DEFINABLE GROUPS AS HOMOMORPHIC IMAGES OF SEMI-LINEAR AND FIELD-DEFINABLE GROUPS

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ABSTRACT. We analyze definably compact groups in o-minimal expansions of ordered groups as a combination of semi-linear groups and groups definable in o-minimal expansions of real closed fields. The analysis involves structure theorems about their locally definable covers. As a corollary, we prove the Compact Domination Conjecture in o-minimal expansions of ordered groups.

1. INTRODUCTION

This is the second of two papers (originally written as one) analyzing groups definable in o-minimal expansions of ordered groups. The ultimate goal of this project is to reduce the analysis of such groups to semi-linear groups and to groups definable in o-minimal expansions of real closed fields. Such a reduction was proposed in Conjecture 2 from [19] and a first step towards it was carried out in [10].

In the first paper ([12]) we established conditions under which locally definable groups have definable quotients of the same dimension. In this paper, we carry out the aforementioned reduction for definably compact groups by first stating a structure theorem for the universal cover $\widehat{G}$ of a definable group G (Theorem 1.1). We describe $\widehat{G}$ as an extension of a locally definable group $\mathcal{U}$ in an o-minimal expansion of a real closed field by a locally definable semi-linear group $\widehat{H}$. We then apply [12, Theorem 3.10] and derive a stronger structure theorem (Theorem 1.3), replacing the above $\mathcal{U}$ by *a definable group*. We expect that the second theorem will be useful when reducing questions for definable groups to groups in the semi-linear and field settings. We illustrate this effect by applying our second theorem to conclude the Compact Domination Conjecture in o-minimal expansions of ordered groups (Theorem 1.4 below).

Let us provide the details.

1.1. **The setting.** We let $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ be an o-minimal expansion of an ordered group. When $\mathcal{M}$ expands a real closed field (with $+$ not necessarily one of the field operations) there is strong compatibility of definable sets with the field structure. For example, each definable function is piecewise differentiable with respect to the field structure. Other powerful tools, such as the triangulation theorem, are available as well ([3]). At the other end, when $\mathcal{M}$ is a linear structure, such as a reduct of an ordered vector space over an ordered division ring, then every definable set is *semi-linear*.

By the Trichotomy Theorem for o-minimal structures there is a third possibility (see [20]), where there is a definable real closed field R on some interval in M, and yet the underlying domain of R is necessarily a bounded interval and not the whole of M. Such a structure is called *semi-bounded* (and non-linear), and definable sets in this case turn out to be a combination of semi-linear sets and sets definable in o-minimal expansions of fields (see [4], [19], [10]). An important example is the expansion of the ordered vector space $\langle \mathbb{R}; <, +, x \mapsto ax \rangle_{a \in \mathbb{R}}$ by all bounded semialgebraic sets. Most of our work is intended for a semi-bounded structure which is non-linear.

We assume in the rest of this paper, and unless stated otherwise, that $\mathcal{M} = \langle M, <, +, \cdots \rangle$ *is a sufficiently saturated o-minimal expansion of an ordered group.*

1.2. **Short sets and long dimension.** Following [19], we call an element $a \in M$ *short* if either $a = 0$ or the interval $(0, a)$ supports a definable real closed field; otherwise a is called *tall*. An element of M^n is called *short* if all its coordinates are short. An interval $[a, b]$ is called *short* if $b - a$ is short, and otherwise it is called *long*. A definable set $X \subseteq M^n$ is called *short* if it is in definable bijection with a subset of I^n for some short interval I. The image of a short set under a definable map is short. As is shown in [4], $\mathcal{M}$ is semi-bounded if and only if all unbounded rays $(a, +\infty)$ are long. However, a semi-bounded and sufficiently saturated $\mathcal{M}$ also has bounded intervals which are long.

Following [10] (see also Section 3 below), we say that *the long dimension* of a definable $X \subseteq M^n$, $\mathrm{lgdim}(X)$, is the maximum k such that X contains a definable homeomorphic image of I^k, for some long interval I (the original definition of $\mathrm{lgdim}(X)$ was given in terms of cones, see Section 3 below, but it is not hard to see the equivalence of the two). The results in [10] show that every definable subset of M^n can be decomposed into "long cones" and as a result it follows that a definable $X \subseteq M^n$ is short if and only if $\mathrm{lgdim}(X) = 0$. We call X *strongly long* if $\mathrm{lgdim}(X) = \dim(X)$; this is for example the case with a cartesian product of long intervals. Note that all these notions are invariant under definable bijections.

Roughly speaking, strongly long sets and short sets are "orthogonal" to each other. The idea is that the structure which $\mathcal{M}$ induces on short sets comes from an o-minimal expansion of a real closed field, while the structure

induced on strongly long sets is closely related to the semi-linear structure. More precisely, if $p(x)$ is a complete type over A such that every formula in $p(x)$ defines a strongly long set then its semi-linear formulas determine the type. This is a result which will not be used in this paper, but its proof is straightforward. Indeed, the aforementioned decomposition from [10] implies, in particular, that every strongly long definable set X of dimension k is a union of a strongly long k-dimensional semi-linear set and a definable set whose long dimension is smaller than k. Both sets are definable over the same set of parameters as X. It follows that $p(x)$ is determined by the semi-linear formulas.

We will see in examples (Section 6) that the analysis of definable groups forces us to use the language of $\bigvee$-definable groups, so we recall some definitions.

1.3. $\bigvee$**-definable and locally definable sets.** Let $\mathcal{M}$ be a κ-saturated, not necessarily o-minimal, structure. By *bounded* cardinality we mean cardinality smaller than κ. We alert the reader that there is a second use of the word "bounded" throughout this paper. Namely, a subset of M^n is *bounded* if it is contained in some cartesian product of bounded intervals. It will always be clear from the context what we mean.

A $\bigvee$-*definable group* is a group $\langle \mathcal{U}, \cdot \rangle$ whose universe is a directed union $\mathcal{U} = \bigcup_{i \in I} X_i$ of definable subsets of M^n for some fixed n (where $|I|$ is bounded) and for every $i, j \in I$, the restriction of group multiplication to $X_i \times X_j$ is a definable function (by saturation, its image is contained in some X_k). Following [5], we say that $\langle \mathcal{U}, \cdot \rangle$ is *locally definable* if $|I|$ is countable. In this paper, we consider exclusively locally definable groups. We are mostly interested in *definably generated* groups, namely $\bigvee$-definable groups which are generated as a group by a definable subset. These groups are of course locally definable. An important example of such groups is the universal cover of a definable group (see [6]). In [16] a similar notion is introduced, of an Ind-definable group.
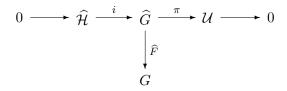
A map $\phi : \mathcal{U} \to \mathcal{H}$ between $\bigvee$-definable (locally definable) groups is called $\bigvee$-*definable (locally definable)* if for every definable $X \subseteq \mathcal{U}$ and $Y \subseteq \mathcal{H}$, $graph(\phi) \cap (X \times Y)$ is a definable set. Equivalently, the restriction of ϕ to any definable set is definable.

In an o-minimal expansion of an ordered group, a $\bigvee$-definable group $\mathcal{U}$ is called *short* if $\mathcal{U}$ is given as a bounded union of definable short sets. If $\mathcal{U} = \bigcup_{i \in I} X_i$ then we let $\mathrm{lgdim}(\mathcal{U}) = max_i(\mathrm{lgdim}(X_i))$. We say that $\mathcal{U}$ is *strongly long* if $\dim(\mathcal{U}) = \mathrm{lgdim}(\mathcal{U})$.

We are now ready to state the main results of this paper. Note that in the special case where $\mathcal{M}$ expands a real closed field, the results below become trivial (since in this case all definable sets are short), and in the case where $\mathcal{M}$ is semi-linear, they reduce to the main theorem from [13] (since in this case every definable short set is finite).

1.4. **The universal cover of a definably compact group.** We first note (see [19, Lemma 7.1]) that every definably compact group in a semi-bounded structure is necessarily bounded; namely, it is contained in some cartesian product of bounded intervals.

Theorem 1.1. *Let G be a definably compact, definably connected group of long dimension k and let $\widehat{F} : \widehat{G} \to G$ be the universal cover of G. Then there exist an open, connected subgroup $\widehat{\mathcal{H}} \subseteq \langle M^k, + \rangle$, generated by a semi-linear set of long dimension k, and a locally definable embedding $i : \widehat{\mathcal{H}} \to \widehat{G}$, with $i(\widehat{\mathcal{H}})$ central in G, such that $\mathcal{U} = \widehat{G}/i(\widehat{\mathcal{H}})$ is generated by a short definable set. Namely, we have the following exact sequence with locally definable maps i, π and $\widehat{F}$:*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\ i\ } & \widehat{G} & \xrightarrow{\ \pi\ } & \mathcal{U} & \longrightarrow & 0 \\
& & & & \downarrow{\scriptstyle \widehat{F}} & & & & \\
& & & & G & & & &
\end{array}
$$

If we let $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$, then $\mathcal{H}$ is the largest connected, strongly long, locally definable subgroup of G, namely it contains every other such group.

Question In Section 6 we present various examples that illustrate this theorem. In all our known examples the universal cover $\widehat{G}$ is the direct sum of the groups $\widehat{\mathcal{H}}$ and $\mathcal{U}$ (rather then just an extension of $\mathcal{U}$ by $\widehat{\mathcal{H}}$). Can $\widehat{G}$ always be realized as a direct sum of $\widehat{\mathcal{H}}$ and $\mathcal{U}$?

Remark 1.2. 1. One immediate corollary of the above theorem is that every definably compact group G which is strongly long is definably isomorphic to a semi-linear group, because in this case $\mathcal{H} = G$.

2. Note that when G is abelian, we have $\ker(\widehat{F}) \simeq \mathbb{Z}^{\dim G}$ (indeed, by [6, Corollary 1.5], we have $ker(\widehat{F}) \simeq \mathbb{Z}^l$, where the k-torsion subgroups of G satisfy $G[k] \simeq (\mathbb{Z}/k\mathbb{Z})^l$. By [19], we have $l = \dim G$).

3. Note that since $\mathcal{U}$ above is generated by a definable short set, there is a definable real closed field R such that $\mathcal{U}$ is locally definable in an o-minimal expansion of R. Indeed, let $X \subseteq \mathcal{U}$ be a definable set which generates $\mathcal{U}$, and let R be a definable real closed field such that, up to an $\mathcal{M}$-definable definable bijection, X is a subset of R^m. Let $\mathcal{N}$ be the structure which $\mathcal{M}$ induces on R. Without loss of generality, $0 \in X$. We let $X_1 = X$ and consider the equivalence relation on $X \times X$ given by $(x, y) \sim (x', y')$ if $x - y = x' - y'$. Clearly, $X \times X/\sim$ is in definable bijection with $X - X$. By definable choice in $\mathcal{N}$, there exists a definable set Y in $\mathcal{N}$ and a definable bijection between $X \times X/\sim$ and Y. Hence, in $\mathcal{M}$ the sets $X - X$ and Y are in definable bijection. Now consider the definable embedding of X into $X - X$ ($x \mapsto x - 0$), which induces an $\mathcal{N}$-definable injection $f_1 : X_1 \to Y$.
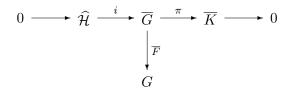
We let

$$X_2 = X_1 \sqcup (Y \setminus f_1(X_1)).$$

The set X_2 is definable in $\mathcal{N}$ and is in definable bijection with Y (so also with $X - X$). We also have $X_1 \subseteq X_2$.

We similarly define X_3 in $\mathcal{N}$ to be in definable bijection with $X - X + X$ and such that $X_1 \subseteq X_2 \subseteq X_3$. We continue in the same way and obtain a locally definable set $\bigcup_{n \in \mathbb{N}} X_n$ in $\mathcal{N}$ that is in locally definable bijection with $\mathcal{U}$.

1.5. **Covers by extensions of definable short groups.** In the next result we want to replace the locally definable group $\mathcal{U}$ from Theorem 1.1 by a definable short group $\overline{K}$. Roughly speaking, it says that G is close to being an extension of a short definable group by a semi-linear group, and the distance from being such a group is measured by the kernel of the map $\overline{F'}$ below.

Theorem 1.3. *Let G be a definably compact, definably connected group of long dimension k. Then G has a locally definable cover $\overline{F} : \overline{G} \to G$ with the following properties: there is an open subgroup $\widehat{\mathcal{H}} \subseteq \langle M^k, + \rangle$, generated by a semi-linear set of long dimension k, and a locally definable embedding $i : \widehat{\mathcal{H}} \to \overline{G}$, with $i(\widehat{\mathcal{H}})$ central in $\overline{G}$, such that $\overline{K} = \overline{G}/i(\widehat{\mathcal{H}})$ is a definably compact* **definable** *short group. Namely, we have the following exact sequence with locally definable maps i, π and $\overline{F}$:*

$$0 \longrightarrow \widehat{\mathcal{H}} \stackrel{i}{\longrightarrow} \overline{G} \stackrel{\pi}{\longrightarrow} \overline{K} \longrightarrow 0$$
$$\downarrow \overline{F}$$
$$G$$

If we take $\mathcal{H} \subseteq G$ as in Theorem 1.1, then there is also a locally definable, central extension G' of $\overline{K}$ by $\mathcal{H}$, with a locally definable homomorphism $F' : G' \to G$.

When G is abelian so is $\overline{G}$ and $\ker(\overline{F}) \simeq \mathbb{Z}^k + F$, for a finite group F.

It is at the passage from the locally definable group $\mathcal{U}$ in Theorem 1.1 to the definable group $\overline{K}$ in Theorem 1.3 that we use [12, Theorem 3.10].

1.6. **Compact Domination.** The relationship between a definable group G and the compact Lie group G/G^{00} has been the topic of quite a few papers. In [9], [15], [17] the related so-called Compact Domination Conjecture was solved for semi-linear groups and for groups definable in expansions of real closed fields. Using the above analysis we can complete the proof of the conjecture for groups definable in arbitrary o-minimal expansions of ordered groups (see Section 7 for the original formulation of the conjecture).

Theorem 1.4. *Let G be a definably compact, definably connected group. Let $\pi : G \to G/G^{00}$ denote the canonical homomorphism. Then, G is compactly dominated by G/G^{00}. That is, for every definable set $X \subseteq G$, the set*

$$\pi(X) \cap \pi(G \setminus X)$$

has **Haar** *measure 0.*

1.7. **Notation.** Let us finish this section with a couple of notational remarks. Given a group $\langle G, \cdot \rangle$ and a set $X \subseteq G$, we denote, for every $n \in \mathbb{N}$,

$$X(n) = \overbrace{(XX^{-1}) \cdots (XX^{-1})}^{n-\text{times}}$$

We assume familiarity with the notion of definable compactness. Whenever we write that a set is definably compact, or definably connected, we assume in particular that it is definable.

Acknowledgements. We thank Alessandro Berarducci, Mario Edmundo and Marcello Mamino for discussions which were helpful during our work. We thank the referee for a very careful reading of the original manuscript.

2. Preliminaries I: locally definable groups, extensions of abelian groups, pushout and pullback

As mentioned in the introduction, we work in a sufficiently saturated o-minimal expansion of an ordered group a $\mathcal{M} = \langle M, <, +, \cdots \rangle$. However, the only use of this assumption is to guarantee a strong version of elimination of imaginaries, which allows us to replace every definable quotient by a definable set. Any structure in which this is true will be just as good here, or, if we are willing to work in $\mathcal{M}^{eq}$, then any o-minimal structure will work.

2.1. **Locally definable groups, compatible subgroups and definable quotients.**

Definition 2.1. (See [5]) For a locally definable group $\mathcal{U}$, we say that $\mathcal{V} \subseteq \mathcal{U}$ is *a compatible subset of $\mathcal{U}$* if for every definable $X \subseteq \mathcal{U}$, the intersection $X \cap \mathcal{V}$ is a definable set (note that in this case $\mathcal{V}$ itself is a countable union of definable sets).

Clearly, the only compatible locally definable subsets of a definable group are the definable ones. Note that if $\phi : \mathcal{U} \to \mathcal{V}$ is a locally definable homomorphism between locally definable groups then $\ker(\phi)$ is a compatible locally definable normal subgroup of $\mathcal{U}$. Compatible subgroups are used in order to obtain locally definable quotients. Together with [5, Theorem 4.2], we have:

Fact 2.2. *If $\mathcal{U}$ is a locally definable group and $\mathcal{H} \subseteq \mathcal{U}$ a locally definable normal subgroup, then $\mathcal{H}$ is a compatible subgroup of $\mathcal{U}$ if and only if there exists a locally definable surjective homomorphism of locally definable groups $\phi : \mathcal{U} \to \mathcal{V}$ whose kernel is $\mathcal{H}$.*

If $\mathcal{M}$ is an o-minimal structure and $\mathcal{U} \subseteq M^n$ is a locally definable group then, by [2, Theorem 4.8], it can be endowed with a manifold-like topology τ, making it into a topological group. Namely, there is a countable collection $\{U_i : i \in I\}$ of definable subsets of $\mathcal{U}$, whose union equals $\mathcal{U}$, such that each U_i is in definable bijection with an open subset of M^k ($k = \dim \mathcal{U}$), and the transition maps are continuous. Moreover the U_i's and the transition maps are definable over the same parameters as $\mathcal{U}$. The group operation and group inverse are continuous with respect to this induced topology. The topology τ is determined by the ambient topology of M^n in the sense that at every generic point of $\mathcal{U}$ the two topologies coincide. From now on, whenever we refer to a topology on G, it is τ we are considering.

Definition 2.3. (See [1]) In an o-minimal structure, a locally definable group $\mathcal{U}$ is called *connected* if there is no locally definable compatible subset $\emptyset \subsetneq \mathcal{V} \subsetneq \mathcal{U}$ which is both closed and open with respect to the group topology.

Remark 2.4. It is easy to see that, in an o-minimal structure, if a locally definable group $\mathcal{U}$ is generated by a definably connected set which contains the identity, then it is connected.

Definition 2.5. Given a locally definable group $\mathcal{U}$ and $\Lambda_0 \subseteq \mathcal{U}$ a normal subgroup, we say that $\mathcal{U}/\Lambda_0$ is *definable* if there is a definable group $\overline{K}$ and a surjective locally definable homomorphism $\mu : \mathcal{U} \to \overline{K}$ whose kernel is Λ_0.

We now quote Theorem 3.10 from [12] (in a restricted case).

Fact 2.6. *Let $\mathcal{U}$ be a connected, abelian locally definable group, which is generated by a definably compact set. Assume that $X \subseteq \mathcal{U}$ is a definable set and $\Lambda \leqslant \mathcal{U}$ is a finitely generated subgroup such that $X + \Lambda = \mathcal{U}$.*

Then there is a subgroup $\Lambda' \subseteq \Lambda$ such that $\mathcal{U}/\Lambda'$ is a definably compact definable group.

2.2. **Pushouts and definability.** *In the following three subsections, all groups are assumed to be abelian and all arrows represent group homomorphisms.*

Several steps of the proof require us to construct extensions of abelian groups with certain maps attached to them. All constructions are standard in the classical theory of abelian groups but because we are concerned here with definability issues we review the basic notions (see [14] for the classical treatment). The proofs of these basic results are given in the appendix. Although we chose to present the constructions below in the more common language of pushouts and pullbacks, it is also possible to carry them out in the less canonical (but possibly more constructive) language of sections and cocycles.

Definition 2.7. Given homomorphisms

$$A \xrightarrow{\alpha} B$$
$$\downarrow{\beta}$$
$$C$$

the triple (D, γ, δ) (or just D) is called *a pushout* (of B and C over A via $\alpha, \beta, \gamma, \delta$) if the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\beta} & & \downarrow{\gamma} \\ C & \xrightarrow{\delta} & D \end{array}$$

and for every commutative diagram

(1)
$$\begin{array}{ccc} A & \xrightarrow{\alpha} & B \\ \downarrow{\beta} & & \downarrow{\gamma'} \\ C & \xrightarrow{\delta'} & D' \end{array}$$

there is a unique $\phi : D \to D'$ such that $\phi\gamma = \gamma'$ and $\phi\delta = \delta'$.

If A, B, C, D and the associated maps are (locally) definable, and if for every (locally) definable D', γ', δ' there is a (locally) definable $\phi : D \to D'$ as required then we say that *the pushout is (locally) definable*.

Proposition 2.8. *Assume that we are given the following diagram*

$$A \xrightarrow{\alpha} B$$
$$\downarrow{\beta}$$
$$C$$

(i) Let (D, γ, δ) be a pushout. Then

$$\ker(\gamma) = \alpha(\ker(\beta)).$$

Moreover, if β is surjective, then so is γ. If α is injective, then so is δ.
(ii) Suppose that all data are definable. Then there exists a definable pushout (D, γ, δ), which is unique up to definable isomorphism.
(iii) Suppose that all data are locally definable and $\alpha(A)$ is a compatible subgroup of B. Then there exists a locally definable pushout (D, γ, δ), which is unique up to locally definable isomorphism.

Assume now that α is injective. If we let $E = B/\alpha(A)$ and $\pi : B \to E$ the projection map then there is a locally definable surjection $\pi' : D \to E$

such that the diagram below commutes and both sequences are exact. In particular, $\ker(\pi') = \delta(C)$ *is a compatible subgroup of* D.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{\alpha}{\longrightarrow} & B & \overset{\pi}{\longrightarrow} & E & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle id_E} & & \\
0 & \longrightarrow & C & \overset{\delta}{\longrightarrow} & D & \overset{\pi'}{\longrightarrow} & E & \longrightarrow & 0
\end{array}
$$

We also need the following general fact, for which we could not find a reference (see appendix for proof):

Lemma 2.9. *Assume that we are given the following commutative diagram*

(2)
$$
\begin{array}{ccc}
A & \overset{\alpha}{\longrightarrow} & B \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\
C & \overset{\delta}{\longrightarrow} & D \\
\downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\mu} \\
E & \overset{\xi}{\longrightarrow} & F
\end{array}
$$

with D *the pushout of* B *and* C *over* A *(via* $\alpha, \beta, \gamma, \delta$*), and* F *the pushout of* B *and* E *over* A *(via* α*,* $\eta\beta$*,* $\mu\gamma$ *and* ξ*). Then* F *is also the pushout of* E *and* D *over* C *(via* η, δ, μ, ξ*).*

2.3. **Pullbacks and definability.**

Definition 2.10. Given homomorphisms

$$
\begin{array}{ccc}
& & B \\
& & \downarrow{\scriptstyle\alpha} \\
C & \overset{}{\underset{\beta}{\longrightarrow}} & A
\end{array}
$$

the triple (D, γ, δ) (or just D) is called *a pullback* (of B and C over A via $\alpha, \beta, \gamma, \delta$) if the following diagram commutes

$$
\begin{array}{ccc}
D & \overset{\gamma}{\longrightarrow} & B \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\alpha} \\
C & \overset{}{\underset{\beta}{\longrightarrow}} & A
\end{array}
$$

and for every commutative diagram

(3)

$$\begin{array}{ccc} D' & \xrightarrow{\gamma'} & B \\ {\scriptstyle\delta'}\downarrow & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow{\beta} & A \end{array}$$

there is a unique $\phi : D' \to D$ such that $\gamma\phi = \gamma'$ and $\delta\phi = \delta'$.

If A, B, C, D and the associated maps are (locally) definable, and if for every (locally) definable D', γ', δ' there is a (locally) definable $\phi : D' \to D$ as required then we say that *the pullback is (locally) definable.*

Proposition 2.11. *Assume that we are given the following diagram*

$$\begin{array}{ccc} & & B \\ & & \downarrow{\scriptstyle\alpha} \\ C & \xrightarrow{\beta} & A \end{array}$$

(i) Let (D, γ, δ) be a pullback. Then

$$\gamma(\ker(\delta)) = \ker(\alpha).$$

Moreover, if β is surjective, then so is γ. If α is injective, then so is δ.
(ii) Suppose that all data are definable. Then there exists a definable pullback (D, γ, δ), which is unique up to definable isomorphism.
(iii) Suppose that all data are locally definable. Then there exists a locally definable pullback (D, γ, δ), which is unique up to locally definable isomorphism.

Assume now that β is surjective. Let $G = \ker(\gamma)$ and $H = \ker(\beta)$. Then G, H are locally definable and compatible in D and C, respectively. Moreover, there is a locally definable isomorphism $j : G \to H$ such that the following diagram commutes and both sequences are exact.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{id_G} & D & \xrightarrow{\gamma} & B & \longrightarrow & 0 \\ & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\alpha} & & \\ 0 & \longrightarrow & H & \xrightarrow{id_H} & C & \xrightarrow{\beta} & A & \longrightarrow & 0 \end{array}$$

2.4. **Additional lemmas.**

Lemma 2.12. *Assume that the sequence*

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} C \longrightarrow 0$$

is exact and that we have a surjective homomorphism $\mu : C \to D$. Let $A' = \ker(\mu\pi) \subseteq B$. Then the following diagram commutes and both sequences are

exact. If all data are locally definable then so is A' and the associated maps.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\ i\ } & B & \xrightarrow{\ \pi\ } & C & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \mu} & & \\
0 & \longrightarrow & A' & \xrightarrow{\ id\ } & B & \xrightarrow{\ \mu\pi\ } & D & \longrightarrow & 0
\end{array}
$$

Proof. This is trivial. $\qquad\square$

Lemma 2.13. *Assume that we have surjective homomorphisms $F : B \to G$ and $F' : B \to G'$ with $\ker(F') \subseteq \ker(F)$. Then there is a canonical surjective homomorphism $h : G' \to G$, given by $h(g') = g$ if and only if there exists $b \in B$ with $F'(b) = g'$ and $F(b) = g$. The kernel of h equals $F'(\ker(F))$ and if all data are locally definable then so is h.*

Proof. Algebraically, this is just the fact that if $B_1 \subseteq B_2 \subseteq B$ then there is a canonical homomorphism $h : B/B_1 \to B/B_2$, whose kernel is B_2/B_1.

As for definability, assume that B, G, G', and F, F' are $\bigvee$-definable, and take definable sets $X \subseteq G$ and $X' \subseteq G'$. We want to show that the intersection of $graph(h)$ with $X' \times X$ is definable. Since F', F are $\bigvee$-definable and surjective, there exists a definable $Y \subseteq B$ such that $F'(Y) \supset X'$ and $F(Y) \supseteq X$. Now, for every $g' \in X'$ there exists $b \in Y$ such that $F'(b) = g'$, and we have $h(g') = F(b)$. Thus, the intersection of $graph(h)$ with $X' \times X$ is definable. $\qquad\square$

Remark 2.14. All statements from Proposition 2.8 to Lemma 2.13 hold under the more general assumption that $\mathcal{M}$ is any sufficiently saturated structure (not necessarily o-minimal) which has strong definable choice. This is because the definability issues in the statements are all based on Fact 2.2, which can be proved for such a more general $\mathcal{M}$.

3. Preliminaries II: Semi-bounded sets

3.1. **Long cones and long dimension.** In this section we recall some notions from [10] and prove basic facts that follow from that paper.

A *k-long cone in M^n* is a set of the form

$$
C = \left\{ b + \sum_{i=1}^{k} \lambda_i(t_i) : b \in B,\ t_i \in J_i \right\},
$$

where B is a short cell, each $J_i = (0, a_i)$ is a long interval (with a_i possibly ∞) and $\lambda_1, \ldots, \lambda_k$ are M-independent partial linear maps from $(-a_i, a_i)$ into M^n (by M-independent we mean: for all $t_1, \ldots, t_k \in M$, if $\lambda_1(t_1) + \cdots + \lambda_k(t_k) = 0$ then $t_1 = \cdots = t_k = 0$). It is required further that for each $x \in C$ there are unique b and t_i's with $x = b + \sum_{i=1}^{k} \lambda_i(t_i)$ (we refer to this as "long cones are normalized"). So $\dim C = \dim B + k$. A *long cone* is a k-long cone for some k. By the normality condition, if C is a k-long cone of dimension k then B must be a singleton.

The *long dimension* of a definable set $X \subseteq M^n$, denoted $\mathrm{lgdim}(X)$, is the maximum k such that X contains a k-long cone. This notion coincides with what we defined as long dimension in the Introduction. We call X *strongly long* if $\mathrm{lgdim}(X) = \dim(X)$.

Note that if C as above is a bounded cone (namely, all a_i's belong to M) then we can take $B' = \{b + (\lambda_1(a_1/2), \ldots, \lambda_k(a_k/2)) : b \in B\}$ and write $C = B' + \langle C \rangle$ where

$$\langle C \rangle = \left\{ \sum_{i=1}^{k} \lambda_i(t_i) : t_i \in (-a_i/2, a_i/2) \right\}.$$

In this paper, we are interested in bounded cones so we replace B with B' and write $C = B + \langle C \rangle$.

As is shown in [10, Section 5] the notion of short and long intervals gives rise to a pregeometry based on the following closure operation:

Definition 3.1. Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Given $A \subseteq M$ and $a \in M$, we say that a is in the short closure of A, $a \in scl(A)$, if there exists an A-definable short interval containing a (in particular, $dcl(A) \subseteq scl(A)$).

We say that $B \subseteq M$ is scl-independent over A if for every $b \in B$, we have $b \notin scl(B \cup A \setminus \{b\})$. We let $\mathrm{lgdim}(B/A)$ be the cardinality of a maximal scl-independent subset of B over A.

Notice that if $\mathcal{M}$ expands a real closed field then every set has long dimension 0 over $\emptyset$. On the other hand if $\mathcal{M}$ is a reduct of an ordered vector space then $scl(-) = dcl(-)$. Thus, this notion is interesting when $\mathcal{M}$ is non-linear and yet does not expand a real closed field (namely, non-linear and semi-bounded).

As for the usual o-minimal dimension, the notion of long dimension for definable sets is compatible with the scl-pregeometry in the following sense (see [10, Corollary 5.10]):

Fact 3.2. *If X is an A-definable set in a sufficiently saturated o-minimal expansion of an ordered group then*

$$\mathrm{lgdim}(X) = \max\{\mathrm{lgdim}(x/A) : x \in X\}.$$

We say that $a \in X$ is *long-generic over* A if $\mathrm{lgdim}(a/A) = \mathrm{lgdim}(X)$.

By [10, Theorem 3.8], if X is A-definable of long dimension k and a is long generic in X over A then a belongs to an A-definable k-long cone in X.

We are now ready to prove two facts which will be used later on.

Fact 3.3. *Let $F : B \times C \to M^l$ be a definable map, where $B \subseteq M^m$ is a short set and $C \subseteq M^n$ is strongly long (namely $\mathrm{lgdim}(C) = dim(C)$). Then there*

is an open subset B_1 of B and a strongly long $X \subseteq C$, with $\dim X = \dim C$, such that F is continuous on $B_1 \times X$.

Proof. We may assume that B, C and F are $\emptyset$-definable. Pick b generic in B and c which is long-generic in C over b. Since B is short we have

$$\mathrm{lgdim}(bc/\emptyset) = \mathrm{lgdim}(c/b) = \mathrm{lgdim}(C) = \mathrm{lgdim}(B \times C).$$

Because $\dim C = \mathrm{lgdim}C$, c is also generic over b and, hence, we have

$$\dim(bc/\emptyset) = \dim B \times C.$$

That is, $\langle b, c \rangle$ is generic in $B \times C$ so there exists a $\emptyset$-definable relatively open set $Y \subseteq B \times C$ containing $\langle b, c \rangle$, on which F is continuous. In particular, there exists a relatively open neighborhood $B_1 \subseteq B$, $b \in B_1$, such that $B_1 \times \{c\} \subseteq Y$. We may assume that B_1 is given as the intersection of a short rectangular neighborhood V_0 and B. By shrinking V_0 if needed, we may assume that the set of parameters A defining V_0 is *scl*-independent from $\langle b, c \rangle$ (and contains short elements). Hence $\mathrm{lgdim}(c/Ab) = \mathrm{lgdim}(c/b)$ so c is still long-generic in C over Ab. By genericity, we can find an Ab-definable set $X \subseteq C$ such that $B_1 \times X \subseteq Y$. Because $c \in X$, the set X must be strongly long of the same (long) dimension as C. $\qquad\square$

Fact 3.4. *Let $h : X \to W$ be a definable map, where $\mathrm{lgdim}X = \dim X > 0$ and $W \subseteq M^m$ is short. Then there exists a definable set $Y \subseteq X$, with $\mathrm{lgdim}Y < \mathrm{lgdim}X$ such that h is locally constant on $X \setminus Y$.*

Proof. Without loss of generality, X, W and h are $\emptyset$-definable. Take x long-generic in X and let $w = h(x)$. Because $w \in W$, we have $\mathrm{lgdim}(w/\emptyset) = 0$ and therefore x is still long-generic in X over w. It follows that there is a w-definable set $X_0 \subseteq X$, such that for every $x' \in X_0$, $h(x') = w$. The set X is strongly long, so x is also generic in X over w. Hence, the set X_0 contains a relative neighborhood of x in X, so h is locally constant at x. This is true for every long-generic element in X so the set of points at which h is not locally constant must have smaller long dimension than that of X. $\qquad\square$

3.2. **A preliminary result about definably compact groups.** We assume that $\langle G, + \rangle$ is a definable abelian group. Recall that $X \subseteq G$ is *generic* if finitely many group translates of X cover G. Using terminology from [18], a definable set $X \subseteq G$ is called *G-linear* if for every $g, h \in X$ there is an open neighborhood U of 0 (here and below, we always refer to the group topology of G), such that $(g - X) \cap U = (h - X) \cap U$. Clearly, every open subset of a definable subgroup of G is a G-linear set. More generally, every group translate of such a set is also G-linear. As is shown in [18], if a G-linear subset contains 0 then it contains an infinitesimal subgroup of G. When the group G is $\langle M^n, + \rangle$ a G-linear subset is also called *affine*. We call a definable G-linear subset $X \subseteq G$ a *local subgroup* of G if it is definably connected and $0 \in X$.

The G-linear set $G_0 \subseteq G$ and the H-linear set $H_0 \subseteq H$ are *definably isomorphic* if there exists a definable bijection $\phi : G_0 \to H_0$ such that for

every $g, h, k \in G_0$, $g - h + k \in G_0$ if and only if $\phi(g) - \phi(h) + \phi(k) \in H_0$, in which case we have $\phi(g - h + k) = \phi(g) - \phi(h) + \phi(k)$. *An isomorphism of local subgroups* $G_0 \subseteq G$ and $H_0 \subseteq H$, is further required to send 0_G to 0_H. If $\phi : G_0 \to H_0$ is an isomorphism of local subgroups then for all $g, k \in G_0$, if $g + k \in G_0$ then $\phi(g) + \phi(h) \in H_0$ and we have $\phi(g + h) = \phi(g) + \phi(h)$.

Our starting point is Proposition 5.4 from [10], which comes out of the analysis of definable sets in semi-bounded structures. Recall our notation $C = B + \langle C \rangle$ from Section 3. Below we use $\oplus$ and $\ominus$ for group addition and subtraction in G and use $+$ and $-$ for the group operations in $\mathcal{M}$.

Fact 3.5. [10, Proposition 5.4] *Let $\langle G, \oplus \rangle$ be a definably compact abelian group of long dimension k. Then G contains a definable, generic, bounded k-long cone C on which the group topology of G agrees with the o-minimal topology. Furthermore, for every $a \in C$ there exists an open neighborhood $V \subseteq G$ of a such that for all $x, y \in V \cap a + \langle C \rangle$,*

$$(4) \qquad\qquad x \ominus a \oplus y = x - a + y.$$

Our goal is to prove:

Proposition 3.6. *Let $\langle G, \oplus \rangle$ be a definably compact, definably connected abelian group. Then there exists a definably connected, k-dimensional local subgroup $H \subseteq G$ and a definable short set $B \subseteq G$, $\dim(B) = \dim(G) - k$, satisfying:*

> *(1) $\langle H, \oplus \rangle$ is definably isomorphic, as a local group, to $\langle H', + \rangle$, where $H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k) \subseteq M^k$, with each $e_i > 0$ tall in M. In particular, $\dim H = lgdim H = k$.*
> *(2) The set $B \oplus H = \{b \oplus h : b \in B \ h \in H\}$ is generic in G.*

Proof. We fix a definably connected short set B and a k-long cone $C = B + \langle C \rangle$ as in Fact 3.5.

For $b \in B$, let C_b be the fiber $b + \langle C \rangle$. Note that for every $x \in C_b$, and a sufficiently small neighborhood V of x, we have $V \cap C_b = V \cap x + \langle C \rangle$. Note also that each C_b is an affine subset of $\langle M^n, + \rangle$. Thus, condition (4) implies that each C_b, locally near every $a \in C_b$, is a G-linear subset of G, and furthermore the identity map is locally an isomorphism of $\langle C_b, + \rangle$ and $\langle C_b, \oplus \rangle$. Because the affine topology and the group topology agree on C (and because C is definably connected in M^n), each fiber C_b is definably connected with respect to the group topology. By [18, Lemma 2.4], each C_b is therefore a G-linear (not only locally) subset of G and the identity map is an isomorphism of the affine set $\langle C_b, + \rangle$ and the G-linear set $\langle C_b, \oplus \rangle$.

Let us summarize what we have so far: On one hand, the set $C = B + \langle C \rangle$ is a generic set in G, which can be written as a disjoint union of affine sets $\bigcup_{b \in B} C_b$. Furthermore, for each $a, b \in B$ the map

$$f_{a,b}(x) = x - a + b$$

is an isomorphism of the affine sets C_a and C_b. On the other hand, each C_b is also a G-linear set, and the same maps $f_{a,b} : C_a \to C_b$ are isomorphisms of G-linear sets (because the identity is an isomorphism of $\langle C_a, + \rangle$ and $\langle C_a, \oplus \rangle$).

Our next goal is to show that, for many a, b in B, each map $f_{a,b}(x)$ is not only a translation in the sense of the group $\langle M^n, + \rangle$ but also a translation in $\langle G, \oplus \rangle$.

We define on B the following equivalence relation: $a \sim b$ if there exists $g \in G$ such that we have $f_{a,b}(x) = x \oplus g$ for all $x \in C_a$. Note that for every $a, b, c \in B$, we have $f_{b,d} \circ f_{a,b} = f_{a,d}$, so it is easy to check that $\sim$ is an equivalence relation.

Claim 3.7. *There are only finitely many $\sim$-equivalence classes in B.*

Proof. Assume towards contradiction that there are infinitely many classes. By definable choice, we can find an infinite definable set of representatives for $B / \sim$. We then replace B by a definably connected component of this set, calling it B again. So, we may assume that any two $a, b \in B$ are in distinct $\sim$-classes and that B is still infinite and definably connected. We fix some $a_0 \in B$ and consider the map $F : B \times C_{a_0} \to G$, given by $F(b, x) = f_{a_0, b}(x)$.

Since C_{a_0} is strongly long, we can find an open subset $B_1 \subseteq B$ and a strongly long set $X \subseteq C_{a_0}$, $\dim X = \dim C_{a_0}$, such that F is continuous on $B_1 \times X$ with respect to the group topology (Fact 3.3). Without loss of generality we can assume that X is definably compact (we first take a bounded X, then shrink it slightly, and take its topological closure).

Let us fix a G-open chart $V \subseteq G$ containing 0_G, and a homeomorphism with an open affine set $\phi : V \to V' \subseteq M^\ell$ ($\ell = dim G$). Without loss of generality $\phi(0_G) = 0 \in M^\ell$. By identifying V and V', we may assume that $V' \subseteq G$ is an open set with respect to both the affine and the G-topology.

By the definable compactness of X, for every neighborhood $W \subseteq M^\ell$ of 0, there is a neighborhood $B_2 \subseteq B_1$ of a_0, such that for all $b', b'' \in B_2$ and $x \in X$, we have $F(x, b') \ominus F(x, b'') \in W$. Indeed, if not then there are definable curves $x(t) \in X$, $b_1(t), b_2(t) \in B_1$, with $b_1(t), b_2(t)$ tending to b and such that for all t,

$$F(x(t), b_1(t)) \ominus F(x(t), b_2(t)) \notin W.$$

Definable compactness of X implies that $x(t) \to x_0 \in X$, so by continuity we have $F(x_0, b) \ominus F(x_0, b) \notin W$, contradiction.

We now fix $W \subseteq M^\ell$ a short neighborhood of 0, and choose B_2 accordingly. If we take distinct b', b'' in B_2 then we obtain a map $h : X \to W$, defined by $h(x) = F(x, b') \ominus F(x, b'')$. Because X is strongly long, and W is short, the map h must be locally constant outside a subset of X of long dimension smaller than k (Fact 3.4). So, we have an open neighborhood $V'' \subseteq C_{a_0}$ and an element $g \in G$, such that for all $x \in V''$, $f_{a_0, b'}(x) \ominus f_{a_0, b''}(x) = g$.

We claim that for all $x \in C_{a_0}$, we have $f_{a_0, b'}(x) \ominus f_{a_0, b''}(x) = g$.

First take $x \in V''$ and choose any $y, z \in C_{a_0}$ which are sufficiently close to each other. Since C_{a_0} is a G-linear set, $x \ominus y \oplus z$ is still in C_{a_0} and still in V''. So we have

$$f_{a_0,b'}(x \ominus y \oplus z) = f_{a_0,b'}(x) \ominus f_{a_0,b'}(y) \oplus f_{a_0,b'}(z)$$

and

$$f_{a_0,b''}(x \ominus y \oplus z) = f_{a_0,b''}(x) \ominus f_{a_0,b''}(y) \oplus f_{a_0,b''}(z).$$

By subtracting the two equations (in G), we obtain

$$g = g \oplus (f_{a_0,b'}(z) \ominus f_{a,b''}(z)) \ominus (f_{a,b'}(y) \ominus f_{a,b''}(y)),$$

so

$$f_{a_0,b'}(z) \ominus f_{a,b''}(z) = f_{a,b'}(y) \ominus f_{a,b''}(y)$$

for all $y, z \in C_{a_0}$ which are sufficiently close to each other. This implies that the function $f_{a_0,b'} \ominus f_{a_0,b''}$ is locally constant on C_{a_0} so by definable connectedness, it must be constant on C_{a_0}. We therefore showed that $f_{a_0,b'} \ominus f_{a_0,b''} = g$, so in fact $b' \sim b''$ contradicting our assumption. Thus $\sim$ has only finitely many classes in B. $\square$

We now return to the relation $\sim$ with its finitely many classes $B_1, \dots, B_m$, and consider the partition of C into $\bigcup_{b \in B_i} C_b$, $i = 1, \dots, m$. Note that for each $i = 1, \dots, m$ and every $b', b'' \in B_i$, there exists $g \in G$ such that $x \mapsto x \oplus g$ is an isomorphism of the G-linear sets $C_{b'}$ and $C_{b''}$.

Since C was generic in G, one of these sets is also generic in G (here we use the definable compactness of G). So we assume from now on that for every $b_1, b_2 \in B$ there exists an element $g \in G$ such that $C_{b_1} = C_{b_2} \oplus g$.

Fix $b_0 \in B$ and for every $b \in B$ choose an element $g(b)$ in G such that $C_b = C_{b_0} \oplus g(b)$. If we let $B' = \{g(b) \oplus b_0 : b \in B\}$ and $H = C_{b_0} \ominus b_0$, then $C = B' \oplus H$.

Let's see that H is as required. Indeed, the map $x \mapsto x \oplus b_0$ is an isomorphism of the local subgroups $\langle H, \oplus \rangle$ and $\langle C_{b_0}, \oplus \rangle$. As we already pointed out, the identity map is an isomorphism of $\langle C_{b_0}, \oplus \rangle$ and $\langle C_{b_0}, + \rangle$. Finally, $y \mapsto y - b_0$ is an isomorphism of the affine sets $\langle C_{b_0}, + \rangle$ and $\langle \langle C \rangle, + \rangle$. The composition of these maps is an isomorphism of the local groups $\langle H, \oplus \rangle$ and

$$H' = \left\langle \left(-\frac{a_1}{2}, \frac{a_1}{2} \right) \times \cdots \times \left(-\frac{a_k}{2}, \frac{a_k}{2} \right), + \right\rangle$$

(it sends 0_G to 0). This ends the proof of Proposition 3.6. $\square$

4. The universal cover of G

4.1. **Proof of Theorem 1.1.** We first prove the abelian case. We proceed with the same notation as in the previous seciton. Namely, $\langle G, \oplus \rangle$ is a definably connected, definably compact abelian group, and $H \subseteq G$ is the definable strongly long set from Proposition 3.6.

Let $f' : \langle H', + \rangle \to \langle H, \oplus \rangle$ be the acclaimed isomorphism of local groups. We let $\mathcal{H} = \langle H \rangle$ be the subgroup of G generated by H. Since H is a local abelian subgroup of G of dimension k, $\mathcal{H}$ is a locally definable abelian

subgroup of G of dimension k (see [18, Lemma 2.18]). One can show that the universal cover of $\mathcal{H}$ is a locally definable subgroup $\widehat{\mathcal{H}}$ of $\langle M^k, + \rangle$. Indeed, let $\widehat{\mathcal{H}} = \langle H' \rangle$ be the subgroup of $\langle M^k, + \rangle$ generated by H'. Then we can extend f' to a map $f : \widehat{\mathcal{H}} \to \mathcal{H}$ with, for every $x_1, \ldots, x_l \in H'$,

$$f(x_1 + \cdots + x_l) = f'(x_1) \oplus f'(x_2) \oplus \cdots \oplus f'(x_l)$$

is a $\bigvee$-definable covering map for $\mathcal{H}$. (The fact that f is well-defined is provided by the same argument as for [13, Lemma 4.27]). Since $\widehat{\mathcal{H}}$ is divisible and torsion-free, it is the universal cover of $\mathcal{H}$.

We let $\mathcal{H}'_0$ be the subset of M^k that consists of all short elements (by this we mean all elements of M^k all of whose coordinates are short). By [19, Lemma 3.4], $\langle \mathcal{H}'_0, + \rangle$ is a subgroup of $\langle M^k, + \rangle$ and moreover, it is a subset of H'. It follows that $\mathcal{H}_0 = f(\mathcal{H}'_0)$ is a subgroup of $\mathcal{H}$ which is isomorphic to $\mathcal{H}'_0$ (note that by [19], $\mathcal{H}_0$ is a $\bigvee$-definable set, but not, in general, a definable one).

From now on, in order to simplify the notation, we will write $+$ for the group operation of G. In few cases we will also use $+$ for the usual operation on M^k, and this will be clear from the context.

We define $\mathcal{B} = \bigcup_{n \in \mathbb{N}} B(n)$, where B is the definable short set from Proposition 3.6, and the notation $B(n)$ is given in Section 1.7. Since each $B(n)$ is a short definable set, $\mathcal{B}$ is a short locally definable subgroup of G.

Claim 4.1. $\mathcal{H} + \mathcal{B} = G$.

Proof. By Proposition 3.6, the set $H + B$ is a generic subset of G and is contained in $\mathcal{H} + \mathcal{B}$ (we use here the fact that $B \subseteq \mathcal{B}$ since $0 \in B$). Since G is definably connected we have $\mathcal{H} + \mathcal{B} = G$. $\square$

The following claim is crucial to the rest of the analysis.

Claim 4.2. *The group $\mathcal{H}_0 \cap \mathcal{B}$ is compatible in $\mathcal{B}$, so in particular locally definable.*

Proof. Let $X \subseteq \mathcal{B}$ be a definable set. The set $\mathcal{B}$ is a bounded union of short definable sets, so X is contained in one of these and must also be short. We prove that, in general, the intersection of any definable short $X \subseteq G$ with $\mathcal{H}_0$ is definable.

Since $\mathcal{H}_0 \subseteq H$ we may assume that X is a subset of H. Let us consider $X' = (f')^{-1}(X) \subseteq M^k$. Because f' is injective the set X' is a finite union of definably connected short subsets of M^k. It is easy to see that if one of these short sets contains a short element then every element of it is short. Thus, if one of these components intersects $\mathcal{H}'_0$ non-trivially then it must be entirely contained in $\mathcal{H}'_0$ (since $\mathcal{H}'_0$ is the collection of all short elements). Hence, $X' \cap \mathcal{H}'_0$ is a finite union of components of X' and therefore definable. Its image under f' is the definable set $X \cap \mathcal{H}_0$. $\square$

Note: It is not true in general that $\mathcal{H} \cap \mathcal{B}$ is a compatible subgroup of $\mathcal{B}$ (see Example 6.1 below).

The decomposition of $\widehat{G}$ is done through a series of steps.

Step 1 By Claim 4.2 and Fact 2.2, the quotient $\mathcal{K} = \mathcal{B}/(\mathcal{H}_0 \cap \mathcal{B})$ is locally definable and hence we obtain the following short exact sequence of locally definable groups:

$$(5) \qquad 0 \longrightarrow \mathcal{H}_0 \cap \mathcal{B} \xrightarrow{i_0} \mathcal{B} \xrightarrow{\pi_{\mathcal{B}}} \mathcal{K} \longrightarrow 0$$

Claim 4.3. $\dim H + \dim \mathcal{K} = \dim G$.

Proof. Because $\mathcal{H} + \mathcal{B} = G$, we have

$$\dim \mathcal{H} + \dim \mathcal{B} - \dim(\mathcal{H} \cap \mathcal{B}) = \dim G.$$

Indeed, this is true for definable groups, and can be proved similarly here by considering a sufficiently small neighborhood of 0 in the locally definable group $\mathcal{H} \cap \mathcal{B}$.

But $\mathcal{H}_0$ is open in $\mathcal{H}$ and therefore $\dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim(\mathcal{H} \cap \mathcal{B})$, so we also have $\dim \mathcal{H} + \dim \mathcal{B} - \dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim G$. Because $\mathcal{K} = \mathcal{B}/(\mathcal{H}_0 \cap \mathcal{B})$, we have $\dim \mathcal{B} - \dim(\mathcal{H}_0 \cap \mathcal{B}) = \dim \mathcal{K}$. We can now conclude $\dim \mathcal{H} + \dim \mathcal{K} = \dim G$. $\qquad\square$

Step 2. Since $\mathcal{H}_0 \cap \mathcal{B}$ embeds into $\mathcal{H}$ and $\mathcal{H}_0 \cap \mathcal{B}$ is a compatible subgroup of $\mathcal{B}$, we can apply Lemma 2.8 and obtain a locally definable group D (the pushout of $\mathcal{H}$ and $\mathcal{B}$ over $\mathcal{H}_0 \cap \mathcal{B}$) with the following diagram commuting

$$(6) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_0 \cap \mathcal{B} & \xrightarrow{i_0} & \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{K} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle id_{\mathcal{K}}} & & \\
0 & \longrightarrow & \mathcal{H} & \xrightarrow{j} & D & \xrightarrow{\pi_D} & \mathcal{K} & \longrightarrow & 0
\end{array}$$

The maps γ and j are injective. Note that since $\mathcal{H}$ and $\mathcal{B}$ are subgroups of G, we also have a commutative diagram (with all maps being inclusions)

$$(7) \qquad
\begin{array}{ccc}
\mathcal{H}_0 \cap \mathcal{B} & \longrightarrow & \mathcal{B} \\
\downarrow & & \downarrow \\
\mathcal{H} & \longrightarrow & G
\end{array}$$

It follows from the definition of pushouts that there exists a locally definable map $\phi : D \to G$ such that $\phi\gamma : \mathcal{B} \to G$ and $\phi j : \mathcal{H} \to G$ are the inclusion maps. The restriction of ϕ to $j(\mathcal{H})$ is therefore injective and furthermore, the set $\phi(D)$ contains $\mathcal{H} + \mathcal{B}$ and hence, by Claim 4.1, ϕ is surjective on G.

Step 3 Consider now the universal cover $f : \widehat{\mathcal{H}} \to \mathcal{H}$ where $\widehat{\mathcal{H}}$ is identified with an open subgroup of $\langle M^k, + \rangle$ as before. As we saw, the group $\widehat{\mathcal{H}}$ has a subgroup $\mathcal{H}'_0$ which is isomorphic via f to $\mathcal{H}_0$. Hence, there is a locally definable embedding $\beta : \mathcal{H}_0 \cap \mathcal{B} \to \widehat{\mathcal{H}}$ such that $f\beta = id_{\mathcal{H}_0 \cap \mathcal{B}}$. Our goal is to use this embedding in order to interpolate an exact sequence between the two sequences in (6) (see (10) below).

We let $\widehat{D}$ be the pushout of $\widehat{\mathcal{H}}$ and $\mathcal{B}$ over $\mathcal{H}_0 \cap \mathcal{B}$. Namely, we have

(8)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_0 \cap \mathcal{B} & \xrightarrow{i_0} & \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{K} & \longrightarrow & 0 \\
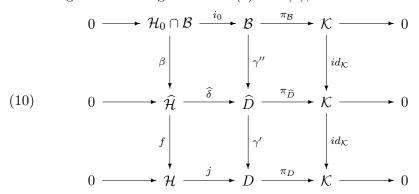& & \downarrow{\beta} & & \downarrow{\gamma''} & & \downarrow{id_{\mathcal{K}}} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\widehat{\delta}} & \widehat{D} & \xrightarrow{\pi_{\widehat{D}}} & \mathcal{K} & \longrightarrow & 0
\end{array}
$$

Step 4 Next, we consider the diagram

(9)
$$
\begin{array}{ccc}
\mathcal{H}_0 \cap \mathcal{B} & \xrightarrow{i_0} & \mathcal{B} \\
\downarrow{\beta} & & \downarrow{\gamma} \\
\widehat{\mathcal{H}} & \xrightarrow{jf} & D
\end{array}
$$

Since $f\beta = id$, it follows from (6) that the above diagram commutes. Since $\widehat{D}$ was a pushout, there exists a locally definable $\gamma' : \widehat{D} \to D$ such that $\gamma'\gamma'' = \gamma$ and $\gamma'\widehat{\delta} = jf$.

Putting the above together with (6) and (8), we obtain

(10)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H}_0 \cap \mathcal{B} & \xrightarrow{i_0} & \mathcal{B} & \xrightarrow{\pi_{\mathcal{B}}} & \mathcal{K} & \longrightarrow & 0 \\
& & \downarrow{\beta} & & \downarrow{\gamma''} & & \downarrow{id_{\mathcal{K}}} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\widehat{\delta}} & \widehat{D} & \xrightarrow{\pi_{\widehat{D}}} & \mathcal{K} & \longrightarrow & 0 \\
& & \downarrow{f} & & \downarrow{\gamma'} & & \downarrow{id_{\mathcal{K}}} & & \\
0 & \longrightarrow & \mathcal{H} & \xrightarrow{j} & D & \xrightarrow{\pi_D} & \mathcal{K} & \longrightarrow & 0
\end{array}
$$

Note that in order to conclude that the above diagram commutes, we still need to verify that the bottom right square commutes, namely, $(id_{\mathcal{K}})\pi_{\widehat{D}} = (\pi_D)\gamma'$.

We now apply Lemma 2.9 and conclude that the group D is the pushout of $\mathcal{H}$ and $\widehat{D}$ over $\widehat{\mathcal{H}}$. As a corollary we conclude, by Lemma 2.8 (and the fact that f is surjective),

(11) (i) $\pi_{\widehat{D}} = (\pi_D)\gamma'$ (ii) $\ker(\gamma') = \widehat{\delta}(\ker f)$ (iii) γ' is surjective.

In particular, (10) commutes.

If we now return to the surjective $\phi : D \to G$ and compose it with γ', we obtain a surjection $\phi\gamma' : \widehat{D} \to G$.

Let us summarize what we have so far:

(12)
$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{\widehat{\delta}} \widehat{D} \xrightarrow{\pi_{\widehat{D}}} \mathcal{K} \longrightarrow 0$$
$$\downarrow {\phi\gamma'}$$
$$G$$

Step 5 Let $\mu : \mathcal{U} \to \mathcal{K}$ be the universal cover of $\mathcal{K}$, (see [6, Theorem 3.11] for its existence and its local definability) and apply the pullback construction from Proposition 2.11 to $\mathcal{U}$, $\mathcal{K}$ and $\widehat{D}$.

We obtain a $\bigvee$-definable group $\widehat{G}$ (the pullback of $\mathcal{U}$ and $\widehat{D}$ over $\mathcal{K}$), with associated $\bigvee$-definable maps such that the following sequences are exact and commute (since the kernels of $\pi_{\widehat{G}}$ and $\pi_{\widehat{G}}$ are isomorphic we identify them both with $\widehat{\mathcal{H}}$ and assume that the map between them is the identity). By Proposition 2.11, we also have

(13)
$$\pi_{\widehat{G}}(\ker(\eta)) = \ker(\mu).$$

(14)
$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{i} \widehat{G} \xrightarrow{\pi_{\widehat{G}}} \mathcal{U} \longrightarrow 0$$
$$id \downarrow \qquad \eta \downarrow \qquad \mu \downarrow$$
$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{\widehat{\delta}} \widehat{D} \xrightarrow{\pi_{\widehat{D}}} \mathcal{K} \longrightarrow 0$$

Because μ is surjective, so is η, so we obtain a surjective homomorphism $\widehat{F} := \phi\gamma'\eta : \widehat{G} \to G$. It can be inferred from what we have so far that $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$.

Note that $\dim \widehat{G} = \dim \mathcal{U} + \dim \widehat{\mathcal{H}}$ and, since $\mathcal{U}$ is the universal cover of $\mathcal{K}$, $\dim \mathcal{U} = \dim \mathcal{K}$. By Claim 4.3, we have $\dim \widehat{G} = \dim G$. Note also that $\mathcal{U}$ and $\widehat{\mathcal{H}}$ are divisible (as connected covers of divisible groups) and torsion-free and therefore so is $\widehat{G}$. It follows that $\widehat{F} : \widehat{G} \to G$ is isomorphic to the universal cover of G.
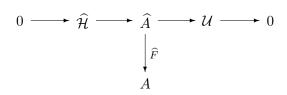
We therefore obtain

(15)
$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{i} \widehat{G} \xrightarrow{\pi_{\widehat{G}}} \mathcal{U} \longrightarrow 0$$
$$\downarrow {\widehat{F}}$$
$$G$$

This ends the proof of the first part Theorem 1.1 for an abelian definably connected, definably compact G.

Assume now that G is an arbitrary definably compact, definably connected group. By [17, Corollary 6.4], the group G is the almost direct product of the definably connected groups $Z(G)^0$ and $[G, G]$, and $[G, G]$ is a semisimple group. The group G is then the homomorphic image of the direct sum $A \oplus S$ with A abelian, S semi-simple, both definably compact, and the kernel of this homomorphism is finite. We may therefore assume that $G = A \oplus S$. By [17, Theorem 4.4 (ii)], the group S is definably isomorphic to a semialgebraic group over a definable real closed field so it must be short. It follows that $\mathrm{lgdim}(G) = \mathrm{lgdim}(A)$. By the abelian case, we obtain the following for the universal cover $\widehat{A}$ of A.

$$0 \longrightarrow \widehat{\mathcal{H}} \longrightarrow \widehat{A} \longrightarrow \mathcal{U} \longrightarrow 0$$
$$\downarrow \widehat{F}$$
$$A$$

Next, we consider $p : \widehat{S} \to S$ the universal cover of S (note that $\widehat{S}$ is also a compact group). By taking the direct product we obtain:
(16)
$$0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{\ i\ } \widehat{G} = \widehat{A} \oplus \widehat{S} \xrightarrow{\ \pi\ } \mathcal{U} \oplus \widehat{S} \longrightarrow 0$$
$$\downarrow \widehat{F} \cdot p$$
$$G = A \oplus S$$

In order to finish the proof of Theorem 1.1 we need to see:

Lemma 4.4. *The group $\mathcal{H} = \widehat{F}(i(\widehat{\mathcal{H}}))$ contains every connected, $\bigvee$-definable strongly long subgroup of G.*

Proof. We first prove the analogous result for the universal cover $\widehat{G}$ of G, namely we prove that $i(\widehat{\mathcal{H}})$ contains every connected, locally definable, strongly long subgroup of $\widehat{G}$. For simplicity, we assume that $\widehat{\mathcal{H}} \subseteq \widehat{G}$.

Assume that $\mathcal{V} \subseteq \widehat{G}$ is a connected, $\bigvee$-definable subgroup with $\dim(\mathcal{V}) = \mathrm{lgdim}(\mathcal{V}) = \ell$. Because $\mathrm{lgdim}(\widehat{G}) = k$ we must have $\ell \leq k$. We will show that the group $\mathcal{V} \cap \widehat{\mathcal{H}}$ has bounded index in $\mathcal{V}$, so by connectedness the two must be equal.

Consider $\mathcal{U}$ from Theorem 1.1. Because $\mathcal{U}$ is short, there exists at least one $u \in \mathcal{U}$ such that $\mathrm{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$ (see [10, Lemma 4.2]). Since $\mathcal{V}$ is a group we can use translation in $\mathcal{V}$ to show that *for every* $u \in \pi(\mathcal{V})$, we must have $\mathrm{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) = \ell$. In particular, $\mathrm{lgdim}(\widehat{\mathcal{H}} \cap \mathcal{V}) = \mathrm{lgdim}(\pi^{-1}(0) \cap \mathcal{V}) = \ell$.

Write $\mathcal{V} = \bigcup_i V_i$ a bounded union of definable sets which we may assume to be all strongly long of dimension ℓ. For every V_i, consider the definable projection $\pi(V_i) \subseteq \mathcal{U}$. By Lemma 9.1 (proved in the appendix), the set F_i

of all $u \in \pi(V_i)$ such that $\operatorname{lgdim}(\pi^{-1}(u) \cap V_i) = \ell$ is definable, so because $\dim(V_i) = \ell$, this set must be finite.

Let $F = \bigcup_i F_i \subseteq \pi(\mathcal{V})$. We claim that $F = \pi(\mathcal{V})$. Indeed, if $u \in \pi(\mathcal{V}) \setminus F$ then by the definition of the F_i's, $\operatorname{lgdim}(\pi^{-1}(u) \cap V_i) < \ell$ for all i, which implies that $\operatorname{lgdim}(\pi^{-1}(u) \cap \mathcal{V}) < \ell$. This is impossible by our above observation, so we must have $F = \pi(\mathcal{V})$.

Because F is a bounded union of finite sets it follows that the index of $\mathcal{V} \cap \widehat{\mathcal{H}}$ in $\mathcal{V}$ is bounded. Since $\mathcal{V}$ is connected it follows that $\mathcal{V} \cap \widehat{\mathcal{H}} = \mathcal{V}$, so $\mathcal{V} \subseteq \widehat{\mathcal{H}}$.

Assume now that $\mathcal{V} \subseteq G$ is a connected, locally definable, strongly long subgroup of G and let $\widehat{\mathcal{V}} \subseteq \widehat{G}$ be the pre-image of $\mathcal{V}$ under $\widehat{F}$. The group $\widehat{\mathcal{V}}$ is strongly long and locally definable, and the connected component of the identity (see [1, Proposition 1]), call it $\widehat{\mathcal{V}}^0$, is still strongly long (since it has the same dimension and long dimension as $\widehat{\mathcal{V}}$). By what we just saw, $\widehat{\mathcal{V}}^0$ is contained in $\widehat{\mathcal{H}}$ and hence $\widehat{F}(\mathcal{V}^0)$ is a $\bigvee$-definable subgroup of $\mathcal{H} \cap \mathcal{V}$, which has bounded index in $\mathcal{V}$. Because $\mathcal{V}$ is connected it follows $\widehat{F}(\mathcal{V}^0) = \mathcal{V} \subseteq \mathcal{H}$. $\square$

This ends the proof of Theorem 1.1.

5. Replacing the locally definable group $\mathcal{U}$ with a definable group

We now proceed to prove Theorem 1.3. We first assume again that G is abelian. The goal is to replace the locally definable group $\mathcal{U}$ in (15) with a *definable* short group. We refer to the notation of (14) and (15).

Step 1 Let $\Lambda = \ker(\widehat{F})$ and let $\Lambda_1 = \pi_{\widehat{G}}(\Lambda) \subseteq \mathcal{U}$.

Claim 5.1. *The universal cover $\mathcal{U}$ of $\mathcal{K}$ from (14), together with Λ_1, satisfy the assumptions of Fact 2.6. Namely, $\mathcal{U}$ is connected, generated by a definably compact set and there is a definable set $X \subseteq \mathcal{U}$ such that $X + \Lambda_1 = \mathcal{U}$. Moreover, Λ_1 is finitely generated.*

Proof. The group $\widehat{G}$ is the universal cover of G. We first find a definable, definably connected, definably compact $X \subseteq \widehat{G}$ which contains the identity, such that $\widehat{F}(X) = G$. We start with a definable $X \subseteq \widehat{G}$ such that $\widehat{F}(X) = G$ and then replace it with $Cl(X)$. We claim that $Cl(X)$ is definably compact. Indeed, if not then by [5, Lemma 5.1 and Theorem 5.2], $\widehat{G}$ has a definable, 1-dimensional subgroup G_0 which is not definably compact. Because μ is locally definable, its restriction to G_0 is definable so $ker(\widehat{F}) \cap G_0$ is finite and therefore trivial. Hence $\widehat{F}(G_0)$ is a definable subgroup of G that is not definably compact, contradicting the fact that G is definably compact. Thus, we can find a definably compact X' with $X' + \ker(\widehat{F}) = \widehat{G}$. By [12, Fact 2.3(2)], X' generates $\widehat{G}$.

By [6, Claim 3.8], $\widehat{G}$ is path connected so we can easily replace X' by $X_1 \supseteq X'$ which is definably compact and path connected (connect any two definably connected components of X' by a definable path). To simplify we call this new set X again.

Also, by [6, Theorem 1.4 and Corollary 1.5], $\ker(\widehat{F})$ is isomorphic to the fundamental group of G, $\pi_1^{def}(G)$, which is finitely generated. It follows that Λ_1 is finitely generated, $\mathcal{U} = \pi_{\widehat{G}}(X) + \Lambda_1$, and $\pi_{\widehat{G}}(X)$ is definably compact and definably connected. Since X generates $\widehat{G}$, the set $\pi_{\widehat{G}}(X)$ generates $\mathcal{U}$. By Remark 2.4, $\mathcal{U}$ is connected. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We can now apply Fact 2.6 and conclude that there is a definably compact group $\overline{K}$ and a $\bigvee$-definable surjection $\widehat{\mu} : \mathcal{U} \to \overline{K}$ with $\ker(\widehat{\mu}) = \Lambda_0 \subseteq \Lambda_1$.

Our goal is to prove: *There are locally definable extensions $\overline{G}$ and G' of $\overline{K}$, by the group $\widehat{\mathcal{H}}$ and $\mathcal{H}$, respectively, and surjective homomorphisms from $\overline{G}$ and G' onto G.*

First, by Lemma 2.12, we have a locally definable group $\widehat{\mathcal{H}}' = \ker(\widehat{\mu}\pi_{\widehat{G}}) = \pi_{\widehat{G}}^{-1}(\Lambda_0) \subseteq \widehat{G}$ such that (we write i for the identity on $\widehat{\mathcal{H}}$ on the top left) the diagram commutes and the following sequences are exact.

$$
(17) \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\ i\ } & \widehat{G} & \xrightarrow{\ \pi_{\widehat{G}}\ } & \mathcal{U} & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle i} & & \Big\downarrow{\scriptstyle id} & & \Big\downarrow{\scriptstyle \widehat{\mu}} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}}' & \xrightarrow{\ id\ } & \widehat{G} & \xrightarrow{\ \widehat{\mu}\pi_{\widehat{G}}\ } & \overline{K} & \longrightarrow & 0
\end{array}
$$

Because $ker(\widehat{\mu}) \subseteq \pi_{\hat{G}}(\Lambda)$, the group $\widehat{\mathcal{H}}'$ is contained in the group $i(\widehat{\mathcal{H}}) + \Lambda$. Since $i(\widehat{\mathcal{H}})$ is a divisible subgroup of $\widehat{\mathcal{H}}'$, there exists a subgroup $\Lambda' \subseteq \Lambda$ such that $\widehat{\mathcal{H}}'$ equals the direct sum of $i(\widehat{\mathcal{H}})$ and Λ'. Because $\ker(\pi_{\widehat{G}}) = i(\widehat{\mathcal{H}})$, the group Λ' is isomorphic, via $\pi_{\widehat{G}}$, to Λ_0, so Λ' is finitely generated. We now have a group homomorphism $p : \widehat{\mathcal{H}}' \to \widehat{\mathcal{H}}$, given via the identification of $\widehat{\mathcal{H}}'$ with $i(\widehat{\mathcal{H}}) \oplus \Lambda'$. Namely, $p(i(h) + \lambda) = h$.

We claim that p is a locally definable map. Indeed, $\widehat{\mathcal{H}}'$ is the union of sets of the form $i(H_i) + F_i$, where H_i is definable and F_i is a finite subset of Λ'. Because the sum of $\widehat{\mathcal{H}}$ and Λ' is direct, each element g of $i(H_i) + F_i$ has a unique representation as $g = i(h) + f$, for $h \in H_i$ and $f \in F_i$. Therefore the restriction of p to $i(H_i) + F_i$ is definable. It follows that p is locally definable.

Step 2. We apply Proposition 2.8 to the diagram

$$
\begin{array}{ccc}
\widehat{\mathcal{H}}' & \xrightarrow{\ id\ } & \widehat{G} \\
\Big\downarrow{\scriptstyle p} & & \\
\widehat{\mathcal{H}} & &
\end{array}
$$

and obtain a locally definable pushout $\overline{G}$, such that the following diagram commutes and the sequences are exact:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}}' & \xrightarrow{id} & \widehat{G} & \xrightarrow{\hat{\mu}\pi_{\widehat{G}}} & \overline{K} & \longrightarrow & 0 \\
& & \downarrow{p} & & \downarrow{\widehat{\alpha}} & & \downarrow{id} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i_1} & \overline{G} & \xrightarrow{\pi_{\overline{G}}} & \overline{K} & \longrightarrow & 0
\end{array}
$$

Because p is surjective the map $\widehat{\alpha} : \widehat{G} \to \overline{G}$ is also surjective. Moreover, by Lemma 2.8, the kernel of $\widehat{\alpha}$ equals $\ker p = \Lambda'$ so is contained in $\Lambda = \ker(\widehat{F})$.

Step 3. We now have surjective maps $\widehat{F} : \widehat{G} \to G$ and $\widehat{\alpha} : \widehat{G} \to \overline{G}$, both $\bigvee$-definable with $\ker(\widehat{\alpha}) \subseteq \ker(\widehat{F})$. By Lemma 2.13 we have a $\bigvee$-definable surjective $\overline{F} : \overline{G} \to G$, with $\ker(\overline{F}) = \widehat{\alpha}(\ker(\widehat{F}))$. We therefore obtained the following diagram:

(18)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{i_1} & \overline{G} & \xrightarrow{\pi_{\overline{G}}} & \overline{K} & \longrightarrow & 0 \\
& & & & \downarrow{\overline{F}} & & & & \\
& & & & G & & & &
\end{array}
$$

Finally, let us calculate $\ker(\overline{F})$: Recall that Λ' is isomorphic to Λ_0 the kernel of the universal covering map $\widehat{\mu} : \mathcal{U} \to \overline{K}$. Because $\overline{K}$ is a short definably compact group, it follows from [8] that $\ker(\widehat{\mu}) = \pi_1^{def}(\overline{K}) = \mathbb{Z}^d$, where $\pi_1^{def}(\overline{K})$ is the o-minimal fundamental group of $\overline{K}$ and

$$
d = \dim(\overline{K}) = \dim(\mathcal{U}) = \dim(G) - k,
$$

for $k = \operatorname{lgdim}(G)$. The map $\widehat{F} : \widehat{G} \to G$ is the universal covering map of G and therefore, as shown in [6, Theorem 1.4, Corollary 1.5], $\ker(\widehat{F}) = \pi_1^{def}(G) = \mathbb{Z}^\ell$, for some ℓ. Furthermore, for every $m \in \mathbb{N}$, the group of m-torsion points $G[m]$ is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^\ell$. By [19, Theorem 7.6], $G[m] = (\mathbb{Z}/m\mathbb{Z})^{\dim(G)}$, hence we can conclude

$$
\Lambda = \ker(\widehat{F}) = \pi_1^{def}(G) = \mathbb{Z}^{\dim G}.
$$

We now have $\ker(\overline{F}) = \widehat{\alpha}(\Lambda) \simeq \Lambda/\Lambda'$, with $\Lambda \simeq \mathbb{Z}^{\dim(G)}$ and $\Lambda' \simeq \mathbb{Z}^{\dim(G)-k}$. Hence, $\ker(\overline{F})$ is isomorphic to the direct sum of $\mathbb{Z}^k$ and a finite group, as required.

Question Can $\overline{K}$ be chosen so that $\ker(\overline{F}) \simeq \mathbb{Z}^k$?

Next, consider $\mathcal{H} \subseteq G$ as in Theorem 1.1. We want to see that we can obtain a similar diagram to (18), with $\mathcal{H}$ instead of $\widehat{\mathcal{H}}$. For simplicity, assume

that i_1 is the identity. First notice that by the last clause of Theorem 1.1, we must have $\overline{F}(\widehat{\mathcal{H}}) \subseteq \mathcal{H}$. However, using exactly the same proof as in Lemma 4.4, we can show that that $\overline{F}(\widehat{\mathcal{H}})$ is also the largest connected strongly long, locally definable, subgroup of G, hence it equals $\mathcal{H}$. We therefore have

$$
\begin{array}{ccc}
\widehat{\mathcal{H}} & \xrightarrow{\ i_1\ } & \overline{G} \\
{\scriptstyle \overline{F}\restriction\widehat{\mathcal{H}}}\Big\downarrow & & \\
\mathcal{H} & &
\end{array}
$$

We can now obtain G', the pushout of $\overline{G}$ and $\mathcal{H}$ over $\widehat{\mathcal{H}}$:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \xrightarrow{\ i_1\ } & \overline{G} & \xrightarrow{\ \pi_{\overline{G}}\ } & \overline{K} & \longrightarrow & 0 \\
& & {\scriptstyle\overline{F}\restriction\widehat{\mathcal{H}}}\big\downarrow & & {\scriptstyle\alpha'}\big\downarrow & & {\scriptstyle id}\big\downarrow & & \\
0 & \longrightarrow & \mathcal{H} & \xrightarrow{\ i'\ } & G' & \xrightarrow{\ \pi_{G'}\ } & \overline{K} & \longrightarrow & 0
\end{array}
$$

Clearly, $\ker(\overline{F} \restriction \widehat{\mathcal{H}}) \subseteq \ker(\overline{F})$, so by Proposition 2.8, $\ker(\alpha') = i(ker\overline{F} \restriction H) \subseteq \ker \overline{F}$. By Lemma 2.13, we have a homomorphism from G' onto G as we want. We therefore have:

(19)
$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{H} & \xrightarrow{\ i'\ } & G' & \xrightarrow{\ \pi_{\overline{G}}\ } & \overline{K} & \longrightarrow & 0 \\
& & & & {\scriptstyle h'}\big\downarrow & & & & \\
& & & & G & & & &
\end{array}
$$

This ends the switch from (18) to (19), and with that the proof of Theorem 1.3 in the case that G is abelian. In order to conclude the same result for arbitrary definably compact, definably connected G, we repeat the same arguments as in the last part of the proof of Theorem 1.1. $\square$

5.1. Special cases.

As was pointed out earlier, we use Fact 2.6 to guarantee that there is a definable group $\overline{K}$ and a $\bigvee$-definable surjection $\widehat{\mu} : \mathcal{U} \to \overline{K}$ with $\Lambda_0 := \ker(\widehat{\mu})$ a subgroup of $\pi_{\widehat{G}}(\ker \widehat{F})$ (see notation of Theorem 1.1). In certain simple cases we can see directly why such Λ_0 exists, without referring to Fact 2.6:

Assume G is abelian. Let $\mathcal{K}$ and $\mathcal{H}$ be as in Section 4.1. Namely, $\mathcal{K}$ is the group obtained as the quotient of the locally definable subgroup $\mathcal{B}$ of G by the compatible subgroup $\mathcal{H}_0 \cap \mathcal{B}$, and $\mathcal{H}$ is the largest locally definable, connected strongly long subgroup of G.

(1) **Assume that $\mathcal{K}$ is definable**.

In this case we take $\Lambda_0 = \ker(\mu)$, where $\mu : \mathcal{U} \to \mathcal{K}$. Obviously, $\mathcal{U}/\Lambda_0$ is definable, so we need only to see that $\Lambda_0 \subseteq \pi_{\widehat{G}}(\ker \widehat{F})$. Let $u \in \ker(\mu)$.

By (13), $u = \pi_{\widehat{G}}(v)$, for some $v \in \ker(\eta)$. But then $\widehat{F}(v) = \phi\gamma'\eta(v) = 0$, so $U \in \pi_{\widehat{G}}(\ker \widehat{F})$.

(2) **Assume that $\mathcal{H}$ is definable**.

We denote by $\overline{K}$ the definable group $G/\mathcal{H}$. From Theorem 1.1 and its proof we obtain the following commutative diagram.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \overset{i}{\longrightarrow} & \widehat{G} & \overset{\pi_{\widehat{G}}}{\longrightarrow} & \mathcal{U} & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widehat{F}} & & & & \\
0 & \longrightarrow & \mathcal{H} & \overset{id}{\longrightarrow} & G & \overset{\pi_{G}}{\longrightarrow} & \overline{K} & \longrightarrow & 0
\end{array}
$$

But now there is a unique map $\mu : \mathcal{U} \to \overline{K}$ which makes the above diagram commute, and it is easy to verify by construction that $\ker(\mu) \subseteq \pi_{\widehat{G}}(\ker(\widehat{F}))$. We now take $\Lambda_0 = \ker(\mu)$.

6. Examples

In this section we provide examples that motivate the statements of Theorem 1.1 and 1.3. More specifically, we give examples of definably compact groups which cannot themselves be written as extensions of short (locally) definable groups by strongly long (locally) definable subgroups. This is what forces us to move our analysis to the level of universal covers.

In the following examples, we fix $\mathcal{M} = \langle M, +, <, 0, R \rangle$ to be an expansion of an ordered divisible abelian group by a real closed field R, whose domain is a bounded interval $(0, a) \subseteq M$. In particular, $\mathcal{M}$ is semi-bounded, o-minimal, and $(0, a)$ is short. Let also $b \in M$ be any tall positive element. In the first two examples, we define semi-linear groups which have the same domain $[0, a) \times [0, b)$ but different operations.

Example 6.1. Pick any $0 < v_1 < a$ such that a and v_1 are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, + \rangle$ generated by the vectors $\langle a, 0 \rangle$ and $\langle v_1, b \rangle$, and let $G = \langle [0, a) \times [0, b), \star, 0 \rangle$ be the group with

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

By [13, Claim 2.7(ii)], G is definable.

Let us see what the various groups of Theorems 1.1 and 1.3 are in this case.

We let $\widehat{G}$ be the subgroup of M^2 generated by $[0, a] \times [0, b]$. The group $\widehat{G}$ is torsion-free and it is easy to see that there is a locally definable covering map $\widehat{F} : \widehat{G} \to G$. Hence, $\widehat{G}$ is the universal cover of G. The group $\widehat{\mathcal{H}} = \{0\} \times \bigcup_n (-nb, nb)$, is a locally definable compatible subgroup of $\widehat{G}$ and the quotient $\widehat{G}/\widehat{\mathcal{H}}$ is isomorphic to the short group $\bigcup_n (-na, na)$.

We have $\mathrm{lgdim}(\widehat{\mathcal{H}}) = \dim(\widehat{\mathcal{H}}) = 1$, so $\widehat{\mathcal{H}}$ is strongly long. As in the proof of Proposition 4.4, the group $\widehat{\mathcal{H}}$ is the largest strongly long, connected, locally definable subgroup of $\widehat{\mathcal{H}}$.

Now, we let $\mathcal{H} = \widehat{F}(\widehat{\mathcal{H}})$. This is the subgroup of G generated by the set $H = \{0\} \times [0,b)$ and we can describe it explicitly. Let $S \subseteq [0,a)$ be the set containing all elements of the form $n(a - v_1) \mod a$. By the choice of v_1, the set S has to be infinite. By the definition of the operation $\star$, it is easy to see that

$$\mathcal{H} = \bigcup_{s \in S} \{s\} \times [0,b),$$

which is not definable (so in particular not compatible in G). This shows the need in Theorem 1.1 to work with the universal cover of G rather than with G itself. Note that $\widehat{F}$ restricted to $\widehat{H}$ is an isomorphism onto $\mathcal{H}$.

In fact, G does not contain any infinite strongly long definable subgroup. Indeed, if it did, then its connected component should be contained in $\mathcal{H}$ and therefore the pre-image of this component under $\widehat{F} \restriction \widehat{\mathcal{H}}$ would be a proper definable subgroup of $\widehat{\mathcal{H}}$ and, thus, of $\langle M, + \rangle$, a contradiction.

Now consider the subgroup $K = \langle [0,a) \times \{0\}, \star, 0 \rangle$ of G and let $\widehat{K}$ be its universal cover. We can write

$$G = \mathcal{H} \star K.$$

Of course $\mathcal{H} \cap K$ is infinite, so this is not a direct sum. However, the universal cover $\widehat{G}$ of G is a direct sum

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K},$$

whereas, if we let

$$\overline{G} = \widehat{\mathcal{H}} \oplus K,$$

then we can define a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}(0,b)$.

We finally observe in this example that $\mathcal{H} \cap K = S$ is not a compatible subgroup of K, which indicates the need for passing to $\mathcal{H}_0$ in the proof of Theorem 1.1 (see Claim 4.2).

Example 6.2. Pick any $0 < u_2 < b$ such that u_2 and b are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, + \rangle$ which is generated by the two vectors $\langle a, u_2 \rangle$ and $\langle 0, b \rangle$, and let again $G = \langle [0,a) \times [0,b), \star, 0 \rangle$ be the group with

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

Here we observe that $H = \{0\} \times [0,b)$ itself is the largest strongly long locally definable subgroup of G and, hence, G is itself an extension of a short definable group by H. However, H does not have a definable complement in G; namely, G cannot be written as a direct sum of H with some definable subgroup of it. The proof of this goes back to [22]. See also [21].

The universal cover $\widehat{\mathcal{H}}$ of H is again the subgroup of M^2 generated by H. Let $\mathcal{K}$ be the subgroup of G generated by $K = [0, a) \times \{0\}$, and $\widehat{K}$ its universal cover. Then we can write

$$G = H \star \mathcal{K},$$

where again $H \cap \mathcal{K}$ is not finite, so this is not a direct sum. The universal cover $\widehat{G}$ of G is again a direct sum

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K}.$$

If we let $\overline{K} = \langle [0, a) \times \{0\}, \star_K, 0 \rangle$ be the group with operation $\star_K = +$ mod a, then we can define a *suitable* extension $\overline{G}$ of $\overline{K}$ by $\widehat{\mathcal{H}}$

$$0 \longrightarrow \widehat{\mathcal{H}} \longrightarrow \overline{G} \longrightarrow \overline{K} \longrightarrow 0$$

and a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}(0, b)$.

We finally give an example for Theorems 1.1 and 1.3 of a definable group G which contains no infinite proper definable subgroup.

Example 6.3. Pick any $0 < v_1 < a$ such that a and v_1 are $\mathbb{Z}$-independent, and any $0 < u_2 < b$ such that u_2 and b are $\mathbb{Z}$-independent. Let L be the subgroup of $\langle M^2, + \rangle$ which is generated by the vectors $\langle a, u_2 \rangle$ and $\langle v_1, b \rangle$. We define the group G with domain

$$\big([0, a) \times [0, b - u_2) \big) \cup \big([v_1, a) \times [b - u_2, b) \big),$$

and group operation again

$$x \star y = z \Leftrightarrow x + y - z \in L.$$

It is not too hard to verify that the above is indeed a definable group - this will appear in a subsequent paper ([11]).

In this case, G does not contain any infinite proper definable subgroup. This again originates in [22]. We let $\mathcal{H}$ the subgroup of G generated by $H = \{0\} \times [0, b - u_2)$, and $\widehat{\mathcal{H}}$ its universal cover. We also let $\mathcal{K}$ be the subgroup of G generated by $K = [0, a) \times \{0\}$, and $\widehat{K}$ its universal cover. Then we have:

$$G = \mathcal{H} \star \mathcal{K},$$

with $\mathcal{H} \cap \mathcal{K}$ infinite, and

$$\widehat{G} = \widehat{\mathcal{H}} \oplus \widehat{K}.$$

Finally, if we let $\overline{K} = \langle [0, a) \times \{0\}, \star_K, 0 \rangle$ be the group with operation $\star_K = +$ mod a, then we can define a suitable extension $\overline{G}$ of $\overline{K}$ by $\widehat{\mathcal{H}}$

$$0 \longrightarrow \widehat{\mathcal{H}} \longrightarrow \overline{G} \longrightarrow \overline{K} \longrightarrow 0$$

and a surjective homomorphism $\overline{F} : \overline{G} \to G$ with $\ker \overline{F} \simeq \mathbb{Z}(v_1, b)$.

7. COMPACT DOMINATION

Let us first recall ([16, Section 7]) that for a definable, or $\bigvee$-definable group $\mathcal{U}$, we write $\mathcal{U}^{00}$ for the smallest, if such exists, type-definable subgroup of $\mathcal{U}$ of bounded index (in particular we require that $\mathcal{U}^{00}$ is contained in a definable subset of $\mathcal{U}$). Note that a type-definable subgroup $\mathcal{H}$ of $\mathcal{U}$ has bounded index if and only if there are no new cosets of $\mathcal{H}$ in $\mathcal{U}$ in elementary extensions of $\mathcal{M}$. A definable $X \subseteq \mathcal{U}$ is called *generic* if boundedly many translates of X cover $\mathcal{U}$. In [12, Theorems 2.9 and 3.9] we established conditions so that $\mathcal{U}^{00}$ and generic sets exist.

Let G be a definably connected, definably compact, abelian definable group and $\pi : G \to G/G^{00}$ the natural projection. We equip the compact Lie group G/G^{00} with the Haar measure, denoted by $m(Z)$, and prove: for every definable $X \subseteq G$, the set of $h \in G/G^{00}$ for which $\pi^{-1}(h) \cap X \neq \emptyset$ and $\pi^{-1}(h) \cap (G \setminus X) \neq \emptyset$ has measure zero. As is pointed out in [16], it is sufficient to prove that

(20)   for every definable $X \subseteq G$, if $\dim X < \dim G$, then $m(\pi X) = 0$.

We say then that G (and π) *satisfy Compact Domination*. When G is locally definable and G^{00} exists then G/G^{00} is locally compact (see [16, Lemma 7.5]) and so admits Haar measure as well. We still say that G satisfies compact domination if (20) holds.

We split the argument into two cases:

I. G is abelian.

Consider the universal covering map $\phi : \widehat{G} \to G$ and the commutative diagram in [12, Proposition 3.8]

(21)
$$
\begin{array}{ccc}
\widehat{G} & \xrightarrow{\ \phi\ } & G \\
{\scriptstyle \pi_{\widehat{G}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_G} \\
\widehat{G}/\widehat{G}^{00} & \xrightarrow{\ \phi'\ } & G/G^{00}
\end{array} \quad .
$$

Using the fact that $\ker \phi$ has dimension zero and $ker\phi'$ is countable, it is not hard to see that G satisfies Compact Domination if and only if $\widehat{G}$ does. Our goal is then to prove (20) for the universal cover $\widehat{G}$.

Recall by Theorem 1.1 the sequence:

$$
0 \longrightarrow \widehat{\mathcal{H}} \xrightarrow{\ i\ } \widehat{G} \xrightarrow{\ f\ } \mathcal{U} \longrightarrow 0
$$

with $\widehat{\mathcal{H}}$ an open subgroup of $\langle M^k, + \rangle$, $\mathrm{lgdim}(\widehat{\mathcal{H}}) = k = \mathrm{lgdim}(\widehat{G})$ and $\mathcal{U}$ a short $\bigvee$-definable group of dimension n. Note that $\widehat{G}$ contains a definable generic set (any definable set which projects onto G), and hence so does $\mathcal{U}$. By [12, Theorem 3.9], $\mathcal{U}$ has a definable, definably compact quotient K, and

the homomorphism from $\mathcal{U}$ onto K has kernel of dimension zero. By [15], the group K, with its map onto K/K^{00} satisfies Compact Domination, and therefore $\pi_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}/\mathcal{U}^{00}$ also satisfies Compact Domination.

We now consider $\widehat{\mathcal{H}}$ and first claim:

(22) $\widehat{\mathcal{H}}^{00}$ exists and contains the set of all short elements in M^k.

Indeed, recall from Section 3.2 that $\widehat{\mathcal{H}}$ is generated by a subset $H' \subseteq M^k$,

$$H' = (-e_1, e_1) \times \cdots \times (-e_k, e_k),$$

with each $e_i > 0$ tall in M. We define, for each $n \in \mathbb{N}$, $H_i = \frac{1}{n} H'$, and claim that

$$\widehat{\mathcal{H}}^{00} = \bigcap_n H_n.$$

Indeed, $\bigcap_n H_n$ is a torsion-free subgroup of $\widehat{\mathcal{H}}$. Moreover, each H_n is generic in $\widehat{\mathcal{H}}$ because we have $\widehat{\mathcal{H}} = H_n + \mathbb{Z}e_1 + \cdots + \mathbb{Z}e_k$. It follows that $\bigcap_n H_n$ has bounded index in $\widehat{\mathcal{H}}$, and thus [12, Proposition 3.6] gives $\widehat{\mathcal{H}}^{00} = \bigcap_n H_n$. Finally, since each e_i is tall, it is easy to verify that each short tuple in M^k must be contained in $\bigcap_n H_n$.

We now claim that $\widehat{G}^{00} \cap i(\widehat{\mathcal{H}}) = i(\widehat{\mathcal{H}}^{00})$. This follows from the fact that $\widehat{G}^{00} \cap i(\widehat{\mathcal{H}})$ has bounded index in $i(\widehat{\mathcal{H}})$ and it is torsion-free ([12, Proposition 3.6]). Next, we claim that $f(\widehat{G}^{00}) = \mathcal{U}^{00}$. Since $f(\widehat{G}^{00})$ has bounded index it must contain $\mathcal{U}^{00}$. Because $\widehat{G}^{00}$ is torsion-free and $\ker(f) = i(\widehat{\mathcal{H}}^{00}) = i(\widehat{\mathcal{H}}) \cap \widehat{G}^{00}$ is divisible ([12, Proposition 3.5]), it follows that $f(\widehat{G}^{00})$ is torsion-free so must equal $\mathcal{U}^{00}$. We therefore have the following commutative diagram of exact sequences:

(23)

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \widehat{\mathcal{H}} & \overset{i}{\longrightarrow} & \widehat{G} & \overset{f}{\longrightarrow} & \mathcal{U} & \longrightarrow & 1 \\
& & \downarrow{\scriptstyle \pi_{\widehat{\mathcal{H}}}} & & \downarrow{\scriptstyle \pi_{\widehat{G}}} & & \downarrow{\scriptstyle \pi_{\mathcal{U}}} & & \\
0 & \longrightarrow & \widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00} & \overset{\widehat{i}}{\longrightarrow} & \widehat{G}/\widehat{G}^{00} & \overset{\widehat{f}}{\longrightarrow} & \mathcal{U}/\mathcal{U}^{00} & \longrightarrow & 0
\end{array}
$$

As in the proof of [12, Proposition 3.8], the map $\widehat{f}$ is continuous.

Assume now that $X \subseteq \widehat{G}$ is a definable set of dimension smaller than $\dim \widehat{G}$. We want to show that $\pi_{\widehat{G}}(X)$ has measure 0. We are going to use several variations of Fubini's theorem so let us see that the setting is correct. By [12], the group $\widehat{G}/\widehat{G}^{00}$ is isomorphic to $\mathbb{R}^k \times \mathbb{R}^n$ and the bottom sequence in the above diagram is just

(24) $0 \longrightarrow \mathbb{R}^k \overset{\widehat{i}}{\longrightarrow} \widehat{G}/\widehat{G}^{00} \overset{\widehat{f}}{\longrightarrow} \mathbb{R}^n \longrightarrow 0$

The above sequence necessarily splits as a Lie group, so by Fubini, a set $Y \subseteq \widehat{G}/\widehat{G}^{00}$ has measure zero if and only if the set

$$\{u \in \mathbb{R}^n : m_{\mathbb{R}^k}(\widehat{f}^{-1}(u) \cap Y) > 0\}$$

has measure zero in $\mathbb{R}^n$. (By $m_{\mathbb{R}^k}(\widehat{f}^{-1}(u) \cap Y)$ we mean the measure after identifying $\mathbb{R}^k \times \{u\}$ with $\mathbb{R}^k$)

We are now ready to start the proof.

Case 1 $\dim f(X) < \dim \mathcal{U}$.

Here we use Compact Domination in expansions of real closed fields (see [15]), so by an earlier observation, $\mathcal{U}$ also satisfies it. Hence, we have $m(\pi_{\mathcal{U}}(f(X))) = 0$, and therefore, by the commutation of the above diagram and Fubini we must have $m(\pi_{\widehat{G}}(X)) = 0$.

Most of the work goes towards the proof of the second case. For simplicity, let us assume that $\widehat{\mathcal{H}} \subseteq \widehat{G}$.

Case 2 $\dim f(X) = \dim \mathcal{U}$.

We first establish two preliminary results.

Claim We may assume that $\mathrm{lgdim}(X) < k = \mathrm{lgdim}(\widehat{G})$.

Indeed, by Lemma 9.1, we can decompose $f(X)$ into two definable sets $Y_1 \cup Y_2$ such that for every $u \in Y_1$, we have $\mathrm{lgdim}(f^{-1}(u) \cap X) < k$ and for every $u \in Y_2$, $\mathrm{lgdim}(f^{-1}(u) \cap X) = k = \dim(f^{-1}(u))$. Because $\dim X < \dim \widehat{G}$ and $\dim f(X) = \dim \mathcal{U}$, the dimension of Y_2 must be smaller than $\dim \mathcal{U}$. By Case (1), we can ignore Y_2 and assume now that for every $u \in f(X)$, $\mathrm{lgdim}(f^{-1}(u) \cap X) < k$. Since $\mathcal{U}$ is short, it follows from [10] that $\mathrm{lgdim}(X) < k$.

In the rest of the argument we prove the more general statement:

Lemma 7.1. *If $X \subseteq \widehat{G}$ is definable and $\mathrm{lgdim}(X) < k$ then the measure of $\pi_{\widehat{G}}(X)$ is zero.*

Proof. We first prove a result for the group $\widehat{\mathcal{H}}$. By [12, Proposition 3.8], the group $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$, equipped with the logic topology, is isomorphic to $\mathbb{R}^k$.

Claim 7.2. *If $Y \subseteq \widehat{\mathcal{H}}$ is definable and $\mathrm{lgdim}(Y) < k$ then $m(\pi_{\widehat{\mathcal{H}}}(Y)) = 0$.*

Proof. Recall that $\widehat{\mathcal{H}}$ is a subgroup of $\langle M^k, + \rangle$ and that the set of all short elements of M^k is contained in $\widehat{\mathcal{H}}^{00}$. Hence, if B is any definably connected short set, then $\pi_{\widehat{\mathcal{H}}}(B) = \{b\}$ is a singleton.

The set Y is a finite union of m-long cones, with $m < k$, hence we may assume that Y is such a cone $C = B + \langle C \rangle$, where $\langle C \rangle = \left\{ \sum_{i=1}^{k} \lambda_i(t_i) : t_i \in I_i \right\}$, for long $I_i = (-a_i, a_i)$ and partial linear maps $\lambda_i : I_i \to M^k$. We have

$$\pi_{\widehat{\mathcal{H}}}(C) = b + \sum_{i=1}^{m} \pi_{\widehat{\mathcal{H}}}(\lambda_i(t_i)).$$

Because $\pi_{\widehat{\mathcal{H}}}$ is a homomorphism from $\langle \widehat{\mathcal{H}}, + \rangle$ onto $\langle \mathbb{R}^k, + \rangle$, it follows that for each $i = 1, \ldots, m$, $t_i \mapsto \pi_{\widehat{\mathcal{H}}}(\lambda_i(t_i))$ is a partial homomorphism from I_i into $\langle \mathbb{R}^k, + \rangle$. Hence, the image of the $\widehat{G}$-linear set $\{ \lambda_i(t) : t \in I_i \}$ is a closed affine subset of $\mathbb{R}^k$ of dimension m. Since $m < k$ we have $m(\pi_{\widehat{\mathcal{H}}}(Y)) = m(\pi_{\widehat{\mathcal{H}}}(C)) = 0$. $\qquad\square$

Claim 7.3. *There exists a definable set $U_0 \subseteq \mathcal{U}$ with $\mathcal{U}^{00} \subseteq U_0$, and a definable section $s : U_0 \to \widehat{G}$ (i.e. $f s(u) = u$ for every $u \in U_0$), such that (i) the function s is continuous with respect to the topologies induced by $\mathcal{U}$ and $\widehat{G}$ and (ii) $s(\mathcal{U}^{00}) \subseteq \widehat{G}^{00}$.*

Proof. Let $U_1 \subseteq \mathcal{U}$ be a definable generic set. By definable choice. there exists a definable partial section $s : U_1 \to \widehat{G}$, namely, $s f(u) = u$ for all $u \in U_1$. The map s is piecewise continuous (with respect to the τ-topologies of $\mathcal{U}$ and $\hat{G}$) and therefore U_1 has a definable, definably connected $U_0 \subseteq U_1$, still generic in $\widehat{G}$ such that s is continuous on U_0. Using Compact Domination for $\mathcal{U}$, it follows from [16, Claim 3, p.590] that the set U_0 contains a coset of $\mathcal{U}^{00}$ so we may assume after translation in U that U_0 contains $\mathcal{U}^{00}$ and $s : U_0 \to \widehat{G}$ is continuous. We may also assume that $s(0) = 0$.

It is left to see that $s(\mathcal{U}^{00})$ is contained in $\widehat{G}^{00}$. Consider the map $\sigma(x, y) = s(x - y) - (s(x) - s(y))$, a definable and continuous map from $U_0 \times U_0$ into $\widehat{\mathcal{H}}$. Because the group topology on $\widehat{\mathcal{H}}$ is the subspace topology of M^k and because $U_0 \times U_0$ is a short definably connected set its image under σ is a short, definably connected subset of $\widehat{\mathcal{H}}$ containing 0. As we pointed out earlier, it must therefore be contained in $\widehat{\mathcal{H}}^{00}$.

We consider the set $\widehat{G}_1 = s(\mathcal{U}^{00}) + \widehat{\mathcal{H}}^{00}$ and claim that $\widehat{G}_1 = \widehat{G}^{00}$. To see first that $\widehat{G}_1$ is a subgroup, we note that

$$(s(u_1) + h_1) - (s(u_2) + h_2) = s(u_1 - u_2) + (h_1 + h_2 - \sigma(u_1, u_2)).$$

When $u_1, u_2 \in \mathcal{U}^{00}$ we have $\sigma(u_1, u_2) \in \widehat{\mathcal{H}}^{00}$ and hence this sum is also in $\widehat{G}_1$. Because s is definable and both $\mathcal{U}^{00}$ and $\widehat{\mathcal{H}}^{00}$ are type-definable the group $\widehat{G}_1$ is also type-definable. Because $\mathcal{U}^{00}$ has bounded index in $\mathcal{U}$ and $\widehat{\mathcal{H}}^{00}$ has bounded index in $\widehat{\mathcal{H}}$ it follows that $\widehat{G}_1$ has bounded index in $\widehat{G}$. Since $\widehat{G}_1$ is torsion-fee it follows from [12, Proposition 3.6] that $\widehat{G}_1 = \widehat{G}^{00}$. In particular, $s(\mathcal{U}^{00})$ is contained in $\widehat{G}^{00}$. $\qquad\square$

Our goal is to show that $m(\pi_{\widehat{G}}(X)) = 0$. By Fubini, it is sufficient to show that for every $u \in \mathcal{U}/\mathcal{U}^{00}$, the fiber $\pi_{\widehat{G}}(X) \cap \widehat{f}^{-1}(u)$ has zero measure in the sense of $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$. Namely, it is the translate in $\widehat{G}/\widehat{G}^{00}$ of a zero measure subset of $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$.

Claim Fix $u \in \mathcal{U}/\mathcal{U}^{00}$. Then there exists a a definable set $Y \subseteq \widehat{\mathcal{H}}$ with lgdim$(Y) < k$, and an element $g \in \widehat{G}$ such that the fiber $\pi_{\widehat{G}}(X) \cap \widehat{f}^{-1}(u)$ is contained in the set

$$\pi_{\widehat{H}}(Y) + \pi_{\widehat{G}}(g).$$

Proof. Fix $\bar{u} \in \mathcal{U}$ such that $\pi_{\mathcal{U}}(\bar{u}) = u$. By translation in $\widehat{G}$ and in $\mathcal{U}$ we may assume that the domain of the partial section s which was defined above, call it still U_0, contains $\bar{u} + \mathcal{U}^{00}$. If we let $g = s(\bar{u})$ then $s(\bar{u} + \mathcal{U}^{00}) \subseteq g + \widehat{G}^{00}$.

Consider the definable map $x \mapsto x - s(f(x))$ from $X \cap f^{-1}(U_0)$ into $\widehat{\mathcal{H}}$ and let Y be its image. Because lgdim$(X) < k$, we must also have lgdim$(Y) < k$. We claim that this is the desired Y. Indeed, we assume that $\widehat{f}(\pi_{\widehat{G}}(x)) = u$ for some $x \in X$ and show that $\pi_{\widehat{G}}(x) \in \pi_{\widehat{H}}(Y) + \pi_{\widehat{G}}(g)$.

By the commuting diagram above, $f(x) \in \bar{u} + \mathcal{U}^{00} \subseteq U_0$ and therefore $x - s(f(x)) \in Y$. Since $s(\bar{u} + \mathcal{U}^{00})$ is contained in $s(\bar{u}) + \widehat{G}^{00}$, we also have $s(f(x)) \in g + \widehat{G}^{00}$. We now have

$$\pi_{\widehat{G}}(x) = \pi_{\widehat{G}}(x - sf(x)) + \pi_{\widehat{G}}(sf(x)) \in \pi_{\widehat{H}}(Y) + \pi_{\widehat{G}}(g).$$
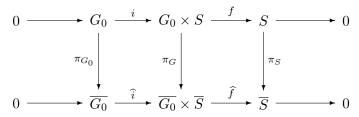
$\square$

We can now complete the proof that $\pi_{\widehat{G}}(X)$ has measure 0. For every $u \in \mathcal{U}/\mathcal{U}^{00}$ we find a definable $Y \subseteq \widehat{\mathcal{H}}$ as above. By Claim 7.2, the set $\pi_{\widehat{\mathcal{H}}}(Y)$ has measure 0 in $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$, hence the fiber $\pi_{\widehat{G}}(X) \cap \widehat{f}^{-1}(u)$ is a translate of a measure zero subset of $\widehat{\mathcal{H}}/\widehat{\mathcal{H}}^{00}$. By Fubini the measure of $\pi_{\widehat{G}}(X)$ is zero.

This ends the proof of Lemma 7.1 and with it that of Compact Domination for abelian G.

II. The general case (G not necessarily abelian).

Assume now that G is an arbitrary definably compact group. By [17], G is the almost direct product of a definably connected abelian group G_0 and a definable semi-simple group S. It is enough to prove the result for a finite cover of G hence we may assume that $G = G_0 \times S$. By [17, Theorem 4.4 (ii)], the group S is definably isomorphic to a semialgebraic group over a definable real closed field so it must be short, and therefore lgdim$G = $ lgdim$G_0 = k$. To simplify the diagram, we use $\overline{G_0} = G_0/G_0^{00}$, $\overline{S} = S/S^{00}$, so we have $G/G^{00} = \overline{G_0} \times \overline{S}$.

We have

$$0 \longrightarrow G_0 \xrightarrow{\ i\ } G_0 \times S \xrightarrow{\ f\ } S \longrightarrow 0$$

$$\begin{array}{ccc} \pi_{G_0}\big\downarrow & \pi_G\big\downarrow & \pi_S\big\downarrow \end{array}$$

$$0 \longrightarrow \overline{G_0} \xrightarrow{\ \widehat{i}\ } \overline{G_0} \times \overline{S} \xrightarrow{\ \widehat{f}\ } \overline{S} \longrightarrow 0$$

Assume now that $X \subseteq G$ is a definable set and $\dim(X) < \dim(G)$. If $\dim(f(X)) < \dim(S)$ then by Compact Domination in expansions of fields, the Haar measure of $\pi_S(f(X))$ in $\overline{S}$ is 0 and therefore $m(\pi_G(X))$ in G/G^{00} is 0.

If $\dim(X) = \dim(S)$ then, as in the abelian case, we may assume, after partition, that for every $s \in S$, $\mathrm{lgdim}(f^{-1}(s) \cap X) < k$. Because S is short, it follows that $\mathrm{lgdim}(X) < k$ and therefore the projection of X into G_0, call it X', has long dimension smaller than k. But now, by Lemma 7.1, the Haar measure in $\overline{G_0}$ of $\pi_{G_0}(X')$ equals to 0. By Fubini, the Haar measure of $\pi_G(X)$ must also be zero.

This ends the proof of Compact Domination for definably compact groups in o-minimal expansions of ordered groups. $\qquad\qquad\square$

8. Appendix A - pullback and pushout

8.1. Pushout.

Proof of Proposition 2.8. We start with

$$A \xrightarrow{\ \alpha\ } B$$
$$\beta\big\downarrow$$
$$C$$

and prove the existence of the pushout D. We first review the standard construction of D (without verifying the algebraic facts). We consider the direct product $B \times C$ and take $D = (B \times C)/H$ where H is the subgroup $H = \{(\alpha(a), -\beta(a)) : a \in A\}$. If we denote by $[b, c]$ the coset of (b, c) mod H then the maps γ, δ are defined by $\gamma(b) = [b, 0]$ and $\delta(c) = [0, c]$. Assume now that we also have

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & B \\ \beta\big\downarrow & & \big\downarrow\gamma' \\ C & \xrightarrow{\ \delta'\ } & D' \end{array}$$

We define $\phi : D \to D'$ by $\phi([b, c]) = \gamma'(b) + \delta'(c)$. Clearly, if all data are definable then so are $B \times C$ and H, and therefore, using definable choice, D and the associated maps are definable.

If α is injective then δ is also injective, and if β is surjective then so is γ (see observation (b) on p. 53 in [14])

Suppose that A, B, C and α, β are $\bigvee$-definable and that $\alpha(A)$ is compatible subgroup of B. Clearly $B \times C$ is $\bigvee$-definable and it is easy to see that H is a $\bigvee$-definable subgroup. We want to show that H is a compatible subgroup of $B \times C$. For that we write $A = \bigcup A_i$, $B = \bigcup B_j$ and $C = \bigcup C_k$. It follows that $B \times C = \bigcup_{j,k} B_j \times C_k$. To show compatibility of H it is enough to show that for every j, k, the intersection $(B_j \times C_k) \cap H$ is definable. Because $\alpha(A)$ is compatible in B, the set $B_j \cap \alpha(A)$ is definable. Hence, there is some i_0 such that $\alpha(A_{i_0}) \supseteq B_j \cap \alpha(A)$. Moreover, because α is injective $\alpha^{-1}(B_j) \subset A_{i_0}$. It follows that the intersection $H \cap (B_j \times C_k)$ equals

$$\{(\alpha(a), -\beta(a)) \in B_j \times C_k : a \in A\} = \{(\alpha(a), -\beta(a)) \in B_j \times C_k : a \in A_{i_0}\}.$$

The set on the right is clearly definable, hence H is a compatible subgroup of $B \times C$, so $D = (B \times C)/H$ is $\bigvee$-definable (see Fact 2.2). It is now easy to check that $\gamma : B \to D$ and $\delta : C \to D$ are $\bigvee$-definable.

If $E = B/\alpha(A)$ then, by the compatibility of $\alpha(A)$, we see that E is $\bigvee$-definable. If $\pi : B \to E$ is the projection then we define $\pi' : D \to E$ by $\pi'([b, c]) = \pi(b)$. It is routine to verify that π' is a well-defined surjective homomorphism whose kernel is $\delta(C)$. It follows, using Fact 2.2, that $\delta(C)$ is a compatible subgroup of D. Finally, it is routine to verify commutation of all maps. $\square$

Proof of Lemma 2.9. We have

(25)
$$
\begin{array}{ccccc}
B & \xrightarrow{\gamma} & D & \xrightarrow{\mu} & F \\
\alpha \uparrow & & \delta \uparrow & & \xi \uparrow \\
A & \xrightarrow{\beta} & C & \xrightarrow{\eta} & E
\end{array}
$$

with D the pushout of B and C over A and F the pushout of B and E over A and we want to see that F is also the pushout of D and E over C.

It is sufficient to show that for every given commutative diagram

(26)
$$
\begin{array}{ccc}
D & \xrightarrow{\mu'} & F' \\
\delta \uparrow & & \xi' \uparrow \\
C & \xrightarrow{\eta} & E
\end{array}
$$

there is a map $\phi' : F \to F'$ such that $\phi'\mu = \mu'$ and $\phi'\xi = \xi'$ (according to the definition we also need to prove uniqueness but this follows).

By commutativity we have $\mu'\delta = \xi'\eta$ and hence $\mu'\delta\beta = \xi'\eta\beta$. Since $\delta\beta = \gamma\alpha$ we also have $(\mu'\gamma)\alpha = (\xi')\eta\beta$. We now use the fact that F is the pushout of B and E over A and conclude that there is $\phi' : F \to F'$ such that

(27) $$(i)\phi'\xi = \xi' \quad \text{and} \quad (ii)\phi'\mu\gamma = \mu'\gamma$$

(i) gives half of what we need to show so it is left to see that $\phi'\mu = \mu'$.

Consider the commutative diagram

(28)

$$
\begin{array}{ccc}
B & \xrightarrow{\mu'\gamma} & F' \\
{\scriptstyle \alpha}\uparrow & & \uparrow{\scriptstyle \xi'\eta} \\
A & \xrightarrow{\beta} & C
\end{array}
$$

Because D is the pushout of B and C over A, there is a unique map $\psi : D \to F'$ with the property

$$(i)\psi\delta = \xi'\eta \quad \text{and} \quad (ii)\psi\gamma = \mu'\gamma.$$

If we can show that both maps μ' and $\phi'\mu$ from D into F' satisfy these properties of ψ then by uniqueness we will get their equality. For $\psi = \mu'$, (i) is part of the assumptions, and (ii) is obvious. For $\psi = \phi'\mu$, we obtain (ii) directly from (27)(ii). To see (i), start from (27)(i), $\phi'\xi = \xi'$, and conclude $\phi'\xi\eta = \xi'\eta$. By commutation, $\xi\eta = \mu\delta$ so we obtain $\phi'\mu\delta = \xi'\eta$, as needed. We therefore conclude that $\mu' = \phi'\mu$ and hence F is the pushout of E and D over C. □

8.2. Pullback.

Proof of Proposition 2.11. Consider the diagram

$$
\begin{array}{ccc}
 & & B \\
 & & \downarrow{\scriptstyle \alpha} \\
C & \xrightarrow{\beta} & A
\end{array}
$$

We again review the algebraic construction of a pullback (which is simpler because we take no quotients). We let

$$D = \{(b,c) \in B \times C : \alpha(b) = \beta(c)\},$$

and the maps are just $\gamma(b,c) = b$ and $\delta(b,c) = c$. Given

$$
\begin{array}{ccc}
D' & \xrightarrow{\gamma'} & B \\
{\scriptstyle \delta'}\downarrow & & \downarrow{\scriptstyle \alpha} \\
C & \xrightarrow{\beta} & A
\end{array}
$$

we define $\phi(d') = (\gamma'(d'), \delta'(d')) \in D$.

Clearly, if all data are definable then so is D and the associated maps. Similarly, if all data are $\bigvee$-definable then so are D and the associated maps.

If $G = \ker(\gamma)$ then

$$G = \{(b,c) \in D : b = 0\} = \{(0,c) \in B \times C : \beta(c) = 0\},$$

and then clearly $j(0,c) = c$ is an isomorphism of G and $H = \ker(\beta)$. If all given data are $\bigvee$-definable then so are G, H and the associated maps.

Furthermore, since G and H are kernels of $\bigvee$-definable maps they are clearly compatible in D, C, respectively.

If β is surjective then so is γ and the sequences in the diagram are exact (and the diagram is commutative). $\qquad\square$

9. Appendix B - Short and long set

We assume that $\mathcal{M}$ is an o-minimal semi-bounded expansion of an ordered group

Lemma 9.1. *Let $S \subseteq M^r$ be a definable short set and let $A \subseteq S \times M^n$ be a definable set. For $s \in S$, we let $A_s = \{x \in M^n : (s, x) \in A\}$. Then, for every $\ell \geq 0$, the set $\ell(A) = \{s \in S : \mathrm{lgdim}(A_s) = \ell\}$ is definable.*

Proof. By [10], the set A can be written as a union of long cones $\bigcup C_i$. Since $\mathrm{lgdim}(X_1 \cup \cdots \cup X_m) = max_i(\mathrm{lgdim}(X_i))$, we may assume that A itself is a long cone $A = B + \sum_{i=1}^k \lambda_i(t_i)$, where $B \subseteq M^{r+n}$ is a short cell, $\lambda_1, \ldots, \lambda_k$ are M-independent partial linear maps $\lambda_i : I_i \to M^{r+n}$ and $I_i = (0, a_i)$ are long intervals. We write $\lambda_i = (\lambda_i^1, \ldots, \lambda_i^{r+n})$, for $i = 1, \ldots, k$, so each λ_i^j is a partial endomorphism from I_i into M.

We claim that for every $s \in S$, $\mathrm{lgdim}(A_s) = k$. This clearly implies what we need.

For $b = (b_1, \ldots, b_{r+n}) \in B$, $i = 1, \ldots, k$ and $t_i \in I_i$, we have $b_i + \lambda_i(t_i) : I_i \to A$. Therefore, we have $(b_1, \ldots, b_r) + (\lambda_i^1(t_i), \ldots, \lambda_i^r(t_i)) \in S$. Each λ_i^j is either injective or constantly 0 and hence, because S is short and each I_i is long, for each $j = 1, \ldots, r$ and $i = 1, \ldots, k$, we have $\lambda_i^j \equiv 0$. It follows that for every $b \in B$, we have $(b_1, \ldots, b_r) \in S$.

For $i = 1 \ldots, k$, we let

$$\hat{\lambda}_i = (\lambda_i^{r+1}, \ldots, \lambda_i^{r+n}) : I_i \to M^n.$$

Because $\lambda_1, \ldots, \lambda_k$ were M-independent, it is still true that $\hat{\lambda}_1, \ldots, \hat{\lambda}_k$ are M-independent. We now have, for every $s \in S$,

$$A_s = \left\{ b + \sum_{i=1}^k \hat{\lambda}_i(t_i) : b \in B_s, t \in I_i \right\}$$

and therefore the set A_s is a k-long cone, so $\mathrm{lgdim}(A_s) = k$. $\qquad\square$

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