COVERINGS BY OPEN CELLS

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Abstract. We prove that in a semi-bounded o-minimal expansion of an ordered group every non-empty open definable set is a finite union of open cells.

1. Introduction

We fix an arbitrary o-minimal expansion $\mathcal{R} = \langle R, <, +, 0, \ldots \rangle$ of an ordered group. In this note we complete the proof of the following theorem.

Theorem 1.1. *Every non-empty open definable set is a finite union of open cells.*

This theorem is already known in two extreme cases. Let us explain the context. By [12] and [3], we can have exactly the following cases:

A. $\mathcal{R} = \langle R, <, +, \cdot, 0, 1, \ldots \rangle$ expands a real closed field.
B. $\mathcal{R}$ does not expand a real closed field, but it contains a definable real closed field whose domain is a bounded interval $I \subseteq R$.
C. No real closed field is definable.

A structure from cases (B) and (C) is called *semi-bounded*. In particular, a structure from case (C), is called *linear*. A typical example of a linear structure is that of an ordered vector space $\mathcal{V} = \langle V, <, +, 0, \{d\}_{\lambda \in D} \rangle$ over an ordered division ring D.

An important example of a semi-bounded structure is the expansion $\mathcal{B}$ of the real ordered vector space $\mathbb{R}_{vect} = \langle \mathbb{R}, <, +, 0, \{d\}_{d \in \mathbb{R}} \rangle$ by all bounded semi-algebraic sets. Every bounded interval in $\mathcal{B}$ admits the structure of a definable real closed field. For example, the field structure on $(-1, 1)$ induced from $\mathbb{R}$ via the semi-algebraic bijection $x \mapsto \frac{x}{\sqrt{1+x^2}}$ is definable in $\mathcal{B}$. By [13, 9, 10], $\mathcal{B}$ is the unique structure that lies strictly between $\mathbb{R}_{vect}$ and the real field. The situation becomes significantly more subtle when $\mathcal{R}$ is non-archimedean, and the study of definable sets and groups in the general semi-bounded setting has recently regained a lot of interest ([4, 6, 7, 11]).

Theorem 1.1 in the field case was proved by Wilkie in [14], and in the linear case by Andrews in [1]. Here we prove it in the semi-bounded non-linear case. We also prove a stronger result in the linear case, which we state next. For the notion of 'linear decomposition' and 'star', see Section 2 below. For the notion of 'stratification', see [2, Chapter 4, (1.11)]. By Lemma 2.6, Corollary 2.11 and Proposition 2.13 below, we have:

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Theorem 1.2. *Assume $\mathcal{R}$ is linear. Let $\mathcal{D}$ be a linear decomposition of R^n. Then there is decomposition $\mathcal{C}$ of R^n that refines $\mathcal{D}$, such that for every $C \in \mathcal{C}$, the star of C is an open (usual) cell. Moreover, $\mathcal{C}$ is a stratification of R^n.*

We expect that our main theorem on coverings by open cells (Theorem 1.1) will find numerous applications in the theory of locally definable manifolds in o-minimal structures. Some of those are exhibited in [5]. As stated in that reference, a strengthened result of coverings would yield further applications. We state the desired result here as a Conjecture:

Conjecture. Every definable set is a finite union of relatively open definable subsets which are definably simply connected.

Structure of the paper. Section 2 contains the stratification result (Theorem 1.2) for the linear case. Section 3 contains the covering by open cells (Theorem 1.1) for the semi-bounded non-linear case.

*N*otation. We recall the standard notation for graphs and "generalized cylinders" of definable maps.

- If $f : X \to R$ is a definable map, we denote by $\Gamma(f)$ the graph of f.
- If $f, g : X \to R$ are definable maps or the constant maps $-\infty$ and $+\infty$ on X with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and set:
  $(f, g)_X = \{(x, y) \in X \times R : f(x) < y < g(x)\}$;
  $[f, g)_X = \{(x, y) \in X \times R : f(x) \le y < g(x)\}$;
  $(f, g]_X = \{(x, y) \in X \times R : f(x) < y \le g(x)\}$;
  $[f, g]_X = \{(x, y) \in X \times R : f(x) \le y \le g(x)\}$.

2. The linear case

We assume in this section that $\mathcal{R}$ is linear. By ([8]), there is an elementary extension of $\mathcal{R}$ which is a reduct of an ordered vector space $\mathcal{V} = \langle V, <, +, 0, \{\lambda\}_{\lambda\in D}\rangle$ over an ordered division ring D. We may thus assume that $\mathcal{R} = \langle R, <, 0, +, \{\lambda\}_{\lambda\in D}\rangle$ is an ordered vector space over an ordered division ring D.

As mentioned in the Introduction, we will prove a 'special linear cell decomposition theorem' in which the 'star' of every cell is an open usual cell.

A *linear (affine) function* on $A \subseteq R^n$ is a function $f : A \to R$ of the form $f(x_1, \dots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n + a$, for some fixed $\lambda_i \in D$ and $a \in R$. For a definable set $X \subseteq R^n$, we set $L(X) = \{f : X \to R : f \text{ is linear}\}$ and $L_\infty(X) = L(X) \cup \{\pm\infty\}$, where we regard $-\infty$ and $+\infty$ as constant functions on X. If $f \in L(X)$, we denote by $\Gamma(f)$ the graph of f. If $f, g \in L_\infty(X)$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and set $(f, g)_X = \{(x, y) \in X \times R : f(x) < y < g(x)\}$. Then,

- a *linear cell in R* is either a singleton subset of R, or an open interval with endpoints in $R \cup \{\pm\infty\}$,
- a *linear cell in R^{n+1}* is a set of the form $\Gamma(f)$, for some $f \in L(X)$, or $(f, g)_X$, for some $f, g \in L_\infty(X)$, $f < g$, where X is a linear cell in R^n.

In either case, X is called *the domain* of the defined cell.

We refer the reader to [2, Chapter 3, (2.10)] for the definition of a *decomposition* of R^n. A *linear decomposition of R^n* is then a decomposition $\mathcal{C}$ of R^n such that each $B \in \mathcal{C}$ is a linear cell. The following can be proved similarly to [2, Chapter 3, (2.11)].

Theorem 2.1 (Linear CDT)**.**

(1) *Given any definable sets* $A_1, \dots, A_k \subseteq R^n$*, there is a linear decomposition* $\mathcal{C}$ *of* R^n *that partitions each* A_i*.*
(2) *Given a definable function* $f : A \to R$*, there is a linear decomposition* $\mathcal{C}$ *of* R^n *that partitions* A *such that the restriction* $f_{|B}$ *to each* $B \in \mathcal{C}$ *with* $B \subseteq A$ *is linear.*

Definition 2.2. Let $\mathcal{C}$ be a linear decomposition of R^n and X a definable subset of R^n. Denote

$$\mathrm{Star}_{\mathcal{C}}(X) = \{D \in \mathcal{C} : X \cap cl(D) \neq \emptyset\}.$$

The *star of* X *with respect to* $\mathcal{C}$, denoted by $\mathrm{st}_{\mathcal{C}}(X)$, is then

$$\mathrm{st}_{\mathcal{C}}(X) = \bigcup \mathrm{Star}_{\mathcal{C}}(X).$$

We just write $\mathrm{Star}(X)$ and $\mathrm{st}(X)$ if $\mathcal{C}$ is fixed.

In what follows, if $k > 0$, then $\pi : R^{k+1} \to R^k$ denotes the usual projection map onto the first k-coordenates, and if $\mathcal{C}$ is a linear decomposition of R^{k+1}, then $\pi(\mathcal{C})$ denotes the linear decomposition $\{\pi(C) : C \in \mathcal{C}\}$ of R^k.

Lemma 2.3. *Let* $\mathcal{C}$ *be a linear decomposition of* R^n *compatible with a definable subset* X *of* R^n*. Then:*
(i) If $n > 1$*, then* $\mathrm{Star}_{\pi(\mathcal{C})}(\pi(X)) = \pi(\mathrm{Star}_{\mathcal{C}}(X))$*.*
(ii) If X *is an open union of cells in* $\mathcal{C}$*, and* $C \in \mathcal{C}$ *with* $C \subseteq X$*, then* $\mathrm{st}(C) \subseteq X$*.*

Proof. (i) $\subseteq$. Let $D \in \mathrm{Star}(\pi(X))$. Since π is open, for any open set U containing X, $\pi(U)$ is an open set containing $\pi(X)$. Thus $D \cap \pi(U) \neq \emptyset$, which implies $\pi^{-1}(D) \cap U \neq \emptyset$. Hence, by Linear CDT, there is some $D' \in \mathrm{Star}(X)$ such that $\pi(D') = D$.

$\supseteq$. Let $D \in \mathrm{Star}(X)$. For any open set U containing $\pi(X)$, $\pi^{-1}(U)$ is an open neighborhood of X. Therefore $\pi^{-1}(U) \cap D \neq \emptyset$, and $U \cap \pi(D) \neq \emptyset$. Hence $\pi(D)$ belongs to $\mathrm{Star}(\pi(X))$.

(ii) Since X is open, for every $B \in \mathrm{Star}(C)$, $B \cap X \neq \emptyset$, and hence $B \subseteq X$. □

One would expect that $\mathrm{st}_{\mathcal{C}}(X)$ is an open set. However, the following example shows that this is not the case.

Example 2.4. Consider points $a_{-1} < a_0 < a_1 < a_2 < a_3$ in R and let $\mathcal{C}$ be a linear decomposition of R^2 that contains the following cells: $(a_{-1}, a_0) \times (a_0, a_2)$, $(a_0, a_1) \times (a_0, a_2)$, $\{a_0\} \times (a_0, a_1)$, $\{a_0\} \times (a_1, a_3)$ and the point (a_0, a_1). Then the star of the point (a_0, a_1) is the union of the above cells, which is not open.

Below we define a special kind of a linear decomposition $\mathcal{C}$ of R^n that remedies the above problem. In fact, such a $\mathcal{C}$ will give us that every $\mathrm{st}_{\mathcal{C}}(X)$ is an open (usual) cell (see Proposition 2.13 below). From this we obtain the version of Theorem 1.1 for the linear case (see Corollary 2.14 below).

For every $h \in L(X)$, where $X \subseteq R^k$, and h of the form $h(x_1, \dots, x_k) = \lambda_1 x_1 + \dots + \lambda_k x_k + c$, we define *the extension of* h *to* R^k to be the linear function $g : R^k \to R$ with $g(x_1, \dots, x_k) = \lambda_1 x_1 + \dots + \lambda_k x_k + c$. We say that g *extends* h. In what follows, if $h \in L(X)$, $X \subseteq R^k$, and $c \in cl(X)$, we denote $h(c) := \lim_{t \to c} h(t)$, which always exists and is equal to $g(c)$, where g extends h.

The next definition is by induction on n.

Definition 2.5. A *special linear decomposition of* R is any linear decomposition of R. A *special linear decomposition of* R^{k+1}*,* $k>0$*,* is a linear decomposition $\mathcal{C}$ of R^{k+1} with the following two properties:

- Let $C, C' \in \mathcal{C}$ be two linear cells of the form
$$C=(f,g)_B \text{ and } C'=(f',g')_{B'},$$
where $B, B' \subseteq R^k$ are disjoint, $f<g$ in $L_\infty(B)$ and $f'<g'$ in $L_\infty(B')$. Then, for every $c \in cl(B)\cap cl(B')$, if $\pi^{-1}(c)\cap cl(C)\cap cl(C')$ is infinite, then
$$\pi^{-1}(c)\cap cl(C)=\pi^{-1}(c)\cap cl(C').$$
Equivalently, for every $c \in cl(B)\cap cl(B')$, if $\pi^{-1}(c)\cap cl(C)\cap cl(C')$ is infinite, then
$$f(c)=f'(c) \text{ and } g(c)=g'(c).$$
- $\pi(\mathcal{C})=\{\pi(D): D\in\mathcal{C}\}$ is a special linear decomposition of R^k.

Before providing the nice consequences of special linear decompositions, we prove that they always exist.

Lemma 2.6. *For any linear decomposition* $\mathcal{D}$ *of* R^n*, there is a special linear decomposition* $\mathcal{C}$ *of* R^n *that refines* $\mathcal{D}$ *(that is, every linear cell in* $\mathcal{D}$ *is a union of linear cells in* $\mathcal{C}$*).*

Proof. By induction on n. For $n=1$, take $\mathcal{C}=\mathcal{D}$. Now assume that $n=k+1$ and the lemma holds for $k>0$. Let $\mathcal{D}$ be a linear decomposition of R^{k+1}. Let $\mathcal{F}$ be the collection of linear maps that appear in the definitions of the linear cells that are contained in $\mathcal{D}$. Now let
$$\overline{\mathcal{F}}=\{g: R^k\to R : g \text{ extends some } f\in\mathcal{F}\}$$
and set
$$\mathcal{G}=\{\Gamma(f)\cap\Gamma(g): f,g\in\overline{\mathcal{F}}\} \text{ and } \mathcal{G}'=\{\pi(A): A\in\mathcal{G}\}.$$
Clearly, $\mathcal{G}'$ is a finite collection of definable subsets of R^k. By (Linear CDT and) the inductive hypothesis, there is a special linear decomposition $\mathcal{C}'$ of R^k that partitions each $B\in\mathcal{G}'$.

Claim 2.7. *For every two distinct* $f,g\in\overline{\mathcal{F}}$*, and* $X\in\mathcal{C}'$*,*
$$f_{|X}<g_{|X} \text{ or } f_{|X}=g_{|X} \text{ or } f_{|X}>g_{|X}.$$

Proof. Indeed, let $A=\Gamma(f)\cap\Gamma(g)\neq\emptyset$. Since $\pi(A)\in\mathcal{G}'$ and $\mathcal{C}'$ partitions $\pi(A)$, we have either $X\subseteq\pi(A)$ or $X\subseteq R^k\setminus\pi(A)$. The former implies $f_{|X}=g_{|X}$, whereas the latter implies one of the two other cases. □

We can thus write $\mathcal{C}'=\{X_1,\dots,X_m\}$, such that for each $i\in\{1,\dots,m\}$, $f_{i1}<\cdots<f_{in(i)}$ are the distinct functions in $L(X_i)$, each being a restriction of some $f\in\overline{\mathcal{F}}$, and exhausting all possible such. Then
$$\mathcal{C}_i=\{(-\infty,f_{1|X_i}),(f_{1|X_i},f_{2|X_i}),\dots,(f_{l|X_i},\infty),\Gamma(f_{1|X_i}),\dots,\Gamma(f_{l|X_i})\}$$
is a partition of $\pi^{-1}(X_i)$, and $\mathcal{C}=\mathcal{C}_1\cup\cdots\cup\mathcal{C}_k$ is a linear decomposition of R^{k+1} which refines $\mathcal{D}$. We show that $\mathcal{C}$ is special. Let $C=(f,g)_B$ and $C'=(f',g')_{B'}$ be as in Definition 2.5. We need to check that
$$f(c)=f'(c) \text{ and } g(c)=g'(c)$$

for every $c \in cl(B) \cap cl(B')$. If not, then since $\pi^{-1}(c) \cap cl(C) \cap cl(C')$ is infinite, we have either

$$f(c) < f'(c) < g(c) \text{ or } f'(c) < g(c) < g'(c).$$

In the first case, the extension h of f' restricted to B satisfies:

$$f < h_{|B} < g.$$

This contradicts the definition of $\mathcal{C}_i$ with i such that $X_i = B$. In the second case we also get a contradiction by considering i such that $X_i = B'$.

Finally, notice that $\pi(\mathcal{C}) = \mathcal{C}'$ is a special linear decomposition of R^k. □

We now aim towards Proposition 2.13 below. We will often use without mentioning the following result which is proved in arbitrary o-minimal expansions of ordered groups for the case of bounded cells in [2, Chapter 6, (1.7)]. It is remarked there that the boundedness assumption is necessary. However, in the linear case, it is not.

Lemma 2.8. *For any linear cell $C \subseteq R^n$, $\pi(cl(C)) = cl(\pi(C))$.*

Proof. We do the case $C = (f, g)_{\pi(C)}$, since the other case is easier. By continuity of π we have $\pi(cl(C)) \subseteq cl(\pi(C))$. Let $a \in cl(\pi(C))$ and suppose $a \notin \pi(C)$. We have to find $b \in cl(C)$ such that $\pi(b) = a$. By curve selection [2, Chapter 6, (1.5)] there is a continuous definable map $\gamma : (0, \epsilon) \to \pi(C)$ such $\gamma(0) := \lim_{t\to 0^+} \gamma(t) = a$. Now define a continuous definable function $\lambda : (0, \epsilon) \to R$ by

$$\lambda(t) = \begin{cases} \big(f(\gamma(t)) + g(\gamma(t))\big)/2 & \text{if } f, g \in L(\pi(C)), \\ f(\gamma(t)) + 1 & \text{if } f \in L(\pi(C)), g = +\infty, \\ g(\gamma(t)) - 1 & \text{if } f = -\infty, g \in L(\pi(C)), \\ 0 & \text{if } f = -\infty, g = +\infty. \end{cases}$$

This gives us continuous definable function $\beta : (0, \epsilon) \to C, t \mapsto (\gamma(t), \lambda(t))$. As we pointed out before, $\beta(0) := \lim_{t\to 0^+} \beta(t)$ exists and is in $\pi(C)$. Since $\pi(\beta(0)) = \gamma(0) = a$, we are done. □

Lemma 2.9. *Let $\mathcal{C}$ be a special linear decomposition of R^n, $n > 1$, $D, E \in \mathcal{C}$ two linear cells of the form*

$$D = \Gamma(f) \text{ and } E = \Gamma(g),$$

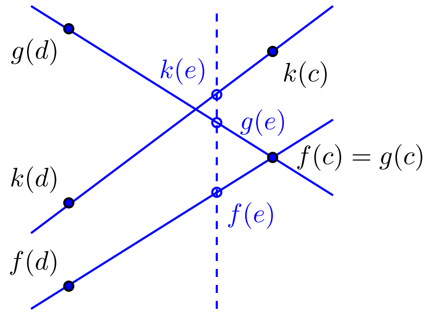
where $f \in L(B)$, $g \in L(B')$, and $A = cl(B) \cap cl(B') \neq \emptyset$. Then:

$$f_{|A} < g_{|A} \text{ or } f_{|A} = g_{|A} \text{ or } f_{|A} > g_{|A}.$$

Proof. Assume not. Then there are points $c, d \in A$, such that $f(c) = g(c)$ and $f(d) \neq g(d)$. Say, $f(d) < g(d)$. Let $F, G \in \mathcal{C}$ be special linear cells of the form $F = (h, k)_B$, $G = (l, m)_{B'}$ such that

$$f_{|A} = h_{|A} < k_{|A} \text{ and } g_{|A} = l_{|A} < m_{|A}.$$

We next claim that there is a point $e \in A$, such that $f(e) < g(e) < k(e)$.



If $g(d) < k(d)$, then let $e = d$. So assume $k(d) \leq g(d)$. We will choose e to be "between" c and d. We first see that there is $q_0 \in (0,1] \cap \mathbb{Q}$, such that

$$q_0 g(d) + (1 - q_0) g(c) < q_0 k(d) + (1 - q_0) k(c)$$

Indeed, if not, then $k(c) \leq g(c)$. But $g(c) = f(c) < k(c)$, a contradiction. On the other hand, since $f(d) < g(d)$ and $f(c) = g(c)$, we have that for every $q \in (0,1] \cap \mathbb{Q}$,

$$qf(d) + (1 - q)f(c) < qg(d) + (1 - q)g(c).$$

Hence, if we let $e = q_0 d + (1 - q_0)c$, we have $f(e) < g(e) < k(e)$, proving our claim.

Now, since $f(e) = h(e)$ and $g(e) = l(e)$, we have $h(e) < l(e) < k(e)$. This implies that $\pi^{-1}(e) \cap cl(F) \cap cl(G)$ is infinite, but $\pi^{-1}(e) \cap cl(F) \neq \pi^{-1}(e) \cap cl(G)$, contradicting the fact that $\mathcal{C}$ is special. $\square$

Lemma 2.10. *Let $\mathcal{C}$ be a special linear decomposition of R^n, $n > 1$, and $D, E \in \mathcal{C}$ such that $D \cap cl(E) \neq \emptyset$. Then:*

$$\pi(D) \subseteq cl(\pi(E)) \Rightarrow D \subseteq cl(E).$$

Proof. The statement trivially holds if $D = E$, hence assume $D \neq E$. Let $E = (f, g)_B$ or $E = \Gamma(f)$, for some $f, g \in L_\infty(B)$. If D has domain B, then $E = (f, g)_B$, and $D = \Gamma(f)$ or $D = \Gamma(g)$. Hence, $D \subseteq cl(E)$. So now assume that D has domain $B' = \pi(D)$, disjoint from $B = \pi(E)$, and, for a contradiction, that $B' \subseteq cl(B)$ but $D \not\subseteq cl(E)$. Let $A = cl(B) \cap cl(B') \neq \emptyset$.

Case A: $D = \Gamma(g')$, for some $g' \in L(B')$. Then one of the pairs f, g' or g, g' must contradict Lemma 2.9.

Case B: $D = (f', g')_{B'}$, for some $f', g' \in L_\infty(B)$. Then, again by Lemma 2.9, applied to each of the four pairs $\{f, f'\}$, $\{f, g'\}$, $\{g, f'\}$, $\{g, g'\}$ that are involved, the only remaining possibilities are the following:

$$f'_{|A} < g_{|A} < g'_{|A} \text{ or } f'_{|A} < f_{|A} < g'_{|A}.$$

In the first case, let $F \in \mathcal{C}$ be a linear cell of the form $F = (h, k)_B$, such that

$$g_{|A} = h_{|A} < k_{|A}.$$

Then for any $c \in A$, $f'(c) < h(c) < g'(c)$. This implies that $\pi^{-1}(c) \cap cl(D) \cap cl(F)$ is infinite, but $\pi^{-1}(c) \cap cl(D) \neq \pi^{-1}(c) \cap cl(F)$, contradicting the fact that $\mathcal{C}$ is special. Similarly for the second case. $\square$

Corollary 2.11. *Let $\mathcal{C}$ be a special linear decomposition of R^n, $n > 0$, and $D, E \in \mathcal{C}$ such that $D \cap cl(E) \neq \emptyset$. Then $D \subseteq cl(E)$.*

In particular, $\mathcal{C}$ is a stratification of R^n.

Proof. The statement trivially holds if $D = E$, hence assume $D \neq E$. We work by induction on n. For $n = 1$, the assumption $D \cap cl(E) \neq \emptyset$ implies that E is an open interval and D is one of its endpoints. So now assume $n > 1$. Clearly, $\pi(D) \cap cl(\pi(E)) \neq \emptyset$ (using Lemma 2.8), and hence by inductive hypothesis, $\pi(D) \subseteq cl(\pi(E))$. By Lemma 2.10, $D \subseteq cl(E)$. □

Lemma 2.12. *Let $\mathcal{C}$ be a special linear decomposition of R^n, $n > 0$. Then, for any definable $X \subseteq R^n$, $\mathrm{st}(X)$ is open.*

Proof. Let $x \in \mathrm{st}(X)$, and assume that x is not in the interior of $\mathrm{st}(X)$. Then there is a linear cell $E \in \mathcal{C}$, $E \subseteq R^n \setminus \mathrm{st}(X)$, such that $x \in cl(E)$. We split cases:

Case A: $x \in X$. Then $x \in X \cap cl(E)$, contradicting the fact that $E \subseteq R^n \setminus \mathrm{st}(X)$.

Case B: $x \in D$, for some linear cell $D \in \mathcal{C}$ such that $X \cap cl(D) \neq \emptyset$. Hence $x \in D \cap cl(E) \neq \emptyset$, and thus by Corollary 2.11, $D \subseteq cl(E)$. Therefore $cl(D) \subseteq cl(E)$, and hence $X \cap cl(E) \neq \emptyset$, contradicting the fact that $E \subseteq R^n \setminus \mathrm{st}(X)$. □

Proposition 2.13. *Let $\mathcal{C}$ be a special linear decomposition of R^n, $n > 0$, and $C \in \mathcal{C}$. Then $U = \mathrm{st}(C)$ is an open (usual) cell.*

Proof. By Lemma 2.12, U is open and thus has dimension n.

Let $\dim(C) = k \leq n$. If $k = n$, then $U = C$ and the statement holds trivially. We may thus assume that $k < n$. We work by induction on n. If $n = 1$, then C is a point and U is an open interval. Now assume that $n = k + 1$ and the Claim holds for $k > 0$.

Assume first that C is a linear cell in $\mathcal{C}$ of dimension k which is the graph of a linear function $h : D \to R$. Then clearly

$$\mathrm{st}(C) = (f, g)_D,$$

for some $f, g \in L_\infty(D)$ with $f < h < g$.

In all other cases, $\dim(\pi(C)) < \dim(\pi(U))$. Since $\mathcal{C}$ is a linear decomposition, for every $B \in \mathrm{Star}(\pi(C))$, $\pi^{-1}(B) \cap U$ is a union of linear cells in $\mathcal{C}$ which are either graphs of linear maps, or cylinders between linear maps, with domain B. By Lemma 2.3(i), $U \subseteq \bigcup\{\pi^{-1}(B) : B \in \mathrm{Star}(\pi(C))\}$, and hence

$$U = \bigcup\{\pi^{-1}(B) \cap U : B \in \mathrm{Star}(\pi(C))\}.$$

Since U is open, we easily obtain that for every $B \in \mathrm{Star}(\pi(C))$,

$$\pi^{-1}(B) \cap U = (f_B, g_B)_B,$$

for some $f_B, g_B \in L_\infty(B)$ with $f_B < g_B$. Let $D = \mathrm{st}(\pi(C))$, $f = \bigcup_{B \in \mathrm{Star}(\pi(C))} f_B$ and $g = \bigcup_{B \in \mathrm{Star}(\pi(C))} g_B$. Then

$$U = (f, g)_D.$$

By inductive hypothesis, D is a usual cell. To show that f, g are continuous, we need to show that for every $B, B' \in \mathrm{Star}(\pi(C))$, and $c \in cl(B) \cap cl(B')$,

$$f_B(c) = f_{B'}(c) \text{ and } g_B(c) = g_{B'}(c).$$

Let $H = (h, g_B)_B$ be the upper-most linear cell in $\mathcal{C}$ contained in $(f_B, g_B)_B$ and $H' = (h', g_{B'})_{B'}$ the upper-most linear cell in $\mathcal{C}$ contained in $(f_{B'}, g_{B'})_{B'}$. By Corollary 2.11, $C \subseteq cl(H) \cap cl(H')$. Hence, if $C = (l, m)_A$, for some $l, m \in L(A)$, then $\pi^{-1}(c) \cap cl(H) \cap cl(H')$ is infinite. On the other hand, if $C = \Gamma(l)$ for some $l \in L(A)$, then by Lemma 2.9,

$$h_{|A} \leq l \text{ and } h'_{|A} \leq l,$$

and hence $\pi^{-1}(c) \cap cl(H) \cap cl(H')$ is again infinite. Since $\mathcal{C}$ is special,

$$h(c) = h'(c) \text{ and } g_B(c) = g_{B'}(c).$$

Similarly, we can show that $f_B(c) = f_{B'}(c)$.

It follows that $U = (f, g)_D$ is a cell. □

Corollary 2.14. *If $\mathcal{R} = (R, <, 0, +, \{\lambda\}_{\lambda \in D})$ is an ordered vector space over an ordered division ring D, then every non-empty open definable set is a finite union of open cells.*

Proof. Let $X \subseteq R^n$ be an open definable subset and take $\mathcal{C}$ a special linear decomposition of R^n that partitions X. By Lemma 2.3(ii),

$$X = \bigcup_{C \in \mathcal{C},\, C \subseteq X} \mathrm{st}(C).$$

Then apply Proposition 2.13. □

3. The semi-bounded non-linear case

We assume in this section that $\mathcal{R}$ is semi-bounded and non-linear. So there exists a definable real closed field $\langle I, 0_I, 1_I, +_I, \cdot_I, <_I \rangle$ on some interval $I \subseteq R$ which, without loss of generality, can be assumed to be of the form $I = (-e, e)$, $0_I = 0$ and $<_I$ is the restriction of $<$ to I. For further details on semi-bounded o-minimal structures we refer the reader to [3].

In this case, we use the existence of a "short" definable real closed field to adapt Wilkie's proof ([14]) in o-minimal expansions of real closed fields. We start with the following preliminary lemmas:

Lemma 3.1. *In a semi-bounded o-minimal expansion $\mathcal{R}$ of an ordered group a definable map $f : X \to R^n$ is continuous if and only if its graph $\Gamma(f)$ is closed in $X \times R^n$.*

Proof. By [2, Chapter 6, (1.15) Exercise 7], in an o-minimal expansion of an ordered group a definable map $f : X \to R^n$ is continuous if and only if its graph $\Gamma(f)$ is closed in $X \times R^n$ and f locally bounded. So it suffices to show that in a semi-bounded o-minimal expansion of an ordered group every definable map $f : X \to R^n$ is locally bounded. Locally bounded means that every $a \in X$ has an open definable neighborhood U in X such that $f(U)$ is bounded. Clearly, every $a \in X$ has a bounded open definable neighborhood U. On the other hand, by [3, Proposition 3.1 (3)], if U is bounded then $f(U)$ is bounded. □

Lemma 3.2 ([14], Lemma 1)**.** *Let C be a cell in R^n. Then there exists an open cell D in R^n with $C \subseteq D$ and a definable retraction $H : D \to C$ (i.e. a continuous map such that $H_{|C} = \mathrm{id}_C$).*

Lemma 3.3. *Let C be a cell in R^n. Suppose that $h : C \to R$ is a continuous definable map and let U be an open definable subset of R^{n+1}. Suppose further that $\Gamma(h) \subseteq U$. Then there exist definable maps $f, g : C \to R$ and cells $C_1, \ldots, C_m \subseteq C$ such that:*

(1) *$f<h<g$;*
(2) *$C=C_1\cup\cdots\cup C_m$;*
(3) *for each i, $f_{|C_i}$ and $g_{|C_i}$ are continuous;*
(4) *for each i, $\Gamma(h_{|C_i})\subseteq[f_{|C_i},g_{|C_i}]_{C_i}\subseteq U$.*

Proof. Since U is open and $\Gamma(h)\subseteq U$, by definable choice ([2, Chapter 6, (1.2)] there exists definable maps $f,g:C\to R$ such that $f<h<g$ and $[f,g]_C\subseteq U$. By cell decomposition, there are cells $C_1,\dots,C_m\subseteq C$ covering C such that for each i, $f_{|C_i}$ and $g_{|C_i}$ are continuous. Now the rest is clear. □

The following is also needed:

Lemma 3.4. *Let C be a cell in R^n. Suppose that $f,g:C\to R$ are continuous definable maps such that $f<g$ and let $V,W\subseteq U$ be open definable subsets of R^{n+1}. Suppose further that $(f,g)_C\subseteq U$, $\Gamma(f)\subseteq V$ and $\Gamma(g)\subseteq W$. Then there exist definable maps $f',g':C\to R$ and cells $C_1,\dots,C_m\subseteq C$ such that:*
(1) *$C=C_1\cup\cdots\cup C_m$;*
(2) *for each i, $f'_{|C_i}$ and $g'_{|C_i}$ are continuous;*
(3) *$f<f'<g'<g$;*
(4) *for each i, $\Gamma(f'_{|C_i})\subseteq V$ and $\Gamma(g'_{|C_i})\subseteq W$;*
(5) *for each i, $(f'_{|C_i},g_{|C_i})_{C_i}\subseteq U$, $(f_{|C_i},g'_{|C_i})_{C_i}\subseteq U$ and $[f'_{|C_i},g'_{|C_i}]_{C_i}\subseteq U$.*

Proof. Since $(f,g)_C\subseteq U$, $\Gamma(f)\subseteq V$ and $\Gamma(g)\subseteq W$ and $V,W\subseteq U$ be open definable subsets of R^{n+1}, by definable choice ([2, Chapter 6, (1.2)] there exists definable maps $f',g':C\to R$ such that
(1) $f<f'<g'<g$;
(2) $\Gamma(f')\subseteq V$ and $\Gamma(g')\subseteq W$;
(3) $(f',g)_C\subseteq U$, $(f,g')_C\subseteq U$ and $[f',g']_C\subseteq U$.

By cell decomposition, there are cells $C_1,\dots,C_m\subseteq C$ covering C such that for each i, $f_{|C_i}$ and $g_{|C_i}$ are continuous. Now the rest is clear. □

Let $d^{(n)}(-,-)$ denote the usual euclidean distance in R^n, where the arguments may be either elements or definable subsets of R^n. Also we let $\pi:R^{n+1}\to R^n$ be the projection onto the first n coordinates. We say that an open definable subset U of R^{n+1} *has I-short height* if for every $\overline{x}\in\pi(U)$ we have

$$\sup\{|t-s|:t,s\in U_{\overline{x}}\}\in I$$

where $U_{\overline{x}}=\{y\in R:(\overline{x},y)\in U\}$.

We now prove the analogue of [14, Lemma 2] for open definable subsets with I-short height. The argument of the proof is similar, one just has to observe that the field operations are used in Wilkie's proof in a uniform way and only along fibers. Since in our case our fibers are I-short, such field operations, in the field I, can also be used in exactly the same way. For completeness we include the details.

Lemma 3.5. *Let C be a cell in R^n. Suppose that $f,g:C\to R$ are continuous definable maps such that $f<g$ and let U be an open definable subset of R^{n+1} with I-short height. Suppose further that $(f,g)_C\subseteq U$ and that $\Gamma(f)\subseteq U$ (respectively $\Gamma(g)\subseteq U$). Then there exists an open definable subset V of R^n and continuous definable maps $F,G:V\to R$ such that:*
(1) *$C\subseteq V$;*
(2) *$F_{|C}=f$ and $\Gamma(F)\subseteq U$ (respectively $\Gamma(G)\subseteq U$);*
(3) *$G_{|C}=g$;*

(4) *$F < G$;*
(5) *for all $\overline{x} \in V$ and all $y \in R$ with $F(\overline{x}) \leq y < G(\overline{x})$, (respectively $F(\overline{x}) < y \leq G(\overline{x})$), $(\overline{x}, y) \in U$.*

Proof. We prove the unparenthesized statement, the parenthetical one being similar.

Applying Lemma 3.2 we obtain an open cell D in R^n, with $C \subseteq D$, and a continuous definable retraction $H : D \to C$.

Let

$$V = \{\overline{x} \in D : d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, f \circ H(\overline{x})), U^c)\},$$

where $U^c = R^{n+1} \setminus U$. Clearly V is open in R^n and (1) holds since $\Gamma(f) \subseteq U$. Putting $F = f \circ H_{|V}$ we see that (2) holds. Also note that for all $\overline{x} \in V$, $F(\overline{x}) < g \circ H(\overline{x})$ and

$$J_{\overline{x}} := [0, g \circ H(\overline{x}) - F(\overline{x})) \subseteq \{t \in R_{\geq 0} : F(\overline{x}) + t \in U_{\overline{x}}\} \subseteq I$$

since U has I-short height.

By o-minimality and the fact that $\Gamma(F) \subseteq U$, there are well defined definable maps $z_0 : V \to I$ and $y_0 : V \to R$ given by

$$z_0(\overline{x}) = \sup\{t \in J_{\overline{x}} : [F(\overline{x}), F(\overline{x}) + t) \subseteq U_{\overline{x}}\}$$

and

$$y_0(\overline{x}) = F(\overline{x}) + z_0(\overline{x}).$$

Now observe that $y_0 : V \to R$ satisfies the conditions (3), (4) and (5) for G ((3) is satisfied because $(f, g)_C \subseteq U$, by hypothesis, and $f = F_{|C}$), but maybe y_0 is not continuous. Thus we need to find a continuous definable map $G : V \to R$ such that $F < G \leq y_0$ and $G_{|C} = y_0$.

Consider the definable set

$$S = \{(\overline{x}, y) \in R^{n+1} : \overline{x} \in V \text{ and } F(\overline{x}) \leq y < g \circ H(\overline{x})\}$$

and the definable continuous maps $\theta_1, \theta_2 : S \to I$ given by

$$\theta_1(\overline{x}, y) = 1_I -_I (y - F(\overline{x})) \cdot_I (g \circ H(\overline{x}) - F(\overline{x}))^{-_I 1_I}$$

where 1_I is the neutral element for the multiplication $\cdot_I$, $-_I$ is the diference and ${}^{-_I 1_I}$ is inversion in the field I, and,

$$\theta_2(\overline{x}, y) = \inf\{d^{(n+1)}((\overline{x}, t), U^c) : F(\overline{x}) \leq t \leq y\}.$$

Note that since U has I-short height we do have $\theta_1(S) \subseteq I$ and $\theta_2(S) \subseteq I$.

Fix $\overline{x} \in V$. Then the continuous definable map $(\theta_1 \cdot_I \theta_2)(\overline{x}, -)$ decreases monotonically and strictly from $d^{(n+1)}(\overline{x}, F(\overline{x}), U^c)$ to $0_I = 0$ on $[F(\overline{x}), y_0(\overline{x})]$ and is identically $0_I = 0$ on $[y_0(\overline{x}), g \circ H(\overline{x})]$. In particular, the restriction $(\theta_1 \cdot_I \theta_2)(\overline{x}, -)_| : [F(\overline{x}), y_0(\overline{x})] \to [0, d^{(n+1)}(\overline{x}, F(\overline{x}), U^c)]$ is injective and so by [2, Chapter 6, (1.12)], this restriction is a homeomorphism. Since $d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, F(\overline{x})), U^c)$ (by definition of V), it follows that there is a unique $y_1(\overline{x}) \in [F(\overline{x}), y_0(\overline{x})]$ such that $(\overline{x}, y_1(\overline{x})) \in S$ and $d^{(n)}(\overline{x}, H(\overline{x})) = (\theta_1 \cdot_I \theta_2)(\overline{x}, y_1(\overline{x}))$. Now observe that $F(\overline{x}) < y_1(\overline{x}) \leq y_0(\overline{x})$. In fact, if not then $F(\overline{x}) = y_1(\overline{x})$ and we obtain $(\theta_1 \cdot_I \theta_2)(\overline{x}, y_1(\overline{x})) = d^{(n+1)}((\overline{x}, F(\overline{x})), U^c)$ contradicting the fact that $d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, F(\overline{x})), U^c)$.

Now let $G : V \to R$ be given by $G(\overline{x}) = y_1(\overline{x})$ for all $\overline{x} \in V$. Then as we saw above, G satisfies (3), (4) and (5). On the other hand, by the uniqueness in the condition determining $y_1(\overline{x})$, we have

$$\Gamma(G) = \{(\overline{x}, y) \in V \times R : (\overline{x}, y) \in S,\ d^{(n)}(\overline{x}, H(\overline{x})) = (\theta_1 \cdot_I \theta_2)(\overline{x}, y)\},$$

showing that $\Gamma(G)$ is closed in $V \times R$. By Lemma 3.1, $G : V \to R$ is continuous as required. □

We need one more lemma:

Lemma 3.6. *Let C be a cell in R^n. Suppose that $f, g : C \to R$ are continuous definable maps such that $f < g$ and let U be an open definable subset of R^{n+1}. Suppose further that $[f, g]_C \subseteq U$. Then there exists an open definable subset W of R^n and continuous definable maps $F, G : W \to R$ such that:*

1. *$C \subseteq W$;*
2. *$F_{|C} = f$ and $\Gamma(F) \subseteq U$;*
3. *$G_{|C} = g$ and $\Gamma(G) \subseteq U$;*
4. *$F < G$;*
5. *for all $\overline{x} \in W$ and all $y \in R$ with $F(\overline{x}) \le y \le G(\overline{x})$, $(\overline{x}, y) \in U$.*

Proof. Applying Lemma 3.2 we obtain an open cell D in R^n, with $C \subseteq D$, and a continuous definable retraction $H : D \to C$.

Let W' be the intersection of

$$\{\overline{x} \in D : d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, f \circ H(\overline{x})), U^c)\}$$

and

$$\{\overline{x} \in D : d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, g \circ H(\overline{x})), U^c)\}$$

where $U^c = R^{n+1} \setminus U$. Clearly W' is open in R^n and (1) holds for W' since $\Gamma(f), \Gamma(g) \subseteq U$. Also (2) and (3) hold for $f \circ H_{|W'}$ and $g \circ H_{|W'}$. Also note that for all $\overline{x} \in W'$, $f \circ H_{|W'}(\overline{x}) < g \circ H_{|W'}(\overline{x})$ so (4) holds for $f \circ H_{|W'}$ and $g \circ H_{|W'}$.

Let $B = [f \circ H_{|W'}, g \circ H_{|W'}]_{|W'} \setminus U$ where

$$[f \circ H_{|W'}, g \circ H_{|W'}]_{|W'} = \{(\overline{x}, y) \in W' \times R : y \in [f \circ H_{|W'}(\overline{x}), g \circ H_{|W'}(\overline{x})]\},$$

and let

$$W = W' \setminus \overline{\pi(B)}.$$

Clearly W is open. We now show that $C \subseteq W$, verifying in this way (1). Suppose not and let $c \in C$ be such that $c \in \overline{\pi(B)}$. Let $\epsilon > 0$ be such that $E = [c - \epsilon, c + \epsilon]^n \subseteq W'$. By definable choice there is a definable map $\alpha : (0, \epsilon) \to \pi(B) \cap E$ such that $\lim_{t \to 0^+} \alpha(t) = c$. By replacing ϵ we may assume that α is continuous. Again by definable choice, we see that there exists a definable map $\beta : (0, \epsilon) \to B \cap [f \circ H_{|E}, g \circ H_{|E}]_{|E}$ such that $\pi \circ \beta = \alpha$. By replacing ϵ we may assume that β is continuous. Since the definable set $B \cap [f \circ H_{|E}, g \circ H_{|E}]_{|E}$ is closed and, by [3, Proposition 3.1 (3)], $\beta((0, \epsilon))$ is bounded, the limit $\lim_{t \to 0^+} \beta(t)$ exists in this set. If d is this limit, then $\pi(d) = c$ since $\pi \circ \beta = \alpha$. So $d \in [f \circ H_{|W'}(c), g \circ H_{|W'}(c)] \cap B \neq \emptyset$ contradicting the fact that $[f \circ H_{|W'}(c), g \circ H_{|W'}(c)] = [f(c), g(c)] \subseteq U$.

If we put $F = f \circ H_{|W}$ and $G = g \circ H_{|W}$ we see that (2), (3) and (4) hold. On the other hand, if $\overline{x} \in W$ and $y \in R$ are such that $F(\overline{x}) \le y \le G(\overline{x})$ and, by absurd, $(\overline{x}, y) \notin U$, then $(\overline{x}, y) \in B$ and so $\overline{x} \in \pi(B) \subseteq \overline{\pi(B)}$ contradicting the fact that $\overline{x} \notin \overline{\pi(B)}$. Thus (5) also holds. □

Combining Lemmas 3.5 and 3.6 we obtain:

Lemma 3.7. *Let C be a cell in R^n. Suppose that $f, g : C \to R$ are continuous definable maps such that $f < g$ and let U be an open definable subset of R^{n+1}. Suppose further that $(f, g)_C \subseteq U$ and that $\Gamma(f) \subseteq U$ (respectively $\Gamma(g) \subseteq U$). Then there exists a cell decomposition $C_1, \ldots, C_l$ of C and for each $i = 1, \ldots, l$ there is an open definable subset V_i of R^n and continuous definable maps $F_i, G_i : V_i \to R$ such that:*

1. *$C_i \subseteq V_i$;*
2. *$F_{i|C_i} = f_{|C_i}$ and $\Gamma(F_i) \subseteq U$ (respectively $\Gamma(G_i) \subseteq U$);*
3. *$G_{i|C_i} = g_{|C_i}$;*
4. *$F < G$;*
5. *for all $\overline{x} \in V_i$ and all $y \in R$ with $F_i(\overline{x}) \leq y < G_i(\overline{x})$, (respectively $F_i(\overline{x}) < y \leq G_i(\overline{x})$), $(\overline{x}, y) \in U$.*

Proof. We prove the unparenthesized statement, the parenthetical one being similar.

Let ϵ be a positive element in I and define $E = (-\epsilon, \epsilon)^{n+1}$,

$$U_f = U \cap (\cup\{(\overline{x}, f(\overline{x})) + E : \overline{x} \in C\})$$

and

$$U_g = U \cap (\cup\{(\overline{x}, g(\overline{x})) + E : \overline{x} \in C\}).$$

Then U_f and U_g are open definable subsets of U with I-short height.

Since $(f, g)_C \subseteq U$, $\Gamma(f) \subseteq U_f$ and $\Gamma(g) \subseteq U_g$, by Lemma 3.4, there exist definable maps $f', g' : C \to R$ and cells $C_1, \ldots, C_m \subseteq C$ such that:

1. $C = C_1 \cup \cdots \cup C_m$;
2. for each i, $f'_{|C_i}$ and $g'_{|C_i}$ are continuous;
3. $f < f' < g' < g$;
4. for each i, $\Gamma(f'_{|C_i}) \subseteq U_f$ and $\Gamma(g'_{|C_i}) \subseteq U_g$;
5. for each i, $(f'_{|C_i}, g_{|C_i})_{C_i} \subseteq U$, $(f_{|C_i}, g'_{|C_i})_{C_i} \subseteq U$ and $[f'_{|C_i}, g'_{|C_i}]_{C_i} \subseteq U$.

Fix $i = 1, \ldots, m$. Then we can apply Lemma 3.5 to the data $(U_f, f_{|C_i}, f'_{|C_i})$ and obtain the data (V_f, F_1, F'_1) satisfying (1) to (5) of that lemma. Similarly, we can apply Lemma 3.5 to the data $(U_g, g'_{|C_i}, g_{|C_i})$ and obtain the data (V_g, G'_1, G_1) satisfying (1) to (5) of that lemma. On the other hand, we can apply Lemma 3.6 to the data $(U, f'_{|C_i}, g'_{|C_i})$ and obtain the data (W, F', G') satisfying (1) to (5) of that lemma .

Take $V_i = V_f \cap V_g \cap W$ and set $F = F_{1|V_i}$, $G = G_{1|C_i}$. Then clearly (1) to (5) hold. □

The following is also required:

Lemma 3.8. *Let C be a cell in R^n. Suppose that $k : C \to R$ is a continuous definable map and let U be an open definable subset of R^{n+1}. Suppose further that $(k, +\infty)_C \subseteq U$ and $\Gamma(k) \subseteq U$ (respectively $(-\infty, k)_C \subseteq U$ and $\Gamma(k) \subseteq U$). Then there exists an open definable subset W of R^n and a continuous definable map $K : W \to R$ such that:*

1. *$C \subseteq W$;*
2. *$K_{|C} = k$ and $\Gamma(K) \subseteq U$;*
3. *for all $\overline{x} \in W$ and all $y \in R$ with $K(\overline{x}) \leq y$ (respectively $y \leq K(\overline{x})$), $(\overline{x}, y) \in U$.*

Proof. We prove the unparenthesized statement, the parenthetical one being similar.

Applying Lemma 3.2 we obtain an open cell D in R^n, with $C \subseteq D$, and a continuous definable retraction $H : D \to C$.

Let

$$W' = \{\overline{x} \in D : d^{(n)}(\overline{x}, H(\overline{x})) < d^{(n+1)}((\overline{x}, k \circ H(\overline{x})), U^c)\}$$

where $U^c = R^{n+1} \backslash U$. Clearly W' is open in R^n and (1) holds for W' since $\Gamma(k) \subseteq U$. Also (2) holds for $k \circ H_{|W'}$.

Let $B = [k \circ H_{|W'}, +\infty)_{|W'} \setminus U$ where

$$[k \circ H_{|W'}, +\infty)_{|W'} = \{(\overline{x}, y) \in W' \times R : k \circ H_{|W'}(\overline{x}) \leq y\},$$

and let

$$W = W' \setminus \overline{\pi(B)}.$$

Clearly W is open. We now show that $C \subseteq W$, verifying in this way (1). Suppose not and let $c \in C$ be such that $c \in \overline{\pi(B)}$. Let $\epsilon > 0$ be such that $E = [c - \epsilon, c + \epsilon]^n \subseteq W'$. By definable choice there is a definable map $\alpha : (0, \epsilon) \to \pi(B) \cap E$ such that $\lim_{t \to 0^+} \alpha(t) = c$. By replacing ϵ we may assume that α is continuous. Again by definable choice, we see that there exists a definable map $\beta : (0, \epsilon) \to B \cap [k \circ H_{|E}, +\infty)_{|E}$ such that $\pi \circ \beta = \alpha$. By replacing ϵ we may assume that β is continuous. Since the definable set $B \cap [k \circ H_{|E}, +\infty)_{|E}$ is closed and, by [3, Proposition 3.1 (3)], $\beta((0, \epsilon))$ is bounded, the limit $\lim_{t \to 0^+} \beta(t)$ exists in this set. If d is this limit, then $\pi(d) = c$ since $\pi \circ \beta = \alpha$. So $d \in [k \circ H_{|W'}(c), +\infty) \cap B \neq \emptyset$ contradicting the fact that $[k \circ H_{|W'}(c), +\infty) = [k(c), +\infty) \subseteq U$.

If we put $K = k \circ H_{|W}$ we see that (2) holds. On the other hand, if $\overline{x} \in W$ and $y \in R$ are such that $K(\overline{x}) \leq y$ and, by absurd, $(\overline{x}, y) \notin U$, then $(\overline{x}, y) \in B$ and so $\overline{x} \in \pi(B) \subseteq \overline{\pi(B)}$ contradicting the fact that $\overline{x} \notin \overline{\pi(B)}$. Thus (3) also holds. □

Corollary 3.9. *If $\mathcal{R} = (R, <, 0, +, \{\lambda\}_{\lambda \in D}, \{B\}_{B \in \mathcal{B}})$ is a semi-bounded non-linear o-minimal expansion of an ordered group, then every non-empty open definable set is a finite union of open cells.*

Proof. This is done by induction on the dimension of the open definable set. For dimension one this is clear. Let U be an open definable subset of R^{n+1}. Let $\mathcal{D}$ be a cell decomposition of R^{n+1} partitioning U. Clearly it is enough to show that each cell $D \in \mathcal{D}$ with $D \subseteq U$ can be covered by finitely many open cells (in R^{n+1}) each of which is contained in U.

Case A: $D = (f, g)_C$ for some cell C in R^n and continuous definable maps $f, g : C \to R$ such that $f < g$.

Let $f' = \frac{2f+g}{3}$ and $g' = \frac{f+2g}{3}$. Then $f', g' : C \to R$ are continuous definable maps such that

- $f < f' < g' < g$;
- $\Gamma(f') \subseteq U$ and $\Gamma(g') \subseteq U$;
- $(f', g)_C \subseteq U$ and $(f, g')_C \subseteq U$.

Then we can apply Lemma 3.7 to the data (C, U, f, g') and obtain the data (C_i, V_i, F_i, G'_i) with $i = 1, \ldots, l$ satisfying (1) to (5) of that lemma. By the inductive hypothesis there exists a finite collection $\mathcal{A}_i$ of open cell in R^n contained in V_i which cover V_i. By (4) and (5) of Lemma 3.7, for each $A \in \mathcal{A}_i$, $(F_{i|A}, G'_{i|A})_A$ is an open cell in R^{n+1} contained in U, and by (1), (2) and (3) of that lemma, $(f_{|C_i}, g'_{|C_i})_{C_i} \subseteq$

$\cup\{(F_{i|A}, G'_{i|A})_A : A \in \mathcal{A}_i\}$. Thus $(f, g')_C \subseteq \cup\{(F_{i|A}, G'_{i|A})_A : A \in \mathcal{A}_i$ and $i = 1, \dots, l\}$.

Similarly, applying the parenthetical statement in Lemma 3.7 to the data (C, U, f', g), we see that $(f', g)_C$ can be covered by finitely many open cells in R^{n+1} each of which is contained in U. Hence the same is true for $(f, g)_C = (f, g')_C \cup (f', g)_C$.

Case B: $D = \Gamma(h)$ for some continuous definable map $h : C \to R$ where C is a cell in R^n. This case reduces to Case A above by Lemma 3.3.

Case C: $D = (k, +\infty)_C$ (respectively $D = (-\infty, k)_C$) for some cell C in R^n and continuous definable map $k : C \to R$.

Then we can apply Lemma 3.8 to the data (C, U, k) and obtain the data (C, W, K) satisfying (1) to (3) of that lemma. By the inductive hypothesis there exists a finite collection $\mathcal{A}$ of open cell in R^n contained in W which cover W. By (3) of Lemma 3.8, for each $A \in \mathcal{A}$, $(K_{|A}, +\infty)_A$ is an open cell in R^{n+1} contained in U, and by (1) and (2) of that lemma, $(k_{|C}, +\infty)_C \subseteq \cup\{(K_{|A}, +\infty)_A : A \in \mathcal{A}\}$.

Similarly for the case $D = (-\infty, k)_C$. □

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