THE DESCENDING CHAIN CONDITION FOR GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

PANTELIS E. ELEFTHERIOU

Our goal is to show the Descending Chain Condition for groups definable in o-minimal structures. Let $\mathcal{M}$ be an o-minimal structure, and $G = \langle G, \cdot, e_G \rangle$ a group definable in $\mathcal{M}$.

Theorem 0.1 (DCC)**.** *Let G be a definable group. Then there is no infinite proper descending chain of definable subgroups of G:*

$$G = H_0 \gneqq H_1 \gneqq H_2 \gneqq \dots$$

The DCC also holds if G is an ω-stable group. The proof in that case is based on the following two properties of the Morley rank and Morley degree, which are always defined and are ordinal valued. Let H be a definable subgroup of G. Then:

(1) $RM(H) = RM(G) \Leftrightarrow [G : H] < \omega$, and
(2) $deg(G) = [G : H]deg(H)$.

Thus, if H_{n+1} is a proper definable subgroup of H_n, then its Morley rank is smaller than that of H_n or its Morley degree is smaller than that of H_n. Since both Morley rank and degree are ordinals, it follows that G has no infinite proper descending chain of definable subgroups.

In case of a group G definable in an o-minimal structure $\mathcal{M}$, we can replace Morley rank by dimension and restate Property (1). But we have to rephrase Property (2), as there is no analogue of Morley degree for definable sets in o-minimal structures.

Proposition 0.2. *Let H be a definable subgroup of G. Then:*

$$\dim H = \dim G \Leftrightarrow [G : H] < \omega.$$

Proposition 0.3. *G contains a smallest definable subgroup G_0 of finite index.*

Proof of DCC based on Propositions 0.2 and 0.3. Assume, towards a contradiction, that

$$G = H_0 \gneqq H_1 \gneqq H_2 \gneqq \dots$$

is a proper descending chain of definable subgroups of G. Since $\forall i, \dim H_{i+1} \leq \dim H_i$, there is some $i \in \mathbb{N}$, such that $\forall k \geq i$, $\dim H_i = \dim H_k$. But then the sequence of $[H_i : H_k]$, for $k \geq i$, is a strictly increasing sequence of natural numbers bounded by $[H_i : (H_i)_0]$, a contradiction. □

In what follows, $\mathcal{M}$ is a sufficiently saturated o-minimal structure, and G a $\emptyset$-definable group. All topological notions concerning subsets of G are taken with respect to the G-topology, unless stated otherwise.

Date: October 2008.
DRAFT lecture notes for the Seminário de Teoria dos Modelos, Universidade de Lisboa.

We are going to prove Propositions 0.2 and 0.3 below. In particular, we define:

$$G_0 = \text{ the definably connected component of } e_G.$$

To prove Proposition 0.2, we need to recall some basic properties of the algebraic closure operator *acl*.

1. Pregeometric theories - an interlude

Definition 1.1. A (finitary) *pregeometry* is a pair (S, cl), where S is a set and $cl : P(S) \to P(S)$ is a *closure operator* satisfying, for all $A, B \subseteq S$ and $a, b \in S$:
(i) $A \subseteq cl(A)$
(ii) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
(iii) $cl\big(cl(A)\big) = cl(A)$
(iv) $cl(A) = \{cl(B) : B \subseteq A \text{ finite}\}$
(v) (Exchange) $a \in cl(bA) \setminus cl(A) \Rightarrow b \in cl(aA)$.

Definition 1.2. If $\mathcal{M}$ is a structure, the *algebraic closure* operator $acl : P(M) \to P(M)$ is defined as:

$$acl(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and } \phi(x, \bar{y}), |\phi(\mathcal{M}, \bar{b})| < \omega \,\&\, \mathcal{M} \vDash \phi(a, \bar{b})\}.$$

A complete theory T is called *pregeometric* if for every model $\mathcal{M} \vDash T$, $(\mathcal{M}, acl)$ is a pregeometry.

Lemma 1.3. *(i) For any structure $\mathcal{M}$, (M, acl) satisfies 1.1(i)-(iv).*
(ii) If T is o-minimal or strongly minimal, then T is a pregeometric theory.

Proof. (i) Easy.
(ii) See [Mac, p.102] and [Hart, p.134], respectively. □

Now let $\mathcal{M}$ be our fixed o-minimal structure.

Definition 1.4. Let $A, B \subseteq M$. We say that B is *A-independent* if for all $b \in B$, $b \notin acl\big(A \cup (B \setminus \{b\})\big)$. A maximal A-independent subset of B is called *a basis for B over A*.

By the Exchange property in a pregeometric theory, any two bases for B over A have the same cardinality. This allows to define the *algebraic dimension*:

$$dim(B/A) = \text{ the cardinality of any basis of } B \text{ over } A.$$

In particular, the dimension of tuples in M satisfies several nice properties, among which we distinguish the following.

Lemma 1.5. *For all $\bar{b}, \bar{c}, A, B \subseteq M$:*
(i) Additivity: $\dim(\bar{b}\bar{c}/A) = \dim(\bar{b}/\bar{c}A) + \dim(\bar{c}/A)$.
(iii) Anti-reflexivity: $\dim(\bar{b}/A) = 0 \Rightarrow \bar{b} \in acl(A)$.

Definition 1.6. Let p be a partial type over $A \subset M$. Then,

$$\dim(p) := \max\{\dim(\bar{c}/A) : \bar{c} \subset M \text{ satisfies } p\}.$$

It can be checked that the above notion is well-defined. The dimension of a definable set is then defined to be the dimension of its defining formula. It can be shown that this dimension coincides with the dimension of definable sets given in [El].

Let $A \subseteq M$. For $a \in X$ and $X \subseteq M^n$ A-definable, we have:

$$\dim(a/A) = \dim(X) \text{ if and only if } a \text{ is generic in } X \text{ over } A.$$

2. The proof of Proposition 0.2

Proof of Proposition 0.2. The right-to-left direction is straightforward, since cosets of H are bijective and thus have the same dimension.

Consider the following definable equivalence relation E on G:

$$g_1 \, E \, g_2 \Leftrightarrow g_1 \in g_2 H.$$

By [Ed, Theorem 7.2], then there is a definable map $\alpha : G \to G$, such that $\alpha(x) = \alpha(y) \Leftrightarrow xEy$. Assume $[G : H]$ is infinite. By o-minimality, the $\dim \big(Im(\alpha)\big) > 0$. Hence there is $a \in Im(\alpha)$ such that $\dim(a) > 0$. Let h be generic in H over a, that is $\dim(h/a) = \dim(H)$. Let $g = ha$. Then $a \in acl(g)$ (since $\alpha(g) = a$). But then also $h \in acl(g, a) = acl(g)$. It follows that:

$$\dim(g) \geq \dim(h, a) = \dim(h/a) + \dim(a) > \dim(H).$$

So $\dim(G) > \dim(H)$, a contradiction. □

3. The proof of Proposition 0.3

Remark 3.1. If $H \leqslant G$ is a definable subgroup of G, then by uniqueness of the definable manifold group topology, the H-topology coincides with the topology on H induced by the G-topology of G. (One has to check that the latter is indeed a definable manifold group topology on H.)

The next two claims are general facts that hold in all topological groups.

Claim 3.2. *If $H \leqslant G$ is a definable subgroup of G, then H is closed in G.*

Proof. Let $\overline{H}$ denote the closure of H in G.

Subclaim 1. $H \leqslant G \Rightarrow \overline{H} \leqslant G$.
Indeed, let $a, b \in \overline{H}$, and let W be an open subset of G containing ab^{-1}. By continuity of $\cdot$, there are open $U, V \subseteq G$ such that $\forall x \in U, \forall y \in V$, $xy^{-1} \in W$. In particular, if $x, y \in H$ then $xy^{-1} \in W \cap H$. Thus, $ab^{-1} \in \overline{H}$.

Subclaim 2. *H is open in G $\Leftrightarrow$ H has non-empty interior in G.*
The left-to-right direction is trivial. For the right-to-left, let $g \in H$ and v any point in the interior of H in G. Then there is some open neighborhood $V \subseteq G$ of v in H. But then $gv^{-1}V$ is an open neighborhood of g in H.

Subclaim 3. *H is open in G $\Rightarrow$ H is closed in G.*
Indeed, $G \setminus H = \bigcup_{g \in G \setminus H} gH$.

Now let $H \leqslant G$ be a definable subgroup of G. By Subclaim 1, $\overline{H} \leqslant G$. By [vdD, Chapter 4, Corollary (1.9)], [El, Lemma 1.5 (ii)] and the remark preceding it, H has a non-empty interior in $\overline{H}$. By Subclaim 2, applied to H and $\overline{H}$, H is open in $\overline{H}$. By Subclaim 3, H is closed in $\overline{H}$. Hence, by Remark 3.1, $H = \overline{H}$ is closed. □

Claim 3.3. $[G : H] < \omega \Rightarrow H$ *is open. (In fact, $\Leftrightarrow$ holds.)*

Proof. Indeed, $G \setminus H = \bigcup_{g \in G \setminus H} gH$. □

Proof of Proposition 0.3. G_0 is a subgroup of G: let $a \in G_0$. Since the map $x \mapsto x^{-1}$ is continuous, G_0^{-1} is also definably connected. Since $G_0 \cap G_0^{-1} \neq \emptyset$ (it contains e_G), it follows that $G_0 \cup G_0^{-1}$ is definably connected, and thus equal to G_0. Hence, $a^{-1} \in G_0 \cup G_0^{-1} = G_0$.

Now let $a, b \in G_0$. Since the map $x \mapsto ax$ is continuous, aG_0 is definably connected. Since $a^{-1} \in G_0$, $G_0 \cap aG_0 \neq \emptyset$, it follows that $G_0 \cup aG_0$ is definably connected, and thus equal to G_0. Hence, $ab \in G_0 \cup aG_0 = G_0$.

G_0 has finite index: G_0 is open, and therefore $\dim(G_0) = \dim(G)$. Then apply Proposition 0.2.

G_0 is the smallest definable subgroup of G of finite index: Let $H \leqslant G$ be a definable subgroup of G of finite index. Then $H \cap G_0$ has finite index in G_0. Hence, $H \cap G_0$ is a closed and open in G_0 (in the G_0-topology). By Remark 3.1, $H \cap G_0$ is closed and open in G_0 in the topology induced on G_0 by the G-topology of G. Since G_0 is definably connected in that topology, $H \cap G_0 = G_0$. □

References

[vdD] L. van den Dries, Tame topology and o-minimal structures, Cambridge University Press, Cambridge, 1998.

[Ed] M.Edmundo, *Solvable groups definable in o-minimal structures*, J. Pure Appl. Algebra 185 (2003), 103-145.

[El] P. Eleftheriou, *Groups definable in o-minimal structures*, Lecture notes (2008).

[Hart] B. Hart, *Stability theory and its variants*, pp. 131-149 in Model theory, algebra, and geometry, edited by D. Haskell, A. Pillay and C. Steinhorn, Math. Sci. Res. Inst. Publ. 39, Cambridge Univ. Press, New York, 2000.

[Mac] D. Macpherson, *Notes on o-minimality and variations*, pp. 97-130 in Model theory, algebra, and geometry, edited by D. Haskell, A. Pillay and C. Steinhorn, Math. Sci. Res. Inst. Publ. 39, Cambridge Univ. Press, New York, 2000.

[Pi] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra 53 (1988), 239-255.

CMAF, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal
E-mail address: pelefthe@ptmat.fc.ul.pt