

SEMI-LINEAR STARS ARE CONTRACTIBLE

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ABSTRACT. Let \mathcal{R} be an ordered vector space over an ordered division ring. We prove that every definable set X is a finite union of relatively open definable subsets which are definably simply-connected, settling a conjecture from [5]. The proof goes through the stronger statement that the star of a cell in a special linear decomposition of X is definably simply-connected. In fact, if the star is bounded, then it is definably contractible.

1. INTRODUCTION

This paper deals with the general problem of covering definable sets in an o-minimal structure \mathcal{R} with topologically nice subsets. It was first proved by Wilkie [11] that if \mathcal{R} expands an ordered field, then every bounded open definable set X is a finite union of open cells. By Andrews [1] and Edmundo-Eleftheriou-Prelli [6], the statement also holds if \mathcal{R} expands an ordered group, but not a field (and, in fact, without assuming X is bounded). In [5], some strong consequences of the above covering statements were derived and applied to the study of locally definable manifolds in \mathcal{R} . Moreover, it was explained there how a positive solution to the following conjecture is crucial in extending those consequences to a much wider context (such as that of locally definable spaces).

Conjecture ([5]). *Every definable set X is a finite union of relatively open definable subsets which are definably simply-connected.*

The virtue of the above statement over the known aforementioned results is that X is not assumed to be open. We indicate below why this is indeed a significant step.

In this paper, we prove the conjecture in the semi-linear setting; that is, when \mathcal{R} is a pure ordered vector space. As pointed out in [5, Section 5], if \mathcal{R} expands an ordered field, then the conjecture can be replaced by (and, perhaps, follows from) the known triangulation theorem. The triangulation theorem fails in the semi-linear setting (see, for example, [8, Example 1.1]). In general, semi-linear geometry exhibits several intricate phenomena that have by now rendered it into a separate subject admitting a totally different set of techniques (see [7] and [9]). In this paper, we develop further semi-linear homotopy theory, featuring the notion of *canonical retractions* of linear cells, which we then use to settle the conjecture. As a note, the conjecture remains open in the ‘intermediate’ semi-bounded case, where \mathcal{R} expands an ordered group only by a bounded field ([4]).

Let us now describe the semi-linear setting. For the rest of this paper, we fix an ordered vector space $\mathcal{R} = \langle R, <, +, 0, \{x \mapsto \lambda x\}_{\lambda \in \Lambda} \rangle$ over an ordered division ring Λ . By ‘definable’ or

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‘semi-linear’ set, we mean a set definable in \mathcal{R} , possibly with parameters. Equivalently, and seen geometrically, semi-linear sets are exactly the Boolean combination of sets of the form

$$\{x \in R^n : f(x) \geq 0\},$$

where $f : R^n \rightarrow R$ is an affine map with coefficients from Λ . The class of semi-linear sets is closed under desirable geometric operations, such as Boolean combinations, cartesian products, and images and fibers under coordinate projections. If \mathcal{R} is the real vector space, then semi-linear geometry amounts to classical PL-topology. Our setting, however, includes far more general structures, whose role becomes very prominent in areas such as that of valued fields and non-archimedean tame topology (Hrushovski-Loeser [10]). This generality owes partially to the fact that we allow Λ to be non-commutative. For example, Λ could be the ordered ring of generalized power series $\mathbb{R}((G))$ with exponents from a non-abelian ordered group G (and \mathcal{R} could be Λ as an ordered vector space over itself). However, this level of generality is not the sole reason for intriguing semi-linear phenomena to occur. To illustrate the last point, even in the case of ordered abelian groups \mathcal{R} (so, vector spaces over \mathbb{Q}), there are semi-linear manifolds that do not admit semi-linear embeddings into the affine space ([7]), as opposed to a classical fact by Whitney [12] that PL-manifolds always do.

At this high level of generality, model theory has proven to be the right approach to semi-linear geometry, since the first-order theory of \mathcal{R} is so well-understood (see, for example, [3, Chapter 1, §7] and [9]). Most notably, one can define the notion of a *linear cell* and prove a linear cell decomposition theorem, which captures the aforementioned closure properties of the class of semi-linear sets, all at once. This theorem alone has also been put into practice in [10, Remark 14.3.4]. We postpone its precise formulation and any further terminology until Section 2, as we now proceed to describe the content of this paper.

Given a linear decomposition \mathcal{C} of R^n and $C \in \mathcal{C}$, the notion of the star $\text{st}_{\mathcal{C}}(C)$ of C was introduced in [5], as follows:

$$\text{st}_{\mathcal{C}}(C) = \bigcup \{D \in \mathcal{C} : C \cap \text{cl}(D) \neq \emptyset\}.$$

It was then proved there that, if \mathcal{C} is *special*, $\text{st}_{\mathcal{C}}(C)$ is an open (usual) cell. This implied the conjecture for an open definable set X . Indeed, if \mathcal{C} partitions X , the collection of all stars of cells in X provide an open covering of X by open cells. To check moreover that every cell D is definably simply-connected, observe that by Berarducci-Fornasiero [2, Lemma 3.2], a bounded cell is definably contractible (and the boundedness assumption is necessary, see Fact 2.15 below). Then it is not hard to see that every definable loop in D is contained in a bounded cell contained in D and hence it is definably homotopic (in D) to a constant loop (see Claim 2.14 below).

In order to apply the same strategy and prove the conjecture for any definable set $X \subseteq R^n$ (not necessarily open), we are led to consider the star $\text{st}_{\mathcal{C}}(C)$ of a cell $C \subseteq X$ in a special linear decomposition \mathcal{C} of X (instead of R^n). For example, \mathcal{C} could be the restriction to X of a special linear decomposition \mathcal{C}' of R^n that partitions X . In that case,

$$\text{st}_{\mathcal{C}}(C) = \text{st}_{\mathcal{C}'}(C) \cap X.$$

The main problem now is that $\text{st}_{\mathcal{C}}(C)$ needs not be a cell anymore, and so it is far from clear if it is contractible. For example, if we consider the definable contraction of $\text{st}_{\mathcal{C}'}(C)$ from [2, Lemma 3.2] (assuming further it is bounded), its restriction to $\text{st}_{\mathcal{C}}(C)$ needs not stay inside $\text{st}_{\mathcal{C}}(C)$. As it turns out, proving the contractibility of $\text{st}_{\mathcal{C}}(C)$ amounts to proving the following general independent statement.

Proposition 1. *Let $Y \subseteq R^n$ be a bounded definable set, $C \subseteq Y$ a linear cell, and \mathcal{D} a special linear decomposition of Y that contains C . Assume that*

$$\forall D \in \mathcal{D}, C \cap \text{cl}(D) \neq \emptyset.$$

Then Y is definably contractible.

Proposition 1 is the heart of this paper and is proved in Section 4. In the rest of this introduction we summarize its consequences and describe how it leads to the solution of the conjecture. We also sketch the main idea behind its proof.

An immediate consequence of Proposition 1 is the following theorem.

Theorem A. *Let $X \subseteq R^n$ be a definable set, \mathcal{C} a special linear decomposition of X , and $C \in \mathcal{C}$ with $C \subseteq X$. If $\text{st}_{\mathcal{C}}(C)$ is bounded, then $\text{st}_{\mathcal{C}}(C)$ is definably contractible.*

Proof. Let $\mathcal{D} = \{D \in \mathcal{C} : C \cap \text{cl}(D) \neq \emptyset\}$. By Proposition 1, $\text{st}_{\mathcal{C}}(C) = \cup \mathcal{D}$ is definably contractible. \square

In order to establish the conjecture, we need to get rid of the boundedness assumption, much alike we did in the case of an open set X . This is achieved in Lemma 4.2, which, together with Proposition 1, implies:

Proposition 2. *Let $Y \subseteq R^n$ be a definable set, $C \subseteq Y$ a linear cell, and \mathcal{D} a special linear decomposition of Y that contains C . Assume that*

$$\forall D \in \mathcal{D}, C \cap \text{cl}(D) \neq \emptyset.$$

Then Y is definably simply-connected.

Proposition 2 implies our second theorem.

Theorem B. *Let X, \mathcal{C} and C be as in Theorem A. Then $\text{st}_{\mathcal{C}}(C)$ is definably simply-connected.*

Proof. Let $\mathcal{D} = \{D \in \mathcal{C} : C \cap \text{cl}(D) \neq \emptyset\}$. By Proposition 2, $\text{st}_{\mathcal{C}}(C) = \cup \mathcal{D}$ is definably simply-connected. \square

As a corollary, we settle the conjecture:

Corollary.

- (1) *Every definable set is a finite union of relatively open definable subsets which are definably simply-connected.*
- (2) *Every bounded definable set is a finite union of relatively open definable subsets which are definably contractible.*

Proof. Let X be a definable set, \mathcal{C}' a special linear decomposition of R^n partitioning it, and \mathcal{C} its restriction to X . Then X is clearly the finite union of all stars $\text{st}_{\mathcal{C}}(C) = \text{st}_{\mathcal{C}'}(C) \cap X$, for $C \in \mathcal{C}$. Since each $\text{st}_{\mathcal{C}'}(C)$ is open, $\text{st}_{\mathcal{C}}(C)$ is relatively open in X . It is also definably simply-connected, by Theorem B. If, moreover, X is bounded, then $\text{st}_{\mathcal{C}}(C)$ is also definably contractible, by Theorem A. \square

The main idea of the proof of Proposition 1. Our strategy is to construct, for each $D \in \mathcal{D}$, a canonical retraction of $\text{cl}(D)$ to the closure of the half-cell $C' \subseteq C$. The virtue of such a retraction is two-fold. First, its restriction to $C \cup D$ is a deformation retraction of $C \cup D$ to C' (Lemma 3.5). Second, if E is another cell in \mathcal{D} , contained in the boundary of D , then

the canonical retraction of $cl(D)$ to $cl(C')$ extends that of $cl(E)$ to $cl(C')$ (Lemma 3.6). As a consequence, we can combine the above retractions together and obtain a deformation retraction of Y to C' (Proposition 4.1). Finally, we observe that C' is definably contractible (Lemma 2.16).

The canonical retraction is in fact given relative to some *corner* c of C . To simplify the presentation, we first define it for a *canonical* linear cell D , a *face* C of D and $c = 0$ (Definition 3.3). For an arbitrary bounded linear cell, the construction is delayed until Definition 3.12. Definition 3.3 is by recursion on n and runs in parallel with Claim 3.4, where we prove that, at the recursive step, the resulting map H_n is indeed a deformation retraction with the required properties. The definition is rather intricate and to facilitate its reading we illustrate it with Example 3.1. The choice of our construction, and especially of retracting Y to C' as opposed to C , is provided with an explanation in Remark 3.7.

Structure of the paper. In Section 2, we introduce our terminology and prove some basic facts. In Section 3, we give the construction of a canonical retraction. In Section 4, we conclude the proofs of Propositions 1 and 2.

2. PRELIMINARIES

Let us first fix some notation of this paper. By 0 we denote the origin of the space at hand. We let $R^0 = \{0\}$. We also denote by $0 : X \rightarrow R$ the map $0(x) = 0$, whereas by 1_X we denote the identity map on X . We write $[a, a]$ for $\{a\}$, and $Im(f)$ and $\Gamma(f)$ for the image and graph, respectively, of a function f . By a *box* we mean a bounded set of the form

$$B = (a_1, b_1) \times \cdots \times (a_n, b_n),$$

where $a_i, b_i \in R$. If $m \leq n$, then $\pi_m : R^n \rightarrow R^m$ denotes the projection onto the first m coordinates. We write π for π_{n-1} . If \mathcal{C} is a collection of sets in R^n , by $\pi_m(\mathcal{C})$ we mean the collection of their projections on R^m . If $Y \subseteq R^n$ is a definable set, then the restriction of \mathcal{C} to Y is the collection of sets $\{C \cap Y : C \in \mathcal{C}\}$. If $\sigma = (j_1, \dots, j_n)$ and $\tau = (i_1, \dots, i_n)$ are in $\{0, 1\}^n$, then $\sigma \leq \tau$ (respectively, $\sigma < \tau$) means that for every m , $j_m \leq i_m$ (respectively, $j_m < i_m$). If $a \in R$ and $X \subseteq R$, then $a < X$ means that $a < x$ for all $x \in X$.

2.1. Special linear decompositions and stars. We recall some basics for semi-linear sets, revisit special linear decompositions and stars from [6], and prove a few simple facts. A function $f : R^n \rightarrow R$ is called *linear* (or *affine*), if it is of the form

$$f(x_1, \dots, x_n) = \lambda_1 x_1 + \dots + \lambda_n x_n + a,$$

where $\lambda_i \in D$ and $a \in R$. For a non-empty set $X \subseteq R^n$, we denote by $L(X)$ the set of restrictions to X of linear functions and by $L_\infty(X)$ the set $L(X) \cup \{\pm\infty\}$, where we regard $-\infty$ and $+\infty$ as constant functions on X . Obviously, if $f \in L(X)$ then it extends uniquely to a linear map, and hence we write $f(a)$ for its value at a , even if $a \notin X$. We also write $f|_Y$ for its restriction to a set Y , and $\Gamma(f)_Y$ for the graph of $f|_Y$. If $f, g \in L_\infty(X)$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and denote

$$(f, g)_X = \{(x, y) \in X \times R : f(x) < y < g(x)\}.$$

The notations $[f, g]_X$, $(f, g]_X$ and $[f, g)_X$ obtain the obvious meanings. By [6, Lemma 2.8], if $C = (f, g)_X$, then $\pi(cl(C)) = cl(\pi(C))$. We use this fact repeatedly.

A *linear cell* in R^n is defined similarly to [3, Chapter 3, (2.3)], recursively, as follows:

- $C \subseteq R$ is a linear cell if it is a singleton, or an open interval with endpoints in $R \cup \{\pm\infty\}$.
- $C \subseteq R^n$, $n > 1$, is a linear cell if it is a set of the form $\Gamma(f)$, for some $f \in L(X)$, or $(f, g)_X$, for some $f, g \in L_\infty(X)$, and X is a linear cell in R^n .

We call $\pi(C)$ the *domain*, and f, g the *cell-maps*, of C . We may attach an index $(i_1, \dots, i_n) \in \{0, 1\}^n$ to each linear cell C , such that $i_m = 0$ if and only if $\pi_m(C)$ is the graph of a function.

We refer the reader to [3, Chapter 3, (2.10)] for the definition of a *decomposition* of R^n . A *linear decomposition of R^n* is then a decomposition \mathcal{C} of R^n such that each $B \in \mathcal{C}$ is a linear cell. The *linear cell decomposition theorem* can be proved similarly to [3, Chapter 3, (2.11)] and has already been observed in [9, Section 3]:

Linear cell decomposition theorem.

- (1) *Given any definable sets $A_1, \dots, A_k \subseteq R^n$, there is a linear decomposition \mathcal{C} of R^n that partitions each A_i .*
- (2) *Given a definable function $f : A \rightarrow R$, there is a linear decomposition \mathcal{C} of R^n that partitions A such that the restriction $f|_B$ to each $B \in \mathcal{C}$ with $B \subseteq A$ is linear.*

The notion of a ‘special linear decomposition’ was introduced in [6] and we recall it here, in a slightly different version. First, let us define a *linear decomposition of a definable set Y* as the restriction to Y of a linear decomposition of R^n that partitions Y . (Equivalently, one could follow [3, Chapter 4 (2.5)], where simply the properties of a decomposition of R^n are required for a partition of Y .)

Definition 2.1. A *special linear decomposition* of a definable set $Y \subseteq R^n$ is defined recursively on n , as follows. Any linear decomposition of Y is special. A linear decomposition \mathcal{C} of Y , $n > 1$, is special if:

- (1) $\pi(\mathcal{C})$ is a special linear decomposition of R .
- (2) For every two cells $\Gamma(f)_S$ and $\Gamma(g)_T$ in \mathcal{C} and $V \in \pi(\mathcal{C})$,

$$f|_V < g|_V \text{ or } f|_V = g|_V \text{ or } f|_V > g|_V.$$

- (3) For every two cells $\Gamma(h)_S$ and $(f, g)_T$ in \mathcal{C} ,

$$\text{there is no } c \in cl(S) \cap cl(T) \text{ such that } f(c) < h(c) < g(c).$$

The above definition differs from [6, Definition 2.5] in that, (a) it is given for any set Y and not just R^n , and (b) it further requires property (2). Point (a) will be handy when stating Lemma 4.2 below, whereas none of (a) or (b) causes serious diverging from [6], since the proof of [6, Lemma 2.6] actually shows:

Fact 2.2. *Let $Y \subseteq R^n$ be a definable set. Then for any linear decomposition \mathcal{D} of Y , there is a special linear decomposition \mathcal{C} of Y that refines \mathcal{D} (that is, every cell in \mathcal{D} is a union of cells in \mathcal{C}).*

We include the proof of the above fact in the Appendix, for completeness. Perhaps redundantly, a special case of property (2) for $Y = R^n$ was proved in [6] in a separate lemma ([6, Lemma 2.12]). Let us recall here another important corollary from [6], which we will use in the proof of Proposition 4.1.

Fact 2.3. *Let $Y \subseteq R^n$ be a definable set, \mathcal{C} a special linear decomposition of Y , and $D, E \in \mathcal{C}$ such that $D \cap cl(E) \neq \emptyset$. Then $D \subseteq cl(E)$.*

Proof. The proof in [6, Corollary 2.15] uses [6, Lemma 2.14]. Both references hold with Y in place of R^n , with identical proofs, after replacing R^n by Y , and [6, Lemma 2.12] by Definition 2.1(2). \square

Remark 2.4. If \mathcal{C}' is a special linear decomposition of R^n that partitions a definable set X , then its restriction \mathcal{C} to X is clearly a special linear decomposition of X . In fact, every special

linear decomposition \mathcal{C} of X can be obtained in this way, but we do not prove or make use of this fact here.

We finally define the notion of a star, which however plays no further role in this paper than it did in the introduction.

Definition 2.5 (Stars). Let X be a definable set, \mathcal{C} a linear decomposition of X and $C \in \mathcal{C}$. The *star of C in X with respect to \mathcal{C}* , denoted by $\text{st}_{\mathcal{C}}(C)$, is the set

$$\text{st}_{\mathcal{C}}(C) = \bigcup \{D \in \mathcal{C} : C \cap \text{cl}(D) \neq \emptyset\}.$$

2.2. Canonical linear cells, faces and half-cells. The canonical retraction in Section 3 is a deformation retraction of a *canonical* linear cell to the closure of the *half-cell* of one of its *faces*. In this section we introduce these three notions.

Definition 2.6. Let $D \subseteq R^n$ be a linear cell. For every $i = 1, \dots, n$, let h_i be either

- the unique linear map $h_i : \pi_{i-1}(D) \rightarrow R$ with $\pi_i(D) = \Gamma(h_i)$, or
- the unique pair of linear maps $h_i = (f_i, g_i)$ with $\pi_i(D) = (f_i, g_i)_{\pi_{i-1}(D)}$.

We call h_1, \dots, h_n the *defining maps* of D . We call D a *canonical linear cell* if it is bounded, and for every i , $h_i = 0$ or $h_i = (0, g_i)$.

Note: since $R^0 = \{0\}$, a canonical linear cell in R is just an interval $(0, a)$, $a \in R$.

Definition 2.7 (Faces). Let $D \subseteq R^n$ be a canonical linear (i_1, \dots, i_n) -cell and h_1, \dots, h_n its defining maps. Let $\sigma = (j_1, \dots, j_n) \in \{0, 1\}^n$ with $\sigma \leq (i_1, \dots, i_n)$. We define the σ -*face* C of D , recursively on n , as follows. Let B be the (j_1, \dots, j_{n-1}) -face of $A = \pi(D)$. Then C is the linear (j_1, \dots, j_n) -cell:

- $\Gamma(0)_B$, if $D = \Gamma(0)_A$,
- $\Gamma(0)_B$, if $D = (0, g)_A$ and $j_n = 0$.
- $(0, g)_B$, if $D = (0, g)_A$ and $j_n = 1$.

A *face* of D is a σ -face of D , for some $\sigma = (j_1, \dots, j_n)$.

We make a few observations. The closure of a face of D contains the origin. A σ -face C of an (i_1, \dots, i_n) -cell of D , with $\sigma < (i_1, \dots, i_n)$, is contained in the boundary of D . The $(0, \dots, 0)$ -face of D consists only of the origin, whereas the (i_1, \dots, i_n) -face of D is D itself. By induction on $n - m$, one can easily see that if $C \subseteq \text{cl}(D)$ is a (j_1, \dots, j_n) -face of D , then for all $m = 1, \dots, n$, $\pi_m(C) \subseteq \text{cl}(\pi_m(D))$ is a (j_1, \dots, j_m) -face of $\pi_m(D)$. Moreover, by induction on n , if E is a face of C and C is a face of D , then E is a face of D . Finally, a face of D is also a canonical linear cell.

The following claim will be used later on.

Claim 2.8. *Let \mathcal{C} be a special linear decomposition of a definable set, and $C, D \in \mathcal{C}$ two canonical linear cells with $0 \in \text{cl}(C) \subseteq \text{cl}(D)$. Then C is a face of D .*

Proof. Let $C \subseteq R^n$ be a (j_1, \dots, j_n) -cell and $D \subseteq R^n$ a (i_1, \dots, i_n) -cell. We prove the claim by induction on n . For $n = 1$, it is immediate. For $n > 1$, we have $0 \in \text{cl}(\pi(C)) \subseteq \text{cl}(\pi(D))$ and thus, by induction, $\pi(C)$ is a face of $\pi(D)$. If $j_n = 0$, then it is immediate that C is a face of D , so assume $i_n = j_n = 1$. Let $D = (f, g)_A$ and $C = (h, k)_B$, with $B \subseteq \text{cl}(A)$. We want to prove that $h = f|_B$ and $k = g|_B$. Since $\text{cl}(C) \subseteq \text{cl}(D)$, we know that for every $t \in C$,

$$f(t) \leq h(t) < k(t) \leq g(t),$$

and by Definition 2.1(3), the first and last inequalities cannot be strict. \square

We now proceed to the notion of a half-cell.

Definition 2.9. Let $A \subseteq R^{n-1}$ and $g \in L(A)$. The *half-map* of g is the map $F \in L(A)$ given by

$$F(x) = \frac{g(x)}{2}.$$

It is clear that $0 < F|_A < g|_A$ and $0 \leq F|_{cl(A)} \leq g|_{cl(A)}$.

Now let C be a canonical linear cell. We define the *half-cell* of C , denoted simply by C' , recursively, as follows:

- (1) $n = 1$. If C is a singleton, then $C' = C$. If $C = (0, a)$, then $C' = (0, \frac{a}{2}]$.
- (2) $n > 1$. Let $A = \pi(C)$ and A' its half-cell.
 - If $C = \Gamma(0)_A$, then $C' = \Gamma(0)_{A'}$.
 - If $C = (0, g)_A$, then $C' = (0, F]_{A'}$, where F is the half-map of g .

By construction, the half-cell of $\pi(C)$ equals $\pi(C')$.

Lemma 2.10. Let $C = (0, g)_B$ and $D = (0, h)_A$ be two canonical linear cells, such that C is a face of D (and so B is a face of A). Let $f \in L(B)$ be the half-map of g and $e \in L(A)$ the half-map of h . Then:

$$f = e|_B.$$

Proof. Clear from the definition. □

It is also clear that for C and D as above, C' is a face of D' , but we will not make use of this fact here.

2.3. Homotopy. We recall the definable analogues of standard notions from algebraic topology, and prepare the ground for the construction of a canonical retraction in Section 3.

Definition 2.11. Let $A \subseteq X$ be two definable sets. We say that X *deformation retracts to* A if there is a definable continuous $H : [0, q] \times X \rightarrow X$ such that:

- (1) $H(0, X) = A$
- (2) $\forall t \in [0, q], H(t, -)|_A = 1_A$
- (3) $H(q, -) = 1_X$.

We call H a *deformation retraction of X to A* . If A above is a singleton $\{c\}$, we say that X is *definably contractible (to c)*, and that H is a *definable contraction of X to c* .

Note that the above notion of a deformation retraction is often regarded as a ‘strong’ one in the literature, because of (2). Note also that we have omitted the word ‘definable’ from our terminology, for simplicity.

Definition 2.12. Let $A \subseteq X \subseteq X'$ be three definable sets and suppose that $H : [0, q] \times X \rightarrow X$ and $H' : [0, q'] \times X' \rightarrow X'$ are deformation retractions of X and X' , respectively, to A . We say that H' *extends* H if $q \leq q'$ and for every $(t, x) \in [0, q] \times X$,

$$H(t, x) = H'(t, x).$$

By a *definable path* we simply mean a definable continuous map $\gamma : [0, p] \rightarrow R^n$. We call γ a *loop* if $\gamma(0) = \gamma(p)$. Given $c \in R^n$, the constant path ε_c is defined by $\varepsilon_c(x) = c$ and its domain can vary according to context. A definable set is called *definably connected* if every two points of it are connected with a definable path.

Definition 2.13. Let X be a definable set and $\gamma, \delta : [0, p] \rightarrow X$ two definable paths with $\gamma(0) = \delta(0)$ and $\gamma(p) = \delta(p)$. We say that γ and δ are *definably homotopic (in X)* if there is a definable continuous $F : [0, q] \times [0, p] \rightarrow X$ such that:

- (1) $F(0, -) = \gamma$.

- (2) $\forall t \in [0, q], F(t, 0) = \gamma(0)$ and $F(t, p) = \gamma(p)$.
(3) $F(q, -) = \delta$.

We call X *definably simply-connected* if it is definably connected and every definable loop in it is definably homotopic to a constant path.

Claim 2.14. *Suppose that X is definably contractible. Then it is definably simply-connected.*

Proof. Given a deformation retraction $H : [0, q] \times X \rightarrow X$ of X to c , and a definable path $\gamma : [0, p] \rightarrow X$, the map $F : [0, q] \times [0, p] \rightarrow X$ defined by

$$F(t, x) = H(t, \gamma(x))$$

witnesses that γ is definably homotopic to the constant path ε_c . Moreover, given any $x \in X$, the map $H(-, x)$ is a definable path from c to x , witnessing that X is also definably connected. \square

By [2, Lemma 3.2], every bounded cell is definably contractible. We also know the converse.

Fact 2.15. *An unbounded definable set is not definably contractible.*

Proof. Let X be an unbounded definable set. Suppose $H : [0, q] \times X \rightarrow X$ is a definable contraction of X to a point $c \in X$. Consider the map $f : [0, q] \rightarrow \mathcal{R}$ given by

$$x \mapsto \sup\{|x_1 + \cdots + x_n| : (x_1, \dots, x_n) \in H(t, X)\}.$$

Then f is a definable map whose image contains an unbounded interval, since X is unbounded. But that is a contradiction, because \mathcal{R} has no poles; that is, there are no definable bijections between bounded and unbounded sets ([4]). \square

We note here that if \mathcal{R} were to expand an ordered field, then unbounded cells could also be shown to be definably contractible. For example, R itself would be definably contractible to 0 via $H : [0, 1] \times R \rightarrow R$ with $H(t, x) = tx$. The lack of multiplication in our setting is obviously one of its main particularities.

Lemma 2.16. *Let C be a canonical linear cell. Then C' is definably contractible.*

Proof. By induction. Let $n = 1$. If C is a singleton, it is trivial, and if $C = (0, a)$, let $H : [0, \frac{a}{2}] \times C' \rightarrow C'$ given by $H(t, x) = \max\{\frac{a}{2} - t, x\}$. Then H is a definable contraction of C' to $\{\frac{a}{2}\}$. Now let $n > 1$. By induction, there is a definable contraction

$$H_1 : [0, q_1] \times \pi(C') \rightarrow \pi(C')$$

of $\pi(C')$ to some $c \in \pi(C')$. If $C = \Gamma(0)_A$, let $H : [0, q_1] \times C' \rightarrow C'$ with

$$H(t, x, y) = (H_1(t, x), 0).$$

Then H is a definable contraction of C' to $(c, 0)$. If $C = (0, g)_A$, let F be the half map of g , and

$$H : [0, q_1 + \sup \text{Im} F] \times C' \rightarrow C'$$

with

$$H(t, x, y) = \begin{cases} (H_1(t, x), F(H_1(t, x))), & \text{if } t < q_1, \\ (x, \max\{F(x) - (t - q_1), y\}), & \text{if } t \geq q_1. \end{cases}$$

It is easy to check that H is a definable contraction of C' to $(c, g(c))$. \square

3. CANONICAL RETRACTIONS

We are now ready to present the construction of a canonical retraction. As mentioned in the introduction, we first give it for canonical linear cells (Definition 3.3) and then for arbitrary bounded linear cells (Definition 3.12); this significantly reduces the notational complexity of the presentation. Definition 3.3 is given recursively on n and runs in parallel with Claim 3.4, where we verify that all necessary properties hold at the recursive step. Before stating the (rather lengthy) definition, we illustrate it with an example.

Example 3.1. Let $D = (0, g)_{(0, a)}$ be a canonical linear cell in R^2 , f the half-map of g , C a face of D , and C' the half-cell of C . We illustrate Cases (II) and (III) of Definition 3.3 below. Let $q = a + \sup \text{Img}$, and define the deformation retraction

$$H : [0, q] \times \text{cl}(D) \rightarrow \text{cl}(D)$$

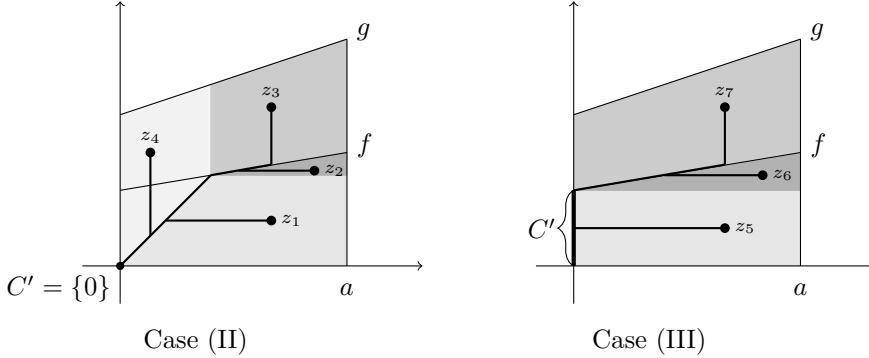
of $\text{cl}(D)$ to $\text{cl}(C')$, as follows.

Case (II): $C = \{0\}$. Then $C' = C$ and:

$$H(t, x, y) = \begin{cases} (t, \min\{y, t, f(t)\}), & \text{if } t < x, \\ (x, \min\{y, t, t - x + f(x)\}), & \text{if } t \geq x. \end{cases}$$

Case (III): $C = \{0\} \times (0, g(0))$. Then $C' = \{0\} \times (0, g(0)/2)$ and:

$$H(t, x, y) = \begin{cases} (t, \min\{y, f(t)\}), & \text{if } t < x, \\ (x, \min\{y, t - x + f(x)\}), & \text{if } t \geq x. \end{cases}$$



The above pictures depict the images of $H(-, z_i)$, for various $z_i = (x, y) \in D$. Depending on the location of z_i , the map $t \mapsto H(t, z_i)_2$ takes the following values:

$$\begin{array}{ll} z_1 : t, y & z_5 : y \\ z_2 : t, f(t), y & z_6 : f(t), y \\ z_3 : t, f(t), t - x + f(x) & z_7 : f(t), t - x + f(x) \\ z_4 : t, t - x + f(x) & \end{array}$$

Our canonical retractions have the additional property of being ‘nice’.

Definition 3.2. Let $H : [0, q] \times X \rightarrow X$ be a deformation retraction of X to $A \subseteq X$. For every $x \in X$, the *fixing point of x under H* is the point $\alpha_x \in [0, q]$ given by

$$\alpha_x = \min\{t \in [0, q] : H(t, x) = x\},$$

which exists by continuity of H . We call H *nice*, if for every $x \in X$ and $t \geq \alpha_x$, $H(t, x) = x$.

Note that there may be more ways to handle Example 3.1. However, the suggested retractions can generalize to an arbitrary n , as follows.

Definition 3.3 (Canonical retraction). Let $D \subseteq R^n$ be a canonical linear (i_1, \dots, i_n) -cell, and C a (j_1, \dots, j_n) -face of D . Let C' be the half-cell of C . The *canonical retraction H of $cl(D)$ to $cl(C')$* ,

$$H_n : [0, q_n] \times cl(D) \rightarrow cl(D),$$

is a nice retraction defined recursively on n , as follows. Let h_1, \dots, h_n be the defining maps of D . So $h_i = 0$ or $h_i = (0, g_i)$.

$n = 1$. First, define

$$q_1 = \begin{cases} 0, & \text{if } i_1 = 0, \\ g_1(0), & \text{if } i_1 = 1. \end{cases}$$

Now let $y \in cl(D)$. If $i_1 = j_1$, then define $H_1(t, y) = y$. If $i_1 > j_1$, define

$$H_1(t, y) = \min\{t, y\}.$$

$n > 1$. Let $H_{n-1} : [0, q_{n-1}] \times cl(\pi(D)) \rightarrow cl(\pi(D))$ be the canonical retraction of $cl(\pi(D))$ to $cl(\pi(C'))$. If $i_n = 1$, we let $f : \pi(D) \rightarrow R$ be the half-map of g_n , and so $C' = (0, f]_{\pi(D)}$. Recall that for every $x \in cl(\pi(D))$, α_x denotes the fixing point of x under H_{n-1} ,

$$\alpha_x = \min\{t \in [0, q_{n-1}] : H_{n-1}(t, x) = x\}.$$

Now let

$$q_n = \begin{cases} q_{n-1}, & \text{if } i_n = 0, \\ q_{n-1} + \sup \text{Im} g_n, & \text{if } i_n = 1, \end{cases}$$

and for every $(x, y) \in cl(D)$, define $H_n(t, x, y)$ by cases, as follows.

(I) If $i_n = j_n = 0$, then $H_n(t, x, y) = (H_{n-1}(t, x), 0)$.

(II) If $i_n > j_n$, then

$$H_n(t, x, y) = \begin{cases} (H_{n-1}(t, x), \min\{y, t, fH_{n-1}(t, x)\}), & \text{if } t < \alpha_x, \\ (H_{n-1}(t, x), \min\{y, t, t - \alpha_x + fH_{n-1}(t, x)\}), & \text{if } t \geq \alpha_x. \end{cases}$$

(III) If $i_n = j_n = 1$, then

$$H_n(t, x, y) = \begin{cases} (H_{n-1}(t, x), \min\{y, fH_{n-1}(t, x)\}), & \text{if } t < \alpha_x, \\ (H_{n-1}(t, x), \min\{y, t - \alpha_x + fH_{n-1}(t, x)\}), & \text{if } t \geq \alpha_x. \end{cases}$$

Note: In Cases (II) and (III), for $t \geq \alpha_x$, $H_{n-1}(t, x) = x$, since H_{n-1} is nice.

We next verify that, at the recursive step, H_n has the required properties.

Claim 3.4. H_n is a nice deformation retraction of $cl(D)$ to $cl(C')$.

Proof. We work by induction. For $n = 1$, all properties below are immediate and we omit their proofs. Let $n > 1$. We denote $g = g_n$.

(a) H_n is a map $H_n : [0, q_n] \times cl(D) \rightarrow cl(D)$.

Let $t \in [0, q_n]$ and $(x, y) \in cl(D)$. In all Cases (I), (II) and (III), the first $n - 1$ coordinates of $H_n(t, x, y)$ equal $H_{n-1}(t, x) \in cl(\pi(D))$, by induction. So in Case (I), we are done. For Cases (II) and (III), we need to check that

$$H_n(t, x, y)_n \leq gH_{n-1}(t, x),$$

In both Cases (II) and (III), if $t < \alpha_x$, we have

$$H_n(t, x, y) \leq fH_{n-1}(t, x) \leq gH_{n-1}(t, x),$$

and if $t \geq \alpha_x$, then

$$H_n(t, x, y)_n \leq y \leq g(x) = gH_{n-1}(t, x).$$

(b) H_n is a deformation retraction of $cl(D)$ to $cl(C')$.

By induction, it is easy to verify that H_n is definable and continuous. We verify the three properties of Definition 2.11.

(1) We prove that for every $(x, y) \in cl(D)$, $H_n(0, x, y) \in cl(C')$. In all Cases (I), (II) and (III), the first $n - 1$ coordinates of $H_n(0, x, y)$ equal $H_{n-1}(0, x) \in cl(\pi(C'))$, by induction. In Case (I), we are clearly done. In Cases (II) and (III), we only need to observe that

$$H_n(0, x, y)_n \leq fH_{n-1}(0, x),$$

which is clear from their definition.

(2) We prove that for every $(x, y) \in cl(C')$ and $t \in [0, q_n]$, $H_n(t, x, y) = (x, y)$. In all Cases (I), (II) and (III), the first $n - 1$ coordinates of $H_n(t, x, y)$ are $H_{n-1}(t, x) = x$, by induction. Moreover, $\alpha_x = 0$. So we only need to prove that

$$H_n(t, x, y)_n = y.$$

In Case (I), it is clear. In Case (II), $(x, y) \in cl(C')$ implies that $y = 0$, and hence $H_n(t, x, y)_n = y = 0$. In Case (III), $(x, y) \in cl(C')$ implies $y \leq f(x)$, and since $\alpha_x = 0$, we have

$$y \leq f(x) = fH_{n-1}(0, x).$$

(3) We prove that for every $(x, y) \in cl(D)$, $H_n(q_n, x, y) = (x, y)$. By induction, $H_{n-1}(q_{n-1}, x) = x$, and hence $q_n \geq q_{n-1} \geq \alpha_x$. Therefore, in all Cases (I), (II) and (III), the first $n - 1$ coordinates of $H_n(q_n, x, y)$ equal $H_{n-1}(q_n, x) = x$. So we only need to prove

$$H_n(q_n, x, y)_n = y.$$

Case (I) is clear, whereas for Cases (II) and (III), we need to show that

$$y \leq q_n \quad \text{and} \quad y \leq q_n - \alpha_x + f(x),$$

respectively. But in both cases, we have:

$$q_n \geq \sup \text{Img} \geq y,$$

and

$$q_n - \alpha_x + f(x) = q_{n-1} + \sup \text{Img} - \alpha_x + f(x) \geq \sup \text{Img} + f(x) \geq y.$$

(c) H_n is nice.

We prove that for every $(x, y) \in cl(D)$ and $t \geq \alpha_{(x,y)}$, $H_n(t, x, y) = (x, y)$. First observe that for $(x, y) \in cl(D)$, $\alpha_x \leq \alpha_{(x,y)}$. Indeed, since $H_n(\alpha_{(x,y)}, x, y) = (x, y)$, we have $H_{n-1}(\alpha_{(x,y)}, x) = x$ and hence $\alpha_x \leq \alpha_{(x,y)}$. Now, Case (I) is clear, whereas for (II) and (III), we need to prove that for $t \geq \alpha_{(x,y)}$,

$$H_n(t, x, y) = y.$$

To that end, observe that for $t \geq \alpha_{(x,y)} \geq \alpha_x$, $fH_{n-1}(t, x) = f(x)$ is fixed. Hence each of

$$t, fH_{n-1}(t, x), t - \alpha_x + fH_{n-1}(t, x)$$

is increasing for $t \geq \alpha_{(x,y)}$. So if y is smaller or equal than some of them at $t = \alpha_{(x,y)}$ then so it is for $t \geq \alpha_{(x,y)}$. By definition of Cases (II) and (III), we are done. \square

In the next two lemmas, if H is the canonical retraction of $cl(D)$ to $cl(C)$, then H_1 denotes the canonical retraction of $cl(\pi(D))$ to $cl(\pi(C))$.

Lemma 3.5. *Let D be a canonical linear cell and C one of its faces. If H is a canonical retraction of $cl(D)$ to $cl(C')$, then $H_{|[0,q] \times (C \cup D)}$ is a deformation retraction of $C \cup D$ to C' .*

Proof. The proof resembles that of Claim 3.4(a). For $n = 1$, it is immediate. Let $n > 1$, $t \in [0, q_n]$ and $(x, y) \in C \cup D$. We need to check that $H(t, x, y) \in C \cup D$. In all Cases (I), (II) and (III), the first $n - 1$ coordinates of $H(t, x, y)$ equal $H_1(t, x) \in \pi_1(C \cup D)$, by induction. So in Case (I), we are done. For Cases (II) and (III), we need to check that

$$H(t, x, y)_n < gH_1(t, x),$$

and in Case (III), we need moreover $0 < H(t, x, y)_n$. The latter is clear since in Case (III), $(x, y) \in C \cup D$ implies that both y and $fH_1(t, x)$ are positive. For the former, in both Cases (II) and (III), if $t < \alpha_x$, we have

$$H(t, x, y) \leq fH_1(t, x) \leq gH_1(t, x).$$

If the last inequality is strict, we are done. Assume $fH_1(t, x) = gH_1(t, x)$. By the definition of half-maps, this can only happen if $H_1(t, x) \in \pi(C)$ and $f_{|\pi(C)} = g_{|\pi(C)} = 0$. However, that would imply $j_n = 0$, and hence

$$H(t, x, y) = (H_1(t, x), 0) \in C.$$

If $t \geq \alpha_x$, then

$$H(t, x, y)_n \leq y \leq g(x) = gH_1(t, x),$$

Again, if the second inequality is strict, we are done. On the other hand, the equation $y = g(x)$ can only happen if $x \in \pi(C)$ and $y = 0$. But that would imply $j_n = 0$, and hence again

$$H(t, x, y) = (H_1(t, x), 0) \in C.$$

\square

Lemma 3.6. *Let $C, E, D \subseteq R^n$ be three canonical linear cells, and assume that C is a face of E , and E is a face of D . Let H and H' be the canonical retractions of $cl(E)$ and $cl(D)$ to $cl(C')$, respectively. Then H' extends H .*

Proof. For $n = 1$, this is immediate. Assume D is a (i_1, \dots, i_n) -cell and E a (j_1, \dots, j_n) -cell, $n > 1$. Let $[0, q]$ and $[0, q']$ be the parameter sets of H and H' , respectively. By induction, it is easy to see that $q' \geq q$. Let $t \in [0, q]$ and $(x, y) \in cl(E)$. We need to prove that $H(t, x, y) = H'(t, x, y)$. By induction, $H_1(t, x) = H'_1(t, x)$. Hence, we only have to show

$$H(t, x, y)_n = H'(t, x, y)_n.$$

If $j_n = 0$, then $y = 0$, and clearly $H(t, x, y)_n = 0$, by Case (I) of Definition 3.3, whereas $H'(t, x, y)_n = 0$, by Cases (I) - (III).

So let $i_n = j_n = 1$. Since H'_1 extends H_1 , the corresponding fixing points α_x and α'_x coincide, whereas by Lemma 2.10, so do the half-maps of g and g' on $cl(\pi(E))$. It follows immediately that $H(t, x, y)_n = H'(t, x, y)_n$. \square

Remark 3.7. It is possible to define a canonical retraction of $cl(D)$ to $cl(C)$, as opposed to $cl(C')$. However, Lemma 3.5 then becomes more difficult to achieve. Indeed, resembling Definition 3.3, we would first need to replace the notion of a half-map by some suitable map which equals g on $\pi(C)$. Then the resulting canonical retraction, restricted to $C \cup D$, would give a retraction of $C \cup D$ to $(0, g]_{\pi(C)}$ instead of $C = (0, g)_{\pi(C)}$. To overcome this issue, one needs to give a more elaborate definition of a canonical retraction, which we avoided doing here. We note that our canonical retraction is *not* the concatenation of two retractions, one from $cl(D)$ to $cl(D')$, and then from $cl(D')$ to $cl(C')$.

3.1. Arbitrary linear cells. We now extend the definition of canonical retractions to arbitrary bounded linear cells. The idea is simply to first map each such cell D to a canonical linear cell $T(D)$, such that if $C \subseteq cl(D)$ is another linear cell and c is a common ‘corner’ of C and D , then $T(c)$ becomes the origin, and $T(C)$ a face of $T(D)$. We then pullback the canonical retraction of $cl(T(D))$ to $cl(T(C))$, to a deformation retraction of $cl(D)$ to $cl(T^{-1}(T(C)'))$, where $T(C)'$ is the half-cell of $T(C)$.

Definition 3.8 (Corners of a linear cell). Let $D \subseteq R^n$ be a linear (i_1, \dots, i_n) -cell. We define, recursively on n , the set of *corners* of D . Let $(l_1, \dots, l_n) \leq (i_1, \dots, i_n)$.

- (1) A point $c \in R$ is an (l_1) -corner of D if
 - D is the singleton $\{c\}$, or
 - if D is an interval, and c is the left endpoint, if $l_1 = 0$, and the right endpoint, if $l_1 = 1$.
- (2) A point $c \in R^n$ is a (l_1, \dots, l_n) -corner of D if $a = \pi(c)$ is a (l_1, \dots, l_{n-1}) -corner of $A = \pi(D)$ and
 - $D = \Gamma(f)_A$ and $c = (a, f(a))$, or
 - $D = (f, g)_A$, and $c = (a, f(a))$, if $l_n = 0$, and $c = (a, g(a))$ if $l_n = 1$.

A *corner* of D is a (l_1, \dots, l_n) -corner for some (l_1, \dots, l_n) .

Observe that if $D = (f, g)_A$ with $f(a) = g(a)$, then its $(l_1, \dots, l_{n-1}, 0)$ -corner and $(l_1, \dots, l_{n-1}, 1)$ -corner coincide.

Lemma 3.9. Let \mathcal{C} be a special linear decomposition of a definable set in R^n , $C, D \subseteq \mathcal{C}$ two linear cells, and c a corner of C . If $C \cap cl(D) \neq \emptyset$, then c is also a corner of D .

Proof. By Fact 2.3, we know that $C \subseteq cl(D)$. We work by induction on n . For $n = 1$, it is immediate. For $n > 1$, let $A = \pi(D)$ and observe by induction that $a = \pi(c)$ is a corner of $A = \pi(D)$. For $D = \Gamma(f)_A$, it is then clear that $(a, f(a))$ is a corner of D . Now let $D = (f, g)_A$. Since $C \subseteq cl(D)$, we have two cases:

Case I. $C = \Gamma(h)_B \subseteq (f, g)_A$, and $B \subseteq cl(A)$. Since \mathcal{C} is special, we have $h = f|_B$ or $h = g|_B$, and hence $c = (a, f(a))$ or $c = (a, g(a))$, respectively, which are both corners of D .

Case II. $C = (k, h)_B \subseteq (f, g)_A$. Since \mathcal{C} is special, we must also have $k = f|_B$ and $h = g|_B$, and hence $c = (a, f(a))$ or $c = (a, g(a))$, which are both corners of D . \square

Definition 3.10 (Canonical transformation of a bounded linear cell). Let $D \subseteq R^n$ be a linear (i_1, \dots, i_n) -cell and c its (l_1, \dots, l_n) -corner. The *canonical transformation* $T_{D,c}$ associated to D and c is a linear map $T = T_{D,c} : cl(D) \rightarrow R^n$, defined recursively as follows.

- (1) For $n = 1$,
 - if $D = \{c\}$, then $T(c) = 0$.
 - if D is an interval, then $T(x) = |x - c|$.
- (2) For $n > 1$, let $A = \pi(D)$ and $a = \pi(c)$.
 - If $D = \Gamma(f)_A$, then $T(x, f(x)) = (T_{A,a}(x), 0)$.

- If $D = (f, g)_A$, then

$$T(x, t) = \begin{cases} (T_{A,a}(x), t - f(x)), & \text{if } l_n = 0, \\ (T_{A,a}(x), g(x) - t), & \text{if } l_n = 1. \end{cases}$$

We call $D_c = T_{D,c}(D)$ the *canonical transformation of D with respect to c* .

Remark 3.11. It is straightforward to check that:

- (1) D_c is a canonical linear (i_1, \dots, i_n) -cell.
- (2) If $cl(C) \subseteq cl(D)$ and c is a common corner of C and D , then $T_{D,c}$ agrees with $T_{C,c}$ on $cl(C)$. Moreover, $0 \in cl(C_c) \subseteq cl(D_c)$, and hence, by Claim 2.8, C_c is a face of D_c .

Definition 3.12 (Canonical retractions of bounded linear cells). Let \mathcal{C} be a special linear decomposition of some definable set, $C, D \in \mathcal{C}$ with $C \subseteq cl(D)$, and c a common corner of C and D . Let C'_c be the half-cell of C_c and $C' = T_{C,c}^{-1}(C'_c)$. We call C' the *c -half-cell* of C .

Now let $H_1 : [0, q] \times cl(D_c) \rightarrow cl(D_c)$ be the canonical retraction of $cl(D_c)$ to $cl(C'_c)$. We define the *c -canonical retraction of $cl(D)$ to C'* to be the map $H_c : [0, q] \times cl(D) \rightarrow cl(D)$, given by:

$$H_c(t, -) = T_{D,c}^{-1} \circ H_1(t, -) \circ T_{D,c}.$$

Claim 3.13. *The c -half-cell of C is definably contractible.*

Proof. By Lemma 2.16. □

Claim 3.14. *Let C, D, c, C' and H_c be as above. Then $(H_c)_{|[0,q] \times (C \cup D)}$ is a deformation retraction of $C \cup D$ to C' .*

Proof. By Lemma 3.5 and Remark 3.11(2). □

Claim 3.15. *Let $C, E, D \subseteq R^n$ be three bounded linear cells satisfying $C \subseteq cl(E) \subseteq cl(D)$, and assume that c a common corner. Let H_c and H'_c be the c -canonical retractions of $cl(E)$ and $cl(D)$ to the c -half-cell C' of C , respectively. Then H'_c extends H_c .*

Proof. By Lemma 3.6 and Remark 3.11(2). □

4. THE PROOFS OF PROPOSITIONS 1 AND 2

We begin with Proposition 1.

Proposition 4.1. *Let $Y \subseteq R^n$ be a bounded definable set, $C \subseteq Y$ a linear cell, and \mathcal{D} a special linear decomposition of Y that contains C . Assume that*

$$\forall D \in \mathcal{D}, C \cap cl(D) \neq \emptyset.$$

Let c be a corner of C , and C' the c -half-cell of C . Then Y deformation retracts to C' . In particular, Y is definably contractible.

Proof. Let $\mathcal{D} = \{D_1, \dots, D_k\}$. By Lemma 3.9, c is also a corner of each D_i . For every i , let

$$H_i^c : [0, q_i] \times cl(D_i) \rightarrow cl(D_i)$$

be the c -canonical retraction of $cl(D_i)$ to $cl(C')$. Let $q = \max_i q_i$ and define

$$H : [0, q] \times Y \rightarrow Y$$

via

$$H(t, x) = H_i^c(\min\{t, q_i\}, x), \text{ where } x \in D_i.$$

By Lemma 3.14, H is indeed a map with image in Y . Moreover, it is clear that H is a definable map that satisfies properties (1) - (3) from Definition 2.11. So we only need to prove that it is

continuous. For that, we need to check that if $D_i, D_j \in \mathcal{D}$ with $D_i \cap cl(D_j) \neq \emptyset$, then H_i^c and H_j^c agree on $D_i \cap cl(D_j)$. But by Fact 2.3, $D_i \subseteq cl(D_j)$, and hence, by Claim 3.15, H_j^c extends H_i^c .

The last clause is by Claim 3.13. \square

For the proof of Proposition 2 we will need the following lemma.

Lemma 4.2. *Let $Y \subseteq R^n$ be a definable set and \mathcal{C} a special linear decomposition of Y . Then for every box $B_1 \subseteq R^n$, there is a bigger box $B \supseteq B_1$ such that the collection*

$$\mathcal{D} = \{B \cap D : D \in \mathcal{C}\}$$

is a special linear decomposition of $B \cap Y$.

Proof. By induction on n . For $n = 1$, let $B = B_1 = (a, b)$, where $a, b \in R$. Then \mathcal{D} is a linear decomposition of $B \cap Y$ and hence is special.

Let $n > 1$, and assume that $B_1 = \pi(B_1) \times (d, e)$. By induction, there is a box $A \subseteq R^{n-1}$ containing $\pi(B_1)$ such that the collection

$$\mathcal{E} = \{A \cap E : E \in \pi(\mathcal{C})\}$$

is a special linear decomposition of $A \cap \pi(Y)$. Let \mathcal{F} be the set of all linear maps that appear in the definitions of cells in \mathcal{C} , with images in R . Now take $d', e' \in R$ such that

$$d' \text{ is smaller than } d \text{ and all } \text{Im}f|_A, \text{ for } f \in \mathcal{F},$$

$$e' \text{ is bigger than } e \text{ and all } \text{Im}f|_A, \text{ for } f \in \mathcal{F},$$

which exist since A is bounded and each $f \in \mathcal{F}$ is linear. Let $B = A \times (d, e)$. So $\mathcal{D} = \{B \cap D : D \in \mathcal{C}\}$ consists of linear cells with domain in \mathcal{E} and cell-maps already in \mathcal{F} , plus some linear cells of the form

$$(*) \quad (d', g)_V \text{ or } (f, e')_V,$$

where $f, g \in \mathcal{F}$ and $V \in \mathcal{E}$. We prove that \mathcal{D} is a special linear decomposition of $B \cap Y$. Since \mathcal{E} is special, we only need to check that:

(**) For every two cells $D_1 = \Gamma(f)_S$ and $D_2 = \Gamma(g)_T$ in \mathcal{D} , and $V \in \pi(\mathcal{D}) = \mathcal{E}$,

$$f|_V < g|_V \text{ or } f|_V = g|_V \text{ or } f|_V > g|_V.$$

(***) For every two cells $D_1 = \Gamma(h)_S$, $D_2 = (f, g)_T \in \mathcal{D}$,

$$\text{there is no } c \in cl(S) \cap cl(T) \text{ such that } f(c) < h(c) < g(c).$$

For (**): since B is open, f, g must belong to \mathcal{F} , whereas $V = A \cap V'$ for some $V' \in \pi(\mathcal{C})$. Hence we can apply Definition 2.1(2) for $f, g \in \mathcal{F}$ and V' .

For (***): again, since B is open, $h \in \mathcal{F}$. So, if $f, g \in \mathcal{F}$, we already know it. If not, then by (*), either $f = d'$ or $g = e'$, say the former. So $D_2 = (d', g)_T = (-\infty, g)_T \cap B$, where $(-\infty, g)_T \subseteq (-\infty, g)_{T'} \in \mathcal{C}$, for some $T' \in \pi(\mathcal{C})$. Also, $\Gamma(h)_S \subseteq \Gamma(h)_{S'}$, for some $S' \in \pi(\mathcal{C})$. By the choice of d' , $f' = d' < h|_S$, whereas applying Definition 2.1(2) for the cells $\Gamma(h)_{S'}$ and $(-\infty, g)_{T'}$ of \mathcal{C} , we obtain (***) \square .

We finally derive Proposition 2.

Proposition 4.3. *Let $Y \subseteq R^n$ be a definable set, $C \subseteq Y$ a linear cell, and \mathcal{D} a special linear decomposition of Y that contains C . Assume that*

$$\forall D \in \mathcal{D}, C \cap cl(D) \neq \emptyset.$$

Then Y is definably simply-connected.

Proof. Consider a definable loop $\gamma : [0, \alpha] \rightarrow Y$. Since $Im(\gamma)$ is bounded, it is contained in some box $B_1 \subseteq R^n$. By Lemma 4.2, there is a bigger box $B \supseteq B_1$ such that

$$\mathcal{D}' = \{B \cap D : D \in \mathcal{D}\}$$

is a special linear decomposition of $B \cap Y$. We now observe that $Im(\gamma) \subseteq \cup \mathcal{D}' \subseteq Y$, and

$$\forall D \in \mathcal{D}', C \cap cl(D) \neq \emptyset.$$

Therefore, by Proposition 4.1, $\cup \mathcal{D}'$ is definably contractible. Hence, by Claim 2.14, γ is definably homotopic in $\cup \mathcal{D}'$ to a constant loop. Therefore it is definably homotopic in Y to a constant loop.

It is also clear that Y is definably connected, since \mathcal{D} contains the linear cell C and for every $D \in \mathcal{D}$, $C \cap cl(D) \neq \emptyset$. \square

5. APPENDIX

Fact 2.2. *Let $Y \subseteq R^n$ be a definable set. Then for any linear decomposition \mathcal{D} of Y , there is a special linear decomposition \mathcal{C} of Y that refines \mathcal{D} (that is, every linear cell in \mathcal{D} is a union of linear cells in \mathcal{C}).*

Proof. The proof is essentially a repetition of that of [6, Lemma 2.6]. By induction on n . For $n = 1$, take $\mathcal{C} = \mathcal{D}$. Now assume that $n > 1$ and the lemma holds for $n - 1$. Let \mathcal{D} be a linear decomposition of R^n . Choose a finite collection \mathcal{F} of linear maps $f : R^{n-1} \rightarrow R$ such that any linear map that appears in the definition of any linear cell from \mathcal{D} is a restriction of a map from \mathcal{F} . Now set

$$\mathcal{G} = \{\Gamma(f) \cap \Gamma(g) : f, g \in \mathcal{F}\} \text{ and } \mathcal{G}' = \{\pi(A) : A \in \mathcal{G}\} \cup \pi(\mathcal{D}).$$

Clearly, \mathcal{G}' is a finite collection of definable subsets of R^{n-1} . By the linear cell decomposition theorem and induction, there is a special linear decomposition \mathcal{C}' of R^{n-1} that partitions each member of \mathcal{G}' .

Claim. For any $f, g \in \mathcal{F}$, either $f < g$ or $f = g$ or $f > g$ on any $V \in \mathcal{C}'$.

Proof of Claim. Let $V \in \mathcal{C}'$ and let $A = \Gamma(f) \cap \Gamma(g)$. Since $\pi(A)$ is a union of members of \mathcal{C}' , we have either $V \subseteq \pi(A)$ or $V \cap \pi(A) = \emptyset$. In the first case $f = g$ on V . In the second case, V is a disjoint union of the open definable subsets $\{b \in V : f(b) < g(b)\}$ and $\{b \in V : g(b) < f(b)\}$. Since V is definably connected, one of the two sets is equal to V . \square

Let \mathcal{C} be the linear decomposition of R^n with $\pi(\mathcal{C}) = \mathcal{C}'$ such that for any $V \in \mathcal{C}'$ the set of cells in \mathcal{C} with domain V is defined by all functions from \mathcal{F} . Since \mathcal{C}' refines $\pi(\mathcal{D})$, the choice of \mathcal{F} and Claim imply that \mathcal{C} refines \mathcal{D} .

To conclude, we need to prove (2) and (3) from Definition 2.1. Item (2), it simply holds by the Claim. For (3), let $(f, g)_T \in \mathcal{C}$. Then $f, g \in \mathcal{F}$ and for any $h \in \mathcal{F}$, again from Claim, we have on T either $h < f$, or $h = f$, or $h = g$ or $h > g$, and so either $h(c) \leq f(c)$ or $g(c) \leq h(c)$, for any $c \in cl(T)$. In particular, for any $\Gamma(h)_S \in \mathcal{C}$ there is no $c \in cl(S) \cap cl(T)$ such that $f(c) < h(c) < g(c)$. \square

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