If X and Y are non-generic, then $X \cup Y$ is still non-generic.*

Proof. This is just a small variation on the work in [16]. Because commutativity plays only a minor role we use multiplicative notation for possible future use.

We may assume that \mathcal{U} contains a definable generic set (otherwise, the conclusion is trivial).

We need to prove that if $X \subseteq \mathcal{U}$ is definable, relatively definably compact and non-generic, and if $Z \supseteq X$ is definable and generic then $Z \setminus X$ is generic.

Fix \mathcal{M}_0 over which all sets are definable. Without loss of generality, X is definably compact (since the closure of a non-generic set is non-generic).

We first prove the result for Z of the form $W \cdot W$, when W is generic. Since X is not generic, no finitely many translates of X cover W (because W is generic). It follows from logical compactness that there is $g \in W$ such that $g \notin \bigcup_{h \in \mathcal{M}_0} hX$. Changing roles, there is $g \in W$ such that $Xg^{-1} \cap \mathcal{M}_0 = \emptyset$. We now apply Fact 2.10 to the definably compact set Xg^{-1} . It follows that there are g_1, \dots, g_r , all realizing the same type as g over \mathcal{M}_0 , so in particular all are in W , such that $Xg_1^{-1} \cap \dots \cap Xg_r^{-1} = \emptyset$. This in turn implies that $\bigcup_{i=1}^r (W \setminus Xg_i^{-1}) = W$. For each $i = 1, \dots, r$ we have

$$W \setminus Xg_i^{-1} = (Wg_i \setminus X)g_i^{-1} \subseteq (WW \setminus X)g_i^{-1}.$$

Therefore, it follows that W is contained in the finite union $\bigcup_{i=1}^r (WW \setminus X)g_i^{-1}$ and since W is generic it follows that $WW \setminus X$ is generic, as needed (it is here that commutativity is used, since left generic sets and right generic sets are the same).

We now consider an arbitrary definable generic set $Z \subseteq \mathcal{U}$, with $X \subseteq Z$ non-generic. Because Z is generic, finitely many translates of Z cover $Z \cdot Z$. Namely, $Z \cdot Z \subseteq \bigcup_{i=1}^t h_i Z$. If $X' = \bigcup_{i=1}^t h_i X$ then X' is still non-generic (and relatively definably compact), so by the case we have just proved, $ZZ \setminus X'$ is generic. However this set difference is contained in

$$\bigcup_{i=1}^t h_i Z \setminus \bigcup_{i=1}^t h_i X \subseteq \bigcup_{i=1}^t h_i (Z \setminus X),$$

hence this right-most union is generic. It follows that $Z \setminus X$ is generic. \square

Corollary 2.12. *Let \mathcal{M} be an o-minimal expansion of an ordered group. Assume that \mathcal{U} is a \forall -definable abelian group which contains a definable generic set and is generated by a definably compact set. Then the definable non-generic subsets of \mathcal{U} form an ideal.*

Proof. Every definable subset of \mathcal{U} must be relatively definably compact, because it is contained in some definably compact set. Then apply Lemma 2.11. \square

3. DIVISIBILITY, GENERICITY AND DEFINABLE QUOTIENTS

In this section, \mathcal{M} is a sufficiently saturated o-minimal expansion of an ordered group.

Proposition 3.1. *If \mathcal{U} is an infinite \forall -definable group of positive dimension, then it has unbounded exponent. In particular, for every n , the subgroup of n torsion points, $\mathcal{U}[n]$, is small.*

Proof. By the Trichotomy Theorem ([15]), there exists a neighborhood of the identity which is in definable bijection with an open subset of R^n for some real closed field R , or of V^n for some ordered vector space V (we use here the definability of a group operation near the identity of \mathcal{U}).

In the linear case, the group operation of \mathcal{U} is locally isomorphic near $e_{\mathcal{U}}$ to $+$ near $0 \in M^n$ (see [10, Proposition 4.1 and Corollary 4.4] for a similar argument). Clearly then the map $x \mapsto kx$ is non-constant.

Assume then that we are in the field case. Namely, we assume that some definable neighborhood W of e is definably homeomorphic to an open subset of R^n , with e identified with $0 \in \mathbb{R}^n$, and that a real closed field whose universe is a subset of W is definable in \mathcal{M} . The following argument was suggested by S. Starchenko. If $M(x, y) = xy$ is the group product of elements near e , then it is R -differentiable and its differential at (e, e) is $x + y$. It follows that the differential of the map $x \mapsto x^n$ is nx . Therefore, for every n , the map $x \mapsto x^n$ is not the constant map.

As for the last clause, note first that $\mathcal{U}[n]$ is a compatible \forall -definable subgroup of \mathcal{U} because its restriction to every definable set is obviously definable (by the formula $nx = 0$). Because $\mathcal{U}[n]$ has exponent at most n , it follows from what we have just proved that its dimension must be zero, so its intersection with every definable set is finite. \square

Remark 3.2. Although we did not write down the details, we believe that the above result is actually true without any assumptions on the ambient o-minimal \mathcal{M} . This can be seen by expressing a neighborhood of $e_{\mathcal{U}}$ as a direct product of neighborhoods, in cartesian powers of orthogonal real closed fields and ordered vector spaces.

Assume that $\mathcal{U} = \bigcup_{i \in I} X_i$ and that \mathcal{U}^{00} exists. Given the projection $\pi : \mathcal{U} \rightarrow \mathcal{U}/\mathcal{U}^{00}$, we define the *logic topology* on $\mathcal{U}/\mathcal{U}^{00}$ by: $F \subseteq \mathcal{U}/\mathcal{U}^{00}$ is closed if and only if for every $i \in I$, $\pi^{-1}(F) \cap X_i$ is type-definable. We first prove a general lemma.

Lemma 3.3. *Let \mathcal{U} be a locally definable group for which \mathcal{U}^{00} exists and let $\pi : \mathcal{U} \rightarrow \mathcal{U}/\mathcal{U}^{00}$ be the projection map. If $K_0 \subseteq \mathcal{U}/\mathcal{U}^{00}$ is a compact set, then $\pi^{-1}(K_0)$ is contained in a definable subset of \mathcal{U} .*

Proof. We write $\mathcal{U} = \bigcup_{n \in \mathbb{N}} X_n$, and we assume that the union is increasing. If the result fails then there is a sequence $k_n \rightarrow \infty$ and $x_n \in X_{k_n} \setminus X_{k_n-1}$ such that $\pi(x_n) \in K_0$. Since K_0 is compact we may assume that the sequence $\pi(x_n)$ converges to some $a \in K_0$. The set $\pi^{-1}(a)$ is a coset of \mathcal{U}^{00} and therefore contained in some definable set $Z \subseteq \mathcal{U}$. Since a can be realized as the intersection of countably many open sets, there is, by logical compactness, some open neighborhood $V \ni a$ in $\mathcal{U}/\mathcal{U}^{00}$ such that $\pi^{-1}(V) \subseteq Z$. But then,

the whole tail of the sequence $\{\pi(x_n)\}$ belongs to V and therefore the tail of $\{x_n\}$ is contained in Z , contradicting our assumption on the sequence. \square

Claim 3.4. *Let \mathcal{U} be an abelian locally definable group. Then there exists a definable torsion-free subgroup $H \subseteq \mathcal{U}$ such that every definable subset of \mathcal{U}/H is relatively definably compact. If, in addition, \mathcal{U} is definably generated, then \mathcal{U}/H can be generated by a definably compact set.*

Proof. As can easily be verified, for a definably generated \forall -definable group \mathcal{V} , the following are equivalent: (a) every definable subset of \mathcal{V} is relatively definably compact, (b) every definable path in \mathcal{V} has limit points in \mathcal{V} . A \forall -definable group with property (b) was called in [6] “definably compact”. In Theorem 5.2 of the same reference, it was shown that if \mathcal{V} is a \forall -definable group which is not definably compact, then \mathcal{V} contains a 1-dimensional torsion-free definable subgroup H_1 . Now, if \mathcal{U} is abelian, then by Fact 1.3, \mathcal{U}/H_1 is definably isomorphic to a locally definable definable group. Using induction on $\dim(\mathcal{U})$, we see that \mathcal{U} contains a definable torsion-free subgroup H such that \mathcal{U}/H is definably compact in the above sense.

If in addition, \mathcal{U} is definably generated then \mathcal{U}/H is also definably generated by some set X . By replacing X with $Cl(X)$ we conclude that \mathcal{U}/H is generated by a definably compact set. \square

Proposition 3.5. *Let \mathcal{U} be a connected abelian \forall -definable group, which is definably generated. If \mathcal{U}^{00} exists, then*

- (1) *The group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k \times K$, for some compact group K . (Later we will see that $K \simeq \mathbb{T}^r$ where \mathbb{T} is the circle group and $r \in \mathbb{N}$).*
- (2) *\mathcal{U} and \mathcal{U}^{00} are divisible.*
- (3) *\mathcal{U}^{00} is torsion-free.*

Proof. (1) Let us denote the group $\mathcal{U}/\mathcal{U}^{00}$ by L . By [4, Lemma 2.6] (applied to \mathcal{U} instead of G there), the image of every definable, definably connected subset of \mathcal{U} under π is a connected subset of L . As in the proof of Theorem 2.9 in [4], the group L is locally connected, and since \mathcal{U} is connected, the group L must actually be connected.

Since \mathcal{U} is generated by a definable set, say $X \subseteq \mathcal{U}$, its image $\pi(\mathcal{U}) = L$ is generated by $\pi(X)$ which is a compact set ($\pi(X)$ is a quotient of X by a type-definable equivalence relation with bounded quotient, see [17]). Hence, the group L is so-called compactly generated. By [11, Theorem 7.57], the group L is then isomorphic, as a topological group, to a direct product $\mathbb{R}^k \times K$, for some compact abelian group K . This proves (2).

In what follows, we use $+$ for the group operation of \mathcal{U} and write \mathcal{U} as an increasing countable union $\bigcup_{k=1}^{\infty} X(k)$ (with $X(k)$ as in the notation from Section 1.6).

(2) Let us see that \mathcal{U} is divisible. Given $n \in \mathbb{N}$, consider the map $z \mapsto nz : \mathcal{U} \rightarrow \mathcal{U}$. For a subset Z of \mathcal{U} , let nZ denote the image of Z under this

map. The kernel of this map is $\mathcal{U}[n]$. By Proposition 3.1, $\mathcal{U}[n]$ must have dimension 0, and therefore by connectedness $\dim(n\mathcal{U}) = \dim(\mathcal{U})$.

Since \mathcal{U} is connected, by [1, Proposition 1] it is sufficient to show that for every n , the group $n\mathcal{U}$ is a compatible subgroup of \mathcal{U} , namely that for every definable $Y \subseteq \mathcal{U}$, the set $Y \cap n\mathcal{U}$ is definable.

We claim that $Y \cap n\mathcal{U}$ is contained in $nX(j)$ for some j . Assume towards a contradiction that this fails. Then for every j there exists $x_j \in \mathcal{U}$ such that $nx_j \in Y \setminus nX(j)$. Hence, $x_j \notin X(j)$ and therefore there is a sequence $k_j \rightarrow \infty$ such that $x_j \in X(k_j) \setminus X(k_j - 1)$ and $nx_j \in Y$. Consider the projection $\pi(Y)$ and $\pi(x_j)$ in L . Because Y is definable the set $\pi(Y)$ is compact.

By Lemma 3.3, because the sequence $\{x_j\}$ is not contained in any definable subset of \mathcal{U} , its image $\{\pi(x_j)\}$ is not contained in any compact subset of L . At the same time, $n\pi(x_j)$ is contained in the compact set $\pi(Y)$. However, since L is isomorphic to $\mathbb{R}^k \times K$, for a compact group K , the map $x \mapsto nx$ is a proper map on L and hence this is impossible. We therefore showed that

$$Y \cap n\mathcal{U} \subseteq nX(j) \subseteq n\mathcal{U},$$

and so $Y \cap n\mathcal{U} = Y \cap nX(j)$ which is a definable set. We can conclude that the group $n\mathcal{U}$ is a compatible subgroup of \mathcal{U} , of the same dimension and therefore $n\mathcal{U} = \mathcal{U}$. It follows that \mathcal{U} is divisible.

Let us see that \mathcal{U}^{00} is also divisible. Indeed, consider the map $x \mapsto nx$ from \mathcal{U} onto \mathcal{U} . It sends \mathcal{U}^{00} onto the group $n\mathcal{U}^{00}$ and therefore $[\mathcal{U} : \mathcal{U}^{00}] \leq [\mathcal{U} : n\mathcal{U}^{00}]$. Since \mathcal{U}^{00} is the smallest type-definable subgroup of bounded index we must have $n\mathcal{U}^{00} = \mathcal{U}^{00}$, so \mathcal{U}^{00} is divisible.

(3) This is a repetition of an argument from [16]. Because \mathcal{U}^{00} exists there is a definable generic set $X \subseteq \mathcal{U}$ which we now fix. By Theorem 2.6, the group $Stab_{ng}(X)$ contains \mathcal{U}^{00} , so it is sufficient to prove that for every n , there is a definable $Y \subseteq \mathcal{U}$ such that $Stab_{ng}(Y) \cap \mathcal{U}[n] = \{0\}$. We do that as follows. Because \mathcal{U} is divisible, the \forall -definable map $h \mapsto nh$ is surjective. By compactness, there exists a definable $Y_1 \subseteq \mathcal{U}$ which maps onto X . However, since $\mathcal{U}[n]$ is compatible and has dimension zero, every element of X has only finitely many pre-images in Y_1 . By definable choice, we can find a definable $Y \subseteq Y_1$ such that the map $h \mapsto nh$ induces a bijection from Y onto X . The set Y is generic in \mathcal{U} as well (since its image is generic and the kernel of the map has dimension zero) and for every $g \in \mathcal{U}[n]$ we have $(g + Y) \cap Y = \emptyset$. Hence, the only element of $\mathcal{U}[n]$ which belongs to $Stab_{ng}(Y)$ is 0. It follows that \mathcal{U}^{00} is torsion-free. \square

As a corollary, we can formulate the following criterion for recognizing \mathcal{U}^{00} , generalizing results from [4] and [12]:

Proposition 3.6. *Let \mathcal{U} be a connected abelian \forall -definable group which is definably generated. Assume that $H \leq \mathcal{U}$ is type-definable of bounded index. Then $H = \mathcal{U}^{00}$ if and only if H is torsion-free.*

In particular, if \mathcal{U} is torsion-free then \mathcal{U}^{00} , if it exists, is the only type-definable subgroup of bounded index.

Proof. Since H is type-definable of bounded index, by [12, Proposition 7.4] \mathcal{U}^{00} exists.

If $H = \mathcal{U}^{00}$, then by Proposition 3.5 it is torsion-free.

For the converse, assume that $H \leq \mathcal{U}$ is torsion-free. We let $L = \mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology. Because $\mathcal{U}^{00} \leq H$, the map $\pi : \mathcal{U} \rightarrow L$ sends the type-definable group H onto a compact subgroup of L . If $\pi(H)$ is non-trivial (namely, $H \neq \mathcal{U}^{00}$) then $\pi(H)$ has torsion. However, $\ker(\pi) = \mathcal{U}^{00}$ is divisible (see Proposition 3.5) and therefore H has torsion. Contradiction. \square

Lemma 3.7. *Let \mathcal{U} be a connected abelian \forall -definable group, which is definably generated. Then the following are equivalent.*

- (1) \mathcal{U} contains a definable generic set.
- (2) \mathcal{U}^{00} exists.
- (3) \mathcal{U}^{00} exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for some $k \in \mathbb{N}$ and a compact group K .
- (4) There exists a definable group G and a \forall -definable surjective homomorphism $\phi : \mathcal{U} \rightarrow G$ with $\ker(\phi) \simeq \mathbb{Z}^{k'}$, for some $k' \in \mathbb{N}$.
- (5) There exists a definable group G and a \forall -definable surjective homomorphism $\phi : \mathcal{U} \rightarrow G$.

Assume now that the above hold. If k is as in (3) and $\phi : \mathcal{U} \rightarrow G$ and k' are as in (4), then $k = k'$.

Proof. (1) \Rightarrow (2). Note first that by Claim 3.4, the group \mathcal{U} has a definable torsion-free subgroup H with \mathcal{U}/H definably generated by a definably compact set. Because \mathcal{U} contains a definable generic set so does \mathcal{U}/H . By Corollary 2.11, the definable non-generic sets in \mathcal{U}/H form an ideal, so by Theorem 2.6, $(\mathcal{U}/H)^{00}$ exists. Its pre-image in \mathcal{U} is a type definable subgroup of bounded index which is also torsion-free (since H and $(\mathcal{U}/H)^{00}$ are both torsion-free). By Proposition 3.6 this pre-image equals \mathcal{U}^{00} .

(2) \Rightarrow (3). By Proposition 3.5.

(3) \Rightarrow (4). Let $L = \mathbb{R}^k \times K$ and $\pi_{\mathcal{U}} : \mathcal{U} \rightarrow L$ be the projection map (whose kernel is \mathcal{U}^{00}).

We now fix generators $z_1, \dots, z_k \in \mathbb{R}^k$ for \mathbb{Z}^k , and find $u_1, \dots, u_k \in \mathcal{U}$ with $\pi_{\mathcal{U}}(u_i) = (z_i, 0)$. If we let $\Gamma \leq \mathcal{U}$ be the subgroup generated by u_1, \dots, u_k then $\pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. Note that since z_1, \dots, z_k are \mathbb{Z} -independent, the restriction of $\pi_{\mathcal{U}}$ to Γ is injective, namely $\Gamma \cap \mathcal{U}^{00} = \{0\}$.

By Lemma 3.3, there is a definable $X \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(K) \subseteq X$. It follows from [4, Lemma 1.7] that the set $\pi_{\mathcal{U}}(X)$ contains not only K but also an open neighborhood of K . But then, there is an m such that $m\pi_{\mathcal{U}}(X) + \mathbb{Z}^k = L$. This implies that $\pi_{\mathcal{U}}(mX + \Gamma) = L$ and hence $mX + \mathcal{U}^{00} + \Gamma \subseteq mX + X + \Gamma = \mathcal{U}$. We let $Y = mX + X$ and then $Y + \Gamma = \mathcal{U}$.

We claim that $Y \cap \Gamma$ is finite. Indeed, if $Y \cap \Gamma$ were infinite then, since $\pi_{\mathcal{U}}$ is injective on Γ , the set $\pi_{\mathcal{U}}(Y) \cap \mathbb{Z}^k$ is infinite, contradicting the compactness of $\pi_{\mathcal{U}}(Y)$. We can now apply Lemma 2.1 and conclude that there is a definable group G and a \vee -definable surjective homomorphism $\phi : \mathcal{U} \rightarrow G$ whose kernel is Γ .

(4) \Rightarrow (5) is clear.

(5) \Rightarrow (1). By logical compactness, there is a definable $X \subseteq \mathcal{U}$ such that $\phi(X) = G$. But then $X + \ker(\phi) = \mathcal{U}$, and since $\ker(\phi) = \mathbb{Z}^{k'}$ is small, X is generic in \mathcal{U} .

Assume now that the conditions hold, k is as in (3), and $\phi : \mathcal{U} \rightarrow G$ and k' are as in (4). We will prove that $k = k'$. Consider the map $\pi_U : U \rightarrow \mathbb{R}^k \times K$ and let Γ be the image of $\ker(\phi)$ under π_U .

We first claim that $k \leq k'$. Let $X \subseteq \mathcal{U}$ be so that $\phi(X) = G$. Then $X + \ker(\phi) = \mathcal{U}$. Thus, $\pi_{\mathcal{U}}(X) + \Gamma = \mathbb{R}^k \times K$. Let Y and Γ' be the projections of $\pi_{\mathcal{U}}(X)$ and Γ , respectively, into \mathbb{R}^k . We have $Y + \Gamma' = \mathbb{R}^k$. The set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \dots, \lambda_{k'}$ be the generators of $\ker(\phi)$ and let $v_1, \dots, v_{k'} \in \mathbb{R}^k$ be their images in Γ' . If $H \subseteq \mathbb{R}^k$ is the real subspace generated by $v_1, \dots, v_{k'}$ then $Y + H = \mathbb{R}^k$, and therefore, since Y is compact, we must have $H = \mathbb{R}^k$. This implies that $k \leq k'$.

Now let us prove that $k' \leq k$. Note first that $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing \mathcal{U}^{00} . Then, since $\phi \upharpoonright X$ is definable, we must have $\ker(\phi) \cap \mathcal{U}^{00} \subseteq \ker(\phi) \cap X$ finite. However, by Proposition 3.5, the group \mathcal{U}^{00} is torsion-free, hence $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$.

It follows that $\Gamma = \pi_{\mathcal{U}}(\ker \phi)$ is of rank k' . It is also discrete. Indeed, using X as above we can find another definable set X' whose image $\pi_{\mathcal{U}}(X')$ contains an open neighborhood of 0 and no other elements of Γ .

Now, since K is compact, no element of Γ can be in K and therefore the projection of Γ onto $\Gamma' \subseteq \mathbb{R}^k$ is an isomorphism. Furthermore, Γ' is also discrete, which implies that $k' \leq k$. \square

At the end of this section, we conjecture that the above conditions always hold.

The result below is proved in [3, Theorem 8.2] for \mathcal{U} the universal covering of an arbitrary definably compact group G in o-minimal expansions of real closed fields.

Proposition 3.8. *Let \mathcal{U} be a connected abelian \vee -definable group, which is definably generated. Let G be a definable group and $\phi : \mathcal{U} \rightarrow G$ a surjective \vee -definable homomorphism with $\ker(\phi) \simeq \mathbb{Z}^k$.*

Then \mathcal{U}^{00} exists, $\ker(\phi) \cap \mathcal{U}^{00} = \{0\}$ and $\phi(\mathcal{U}^{00}) = G^{00}$. Furthermore there is a topological covering map $\phi' : \mathcal{U}/\mathcal{U}^{00} \rightarrow G/G^{00}$, with respect to the

logic topologies, such that the following diagram commutes.

$$(1) \quad \begin{array}{ccc} \mathcal{U} & \xrightarrow{\phi} & G \\ \pi_{\mathcal{U}} \downarrow & & \downarrow \pi_G \\ \mathcal{U}/\mathcal{U}^{00} & \xrightarrow{\phi'} & G/G^{00} \end{array}$$

The group $\mathcal{U}/\mathcal{U}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k \times \mathbb{T}^r$, for \mathbb{T} the circle group and $r \in \mathbb{N}$. If \mathcal{U} is generated by a definably compact set, then $k + r = \dim(\mathcal{U})$. If, moreover, \mathcal{U} is torsion-free, then $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^{\dim \mathcal{U}}$.

Proof. By Lemma 3.7, \mathcal{U}^{00} exists. Let $\Gamma = \ker(\phi)$. We first claim that $\Gamma \cap \mathcal{U}^{00} = \{0\}$. Indeed, take any definable set $X \subseteq \mathcal{U}$ containing \mathcal{U}^{00} . Then, since $\phi \upharpoonright X$ is definable, we must have $\Gamma \cap \mathcal{U}^{00} \subseteq \Gamma \cap X$ finite. However, by Proposition 3.5, the group \mathcal{U}^{00} is torsion-free, hence $\Gamma \cap \mathcal{U}^{00} = \{0\}$.

We claim that $\phi(\mathcal{U}^{00}) = G^{00}$. First note that since \mathcal{U}^{00} has bounded index in \mathcal{U} and ϕ is surjective, the group $\phi(\mathcal{U}^{00})$ has bounded index in G . Because $\Gamma \cap \mathcal{U}^{00} = \{0\}$ the restriction of ϕ to \mathcal{U}^{00} is injective and hence $\phi(\mathcal{U}^{00})$ is torsion-free. By [4], we must have $\phi(\mathcal{U}^{00}) = G^{00}$.

By [17], we have

$$G/G^{00} \simeq \mathbb{T}^l,$$

for some $l \in \mathbb{N}$. We now consider $\pi_G : G \rightarrow G/G^{00}$ and define $\phi' : \mathcal{U}/\mathcal{U}^{00} \rightarrow G/G^{00}$ as follows: For $u \in \mathcal{U}$, let $\phi'(\pi_{\mathcal{U}}(u)) = \pi_G(\phi(u))$. Since $\phi(\mathcal{U}^{00}) = G^{00}$ this map is a well-defined homomorphism which makes the above diagram commute. It is left to see that ϕ' is a covering map.

It follows from what we established thus far that $\ker(\phi') = \pi_{\mathcal{U}}(\Gamma) = \mathbb{Z}^k$. Let us see that this is a discrete subgroup of $\mathcal{U}/\mathcal{U}^{00}$. Indeed, as we already saw, for every compact neighborhood $W \subseteq \mathcal{U}/\mathcal{U}^{00}$ of 0, there is a definable set $Z \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(W) \subseteq Z$. But we already saw that $Z \cap \Gamma$ is finite and hence $W \cap \ker(\phi')$ must be finite. It follows that $\ker(\phi')$ is discrete.

By Lemma 3.7, $\mathcal{U}/\mathcal{U}^{00}$, equipped with the Logic topology, is locally compact. Since $\phi' : \mathcal{U}/\mathcal{U}^{00} \rightarrow G/G^{00}$ is a surjective homomorphism with discrete kernel it is sufficient to check that it is continuous as a map between topological groups. If $W \subseteq G/G^{00}$ is open then $V = \pi_G^{-1}(W)$ is a \vee -definable subset of G and hence $\phi^{-1}(V)$ is a \vee -definable subset of \mathcal{U} (because $\ker \phi$ is a small group). By commutation, this last set equals $\pi_{\mathcal{U}}^{-1}(\phi'^{-1}(W))$ and therefore $\phi'^{-1}(W)$ is open in $\mathcal{U}/\mathcal{U}^{00}$.

By Lemma 3.7, $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times K$, for a compact group K . We now have a covering map $\phi' : \mathbb{R}^k \times K \rightarrow G/G^{00} = \mathbb{T}^{k+r}$, with $\ker(\phi') = \mathbb{Z}^k \subseteq \mathbb{R}^k$. It follows that $K \simeq \mathbb{T}^r$.

If \mathcal{U} is generated by a definably compact set, G will be definably compact. In this case, by the work in [10], [12] and [14],

$$G/G^{00} \simeq \mathbb{T}^{\dim(G)}$$

and, hence, $k + r = \dim(G) = \dim(\mathcal{U})$.

If, moreover, \mathcal{U} is torsion-free, we have $r = 0$. \square

We summarize the above results in the following theorem.

Theorem 3.9. *Let \mathcal{U} be a connected abelian \forall -definable group which is definably generated. Then there is $k \in \mathbb{N}$ such that the following are equivalent:*

- (i) \mathcal{U} contains a definable generic set.
- (ii) \mathcal{U}^{00} exists.
- (iii) \mathcal{U}^{00} exists and $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, for some $r \in \mathbb{N}$.
- (iv) There is a definable group G , with $\dim G = \dim \mathcal{U}$, and a \forall -definable surjective homomorphism $\phi : \mathcal{U} \rightarrow G$.

If in addition \mathcal{U} is generated by a definably compact set, then (ii) is strengthened by the condition that $k + r = \dim \mathcal{U}$.

Proof. By Lemma 3.7 and Proposition 3.8. \square

Theorem 3.10. *Let \mathcal{U} be a connected abelian \forall -definable group which is definably generated. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leq \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that \mathcal{U}/Λ' is a definable group.

If \mathcal{U} generated by a definably compact set, then \mathcal{U}/Λ' is moreover definably compact.

Proof. Since $X + \Lambda = \mathcal{U}$, X is generic. By Theorem 3.9, $\mathcal{U}/\mathcal{U}^{00} \simeq \mathbb{R}^k \times \mathbb{T}^r$, for some $k, r \in \mathbb{N}$. We now consider $\Delta = \pi_{\mathcal{U}}(\Lambda) \subseteq \mathbb{R}^k \times \mathbb{T}^r$ and let $\Delta' \subseteq \mathbb{R}^k$ be the projection of Δ into \mathbb{R}^k . Since $X + \Lambda = \mathcal{U}$, we have $\pi_{\mathcal{U}}(X) + \Delta = \mathbb{R}^k \times \mathbb{T}^r$. Hence, if Y is the projection of $\pi_{\mathcal{U}}(X)$ into \mathbb{R}^k then we have $Y + \Delta' = \mathbb{R}^k$. The set $\pi_{\mathcal{U}}(X)$ is compact and so Y is also compact.

We let $\lambda_1, \dots, \lambda_m$ be generators of Λ and let $v_1, \dots, v_m \in \mathbb{R}^k$ be their images in Δ' . If $H \subseteq \mathbb{R}^k$ is the real subspace generated by v_1, \dots, v_m then $Y + H = \mathbb{R}^k$, and therefore, since Y is compact, we must have $H = \mathbb{R}^k$. This implies that among v_1, \dots, v_m there are elements v_{i_1}, \dots, v_{i_k} which are \mathbb{R} -independent. It follows that $\lambda_{i_1}, \dots, \lambda_{i_k} \in \Delta$ are \mathbb{Z} -independent. If we let Λ' be the group generated by $\lambda_{i_1}, \dots, \lambda_{i_k}$ then we immediately see that the restriction of $\pi_{\mathcal{U}}$ to Λ' is injective. We claim that \mathcal{U}/Λ' is definable.

First, let us see that for every definable $Z \subseteq \mathcal{U}$, the set $Z \cap \Lambda'$ is finite. Indeed, $\pi_{\mathcal{U}}(Z)$ is a compact subset of $\mathbb{R}^k \times \mathbb{T}^r$ and hence $\pi_{\mathcal{U}}(Z) \cap (\mathbb{Z}v_{i_1} + \dots + \mathbb{Z}v_{i_k})$ is finite. Because $\pi_{\mathcal{U}}|_{\Lambda'}$ is injective it follows that $Z \cap \Lambda'$ is also finite.

We can now take a compact set $K \subseteq \mathbb{R}^k \times \mathbb{T}^r$ such that $K + \mathbb{Z}^k = \mathbb{R}^k \times \mathbb{T}^r$. It follows that $\pi_{\mathcal{U}}^{-1}(K) + \Lambda' = \mathcal{U}$. By Lemma 3.3, there is a definable set $Z \subseteq \mathcal{U}$ such that $\pi_{\mathcal{U}}^{-1}(K) \subseteq Z$. We now have $Z + \Lambda' = \mathcal{U}$ and $Z \cap \Lambda'$ finite. By Lemma 2.1, \mathcal{U}/Λ' is definable.

For the last clause, let $f : \mathcal{U} \rightarrow \mathcal{U}/\Lambda'$ be the quotient map, and X' a definable subset of \mathcal{U} such that $f(X') = \mathcal{U}/\Lambda'$. Since \mathcal{U} is generated by a definably compact set, the closure of X' in \mathcal{U} must be a subset of a definably compact set and, hence, itself definably compact. But then it is easy to verify that $\mathcal{U}/\Lambda' = f(X')$ is definably compact. \square

We end this section with a conjecture.

Conjecture A. *Let \mathcal{U} be a connected abelian \mathbb{V} -definable group which is definably generated. Then*

- (i) \mathcal{U} contains a definable generic set.
- (ii) \mathcal{U} is divisible.

Although we cannot prove the above conjecture, we can reduce it to proving (i) under additional assumptions.

Conjecture B. *Let \mathcal{U} be a connected abelian \mathbb{V} -definable group, generated by a definably compact set. Then \mathcal{U} contains a definable generic set.*

Claim 3.11. *Conjecture B implies Conjecture A.*

Proof. We assume that Conjecture B is true.

Let \mathcal{U} be a connected abelian \mathbb{V} -definable group which is definably generated. Let \mathcal{V} be the universal cover of \mathcal{U} (see [7]). Because \mathcal{U} is the homomorphic image of \mathcal{V} under a \mathbb{V} -definable homomorphism whose kernel is a set of dimension 0, it is sufficient to prove that \mathcal{V} contains a generic set and that \mathcal{V} is divisible.

The group \mathcal{V} is connected, torsion-free and generated by a definable set $X \subseteq \mathcal{V}$. We work by induction on $\dim(\mathcal{V})$.

Let Y be the closure of X with respect to the group topology of \mathcal{V} .

Case 1 The set Y is definably compact.

Since \mathcal{V} is generated by Y , then by our standing assumption we may conclude that \mathcal{V} contains a definable generic set. By Theorem 3.9 and Proposition 3.5, \mathcal{V} is divisible.

Case 2 The set Y is not definably compact.

In this case, we can apply [6, Theorem 5.2] and obtain a definable 1-dimensional, definably connected, divisible, torsion-free subgroup of \mathcal{V} , call it H . Clearly, H is a compatible subgroup of \mathcal{V} , hence the group \mathcal{V}/H is \mathbb{V} -definable, connected ([6, Corollary 4.8]), torsion-free and definably generated (by the image of X under the projection map). We have $\dim(\mathcal{V}/H) < \dim \mathcal{V}$, so by induction, the conjecture holds for \mathcal{V}/H , hence it is divisible and contains a definable generic set Z . Because H is divisible as well, it follows that \mathcal{V} is divisible. It is easy to see that the pre-image of Z in \mathcal{V} is a definable generic subset of \mathcal{V} . \square

Finally, although we know that \mathcal{U} needs to be definably generated in order to guarantee (i) (by Fact 2.3(2)), we do not know if the same is true for (ii).

Conjecture C. *Let \mathcal{U} be a connected abelian \forall -definable group. Then \mathcal{U} is divisible.*

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CMAF, UNIVERSIDADE DE LISBOA, AV. PROF. GAMA PINTO 2, 1649-003 LISBOA,
PORTUGAL

E-mail address: `pelefthe@uwaterloo.ca`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HAIFA, HAIFA, ISRAEL

E-mail address: `kobi@math.haifa.ac.il`