GROUPS DEFINABLE IN LINEAR O-MINIMAL STRUCTURES: THE NON-COMPACT CASE

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Abstract. Let $\mathcal{M} = \langle M, +, <, 0, S\rangle$ be a linear o-minimal expansion of an ordered group, and $G = \langle G, \oplus, e_G\rangle$ an n-dimensional group definable in $\mathcal{M}$. We show that if G is definably connected with respect to the t-topology, then it is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup U of $\langle M^n, +\rangle$ and a lattice L of rank equal to the dimension of the 'compact part' of G.

1. Introduction

This paper is a natural continuation of [ElSt]. Let $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda\in D}\rangle$ be an ordered vector space over an ordered division ring D. It was shown in [ElSt, Theorem 1.4] that a definably compact group G definable in $\mathcal{M}$, satisfying the assumptions of the above abstract, is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant \langle M^n, +\rangle$ and a lattice L of rank n. Here we consider the case where G is not necessarily definably compact, and generalize [ElSt, Theorem 1.4] towards a structure theorem analogous to the following classical theorem (see, for example, [Bour]).

Fact 1.1. *Every connected abelian real Lie group is isomorphic to a direct sum of copies of the additive group $\langle \mathbb{R}, +\rangle$ of the reals and the torus S^1.*

Moreover, we prove our theorem in the more general setting where $\mathcal{M}$ is any linear o-minimal expansion of an ordered group.

Definition 1.2 ([LP])**.** An o-minimal expansion $\mathcal{M} = \langle M, +, <, 0, \dots\rangle$ of an ordered group is called *linear* if for every $\mathcal{M}$-definable function $f : A \subseteq M^n \to M$, there is a partition of A into finitely many A_i, such that for each i, if $x, y, x+t, y+t \in A_i$, then

$$f(x + t) - f(x) = f(y + t) - f(y).$$

For the rest of this introduction, let $\mathcal{M}$ be a linear o-minimal expansion of an ordered group. By 'definable' we mean 'definable in $\mathcal{M}$ with parameters'.

By [Pi1], we know that every definable group G can be equipped with a unique definable manifold topology that makes it into a topological group, called t-topology. In the rest of this introduction, all topological notions about such a G are referring to this t-topology.

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It is known that a definably connected definable group G is abelian. (See, for example, [PeSt, Corollary 5.1].)

Fact 1.3 ([EdEl]). *Every definable group G is a definable extension of a definably compact definable group B by $M^r = \langle M^r, +, 0\rangle$, for some $r \in \mathbb{N}$. That is, there is a short exact sequence*

$$0 \to M^r \to G \xrightarrow{q} B \to 0,$$

where all maps involved are definable homomorphisms.

We let $\sigma : B \to G$ be a definable global section; that is, a definable map such that $q \circ \sigma = id_B$. Let $K = \sigma(B)$ be the topological group with the structure induced by σ. We call K the *compact part of G*. Clearly, G is in definable bijection with $M^r \times K$ as abstract sets. As we know by examples in [PeS] and [Str], however, we cannot always expect G to be definably isomorphic to the direct sum of M^r and K. We show:

Theorem 1.4. *Let G be a definably connected definable group of dimension n. Assume that the compact part of G has dimension s. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant \langle M^n, +, 0\rangle$, and a lattice L of rank s.*

The terminology in Theorem 1.4 was introduced in [ElSt], and we briefly recall it in Section 2 below. We obtain two corollaries. The first one is a generalization of Pillay's Conjecture in the present context.

Proposition 4.1. *Assume that $\mathcal{M}$ is sufficiently saturated, and let G be as in Theorem 1.4. Then there is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s.*

Proposition 5.8. *Let G be as in Theorem 1.4. Then the o-minimal fundamental group of G is isomorphic to L.*

Structure of the paper.

In Sections 2, 3 and 4, we handle the case where $\mathcal{M}$ is an ordered vector space over an ordered division ring.

Section 2 contains some definitions and basic results that were proved in [ElSt] without (using) the assumption that G is definably compact.

Section 3 contains the proof of Theorem 1.4, which we outline here. In analogy with [ElSt, Theorem 1.4], the proof consists of three steps. In Step I, we compare the two group operations $\oplus$ and $+$. In Step II, we find a suitable generic open s-parallelogram H in K. We then let U be the subgroup of $\langle M^n, +, 0\rangle$ generated by the set $H^G = M^r \times H$, and define a surjective group homomorphism $\phi : U \to G$. In Step III, we prove that the kernel $L = \ker(\phi)$ is a lattice of rank s.

In Section 4, we prove Proposition 4.1.

In Section 5, we extend our results to the case where $\mathcal{M}$ is any linear o-minimal expansion of an ordered group, and prove Proposition 5.8.

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2. Preliminaries

Until Section 5, we fix a big sufficiently saturated ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D}\rangle$ over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1\rangle$.

The terminology and notation of this paper was introduced in [ElSt]. We recall the definitions that play an important role here, and refer the reader to [ElSt] for the rest.

Definition 2.1 ([ElSt], Section 1)**.** Let $L \leqslant U \leqslant \langle M^n, +\rangle$. The group U/L is called *a definable quotient group* if there is a definable complete set $S \subseteq U$ of representatives for U/L, such that the induced group structure $\langle S, +_S\rangle$ is definable. In this case, we identify U/L with $\langle S, +_S\rangle$.

In [ElSt, Section 2], the t-topology of a definable group G and several notions relevant to it were fixed. In what follows, an index 't-' will indicate that the corresponding notion is taken with respect to the t-topology of G, if ambiguity would otherwise arise. In [ElSt, Section 3], basic facts about the definable structure of $\mathcal{M}$ were established. In particular, the Linear Cell Decomposition Theorem (Linear CDT) was stated therein. Another important notion was that of an 'open parallelogram', which we now recall. For $\lambda = (\lambda_1, \dots, \lambda_n) \in D^n$ and $x \in M$, let us denote $\lambda x := (\lambda_1 x, \dots, \lambda_n x)$.

Definition 2.2 ([ElSt], Definition 3.5)**.** Let $0 < m \leq n$ and $c \in M^n$. An *open m-parallelogram H with center c* is a definable subset $H \subseteq M^n$ of the form

$$H = c + \{\lambda_1 t_1 + \cdots + \lambda_m t_m : -e_1 < t_i < e_i\},$$

for some fixed $e_i > 0$ in M and $\lambda_i \in D^n$, $1 \leq i \leq m$.

For the rest of this section, let $G = \langle G, \oplus, e_G\rangle$ be a definable group, with $G \subseteq M^n$. We recall some basic results which were proved in [ElSt] without (using) the assumption that G is definably compact or t-connected.

Definition 2.3 ([ElSt], Definition 2.4)**.** Let W^G be a fixed definable large t-open subset of G on which the $\mathcal{M}$- and t- topologies coincide. Let

$$V^G = \{a \in G : \text{there is a } t\text{-open neighborhood } V_a \text{ of } a \text{ in } G,$$

$$\text{such that } \forall x, y \in V_a,\ x \ominus a \oplus y = x - a + y\} \cap W^G.$$

By [ElSt, Lemma 2.5], V^G is a definable, large, open and t-open subset of G. By cell decomposition, V^G is the disjoint union of finitely many open (t-)connected components.

Fact 2.4 ([ElSt], Lemma 4.7)**.** *For all u, v in the same definably connected component of V^G, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$, we have $u + \varepsilon, v + \varepsilon \in G$, and*

$$(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v.$$

It will often be convenient to assume that $e_G = 0 \in V^G$. In [ElSt, Lemma 4.9] it was shown that we may do so. Namely, it was shown there that $(G, \oplus, e_G)$ is definably isomorphic to a topological group $(G', +_1, 0)$ with $0 \in V^{G'}$. The critical lemma is the following fact, which we will also use here later.

Fact 2.5. *Let $b \in V^G$ and $f : G \to M^n$ with*

$$f(x) = (x \oplus b) - b.$$

If $G' = \langle f(G), +_1, 0\rangle$ is the induced group, then $V^{G'} = V^G - b$.

Fact 2.5 was proved in [ElSt, Lemma 4.9]. In that proof, a generic element b was chosen, but only the property that $b \in V^G$ was used.

Finally, we state the main theorem from [ElSt], in a form that will be useful in the present paper (and whose proof can easily be extracted from [ElSt]).

Fact 2.6. *Let $K = \langle K, \oplus_K, 0\rangle \subseteq M^s$ be a definably compact, t-connected, definable group of dimension s. Assume that H is an open s-parallelogram, generic in K, such that*

- *H has center 0,*
- *H is contained in V^K.*

Let U_H be the subgroup of $\langle M^s, +\rangle$ generated by H:

$$U_H =< H >= \bigcup_{k<\omega} \underbrace{H + \cdots + H}_{k-times} \leqslant M^s.$$

Then:

- *the following map $\phi^K : U_H \to K$ is a well-defined, continuous, surjective group homomorphism: for all $x_1, \ldots, x_k \in H$, if $x = x_1 + \cdots + x_k$, then*
$$\phi^K(x) = x_1 \oplus_K \cdots \oplus_K x_k.$$
- *$L^K = \ker(\phi^K)$ is a lattice of rank s,*
- *$U_H/L^K = \langle S, +_S\rangle$ is a definable quotient group and $\phi_{H\restriction S} : \langle S, +_S\rangle \to K$ is a definable isomorphism.*

That is, $K \cong_{def} U_H/L^K$.

The following fact can also be extracted from [ElSt, Step II], but we include its proof for completeness.

Fact 2.7. *Let $K \subseteq M^s$ be a definably compact group of dimension s. Let W be a large definable subset of K. Then there is an open s-parallelogram $H \subseteq W$ which is generic in K.*

Proof. Since W is large in K, it is also generic. By Linear CDT ([ElSt]), W is a finite union of linear cells, and by [PePi, Corollary 3.9], one of them, call it Y, must be generic. By [ElSt, Lemma 3.10], Y has dimension s. By [ElSt, Lemma 3.6], $\overline{Y}$ is a finite union of closed s-parallelograms, say $W_1, \ldots, W_l$. For $i \in \{1, \ldots, l\}$, let $Y_i = Y \cap W_i$. Then $Y = Y_1 \cup \cdots \cup Y_l$. By [ElSt, Lemma 3.10] again, one of the Y_i's must be generic, say Y_1. Let $H = \mathrm{Int}(Y_1)$. By [ElSt, Lemma 3.10], H is generic. □

3. The proof of Theorem 1.4

As described in the introduction, the proof runs in three steps. Let $G = \langle G, \oplus, e_G\rangle$ be a definable group, with $G \subseteq M^n$.

Step I. Comparing $\oplus$ with $+$.

Lemma 3.1. *Assume that $e_G = 0$. Assume that H is an open m-parallelogram, $m \leq n$, contained in G such that:*

- *H has center 0,*
- *H is contained in V^G.*

Then, for every $x, y \in H$ such that $x + y \in H$, we have:

$$x \oplus y = x + y.$$

Proof. We first notice that:

Claim 1. *For all $u, v \in H$, such that $u \oplus v \in H$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$, $v + \varepsilon \in H$ and*

$$u \oplus (v + \varepsilon) = (u \oplus v) + \varepsilon.$$

Indeed, by Fact 2.4, there is $r > 0$ in M, such that $\forall \varepsilon \in (-r, r)^n$,

$$(v + \varepsilon) \ominus v = [(u \oplus v) + \varepsilon] \ominus (u \oplus v).$$

Claim 2. *Let $\varepsilon(t) : [0, p] \to H$, $\varepsilon(0) = 0$, be a path, such that $\forall t \in [0, p]$, $u + \varepsilon(t) \in H$. Then:*

$$u \oplus \varepsilon(p) = u + \varepsilon(p).$$

Indeed, consider the function $f : G \to M^n$ with $x \mapsto (u \oplus x) - x$. We show that f is locally constant on $\mathrm{Im}(\varepsilon)$. Indeed, by Claim 1, $\forall s \in [0, p]$, $\exists z > 0$, such that $\forall t \in [s - z, s + z] \cap [0, p]$,

$$u \oplus \varepsilon(t) = u \oplus \big(\varepsilon(s) + \varepsilon(t) - \varepsilon(s)\big) = \big(u \oplus \varepsilon(s)\big) + \varepsilon(t) - \varepsilon(s).$$

Thus $\forall t \in [s-z, s+z]$, $f\big(\varepsilon(t)\big) = \big(u \oplus \varepsilon(t)\big) - \varepsilon(t) = \big(u \oplus \varepsilon(s)\big) - \varepsilon(s) = f\big(u + \varepsilon(s)\big)$. It follows that f is constant on $\mathrm{Im}(\varepsilon)$ and equal to $(u \oplus 0) - 0 = u$. Thus, $\forall t \in [0, p]$, $\big(u \oplus \varepsilon(t)\big) - \varepsilon(t) = u$, that is, $u \oplus \varepsilon(t) = u + \varepsilon(t)$. This proves Claim 2.

Now, let $x, y \in H$, such that $x + y \in H$. By [ElSt, Lemma 4.25], there is a path $\varepsilon(t)$ in H from 0 to y, such that the path $x + \varepsilon(t)$ lies entirely in H, as well. By Claim 2 for $u = x$, we have: $x \oplus y = x + y$. □

Corollary 3.2. *Let H be as in Lemma 3.1. Let $x_1, \dots, x_l \in H$ be such that for any subset σ of $\{1, \dots, l\}$, $\sum_{j \in \sigma} x_j \in H$. Then, $x_1 + \dots + x_l = x_1 \oplus \dots \oplus x_l$.*

Proof. By induction on l.

$\mathbf{l = 2}$. By Lemma 3.1.

$\mathbf{l > 2}$. $x_1 + \dots + x_l = x_1 + (x_2 + \dots + x_l) = x_1 \oplus (x_2 \oplus \dots \oplus x_l) = x_1 \oplus \dots \oplus x_l$. □

Corollary 3.3. *Let H be as in Lemma 3.1. For every $x_1, \dots, x_l, y_1, \dots, y_m \in H$, if $x_1 + \dots + x_l = y_1 + \dots + y_m$, then $x_1 \oplus \dots \oplus x_l = y_1 \oplus \dots \oplus y_m$.*

Proof. This is [ElSt, Lemma 4.27], with identical proof. The assumption that G is definably compact in that proof is not used. □

Step II. A generic open s-parallelogram of K. As we saw in the introduction, G is a definable extension of a definably compact definable group B by $M^r = \langle M^r, +, 0\rangle$, for some $r \in \mathbb{N}$:

$$0 \to M^r \xrightarrow{i} G \xrightarrow{q} B \to 0.$$

Let $\sigma : B \to G$ be a definable global section and $K = \sigma(B)$. Then $K \subseteq G$ is a definable complete set of representatives for G/M^r. We may choose σ so that

$e_G \in K$. We let $K = \langle K, \oplus_K, e_G\rangle$ be the topological group with the structure induced by $\sigma : B \to K$. We have:

$$G = \{i(a) \oplus u : a \in M^r,\, u \in K\}.$$

Clearly, $i(M^r) \cap K = \{e_G\}$. Thus there is a definable bijection $G \to M^r \times K$, with $i(a) \oplus u \mapsto (a, u)$. Moreover, since K is definably compact of dimension $n - r$, using [ElSt, Lemma 3.7] we can find a definable injective map $f : K \to M^{n-r}$ with $f_2(e_G) = 0$. By taking the composition $i(a) \oplus u \mapsto \big(a, f(u)\big)$, we may assume that:

(1) $G = M^r \times K \subseteq M^n$
(2) $M^r \leqslant G$
(3) $G = \langle G, \oplus, 0\rangle$
(4) $K = \langle K, \oplus_K, 0\rangle$ is a definable complete set of representatives for G/M^r equipped with the induced group structure
(5) for every $a \in M^r$ and $u \in K$, $a \oplus u = (a, u)$.

Furthermore, since V^G is a large definable subset of G, we may assume that

$$V^G \cap K \text{ is large in } K. \tag{6}$$

Indeed, if needed, we may take another section $\tau : K \to G$, $\tau(0) = 0$, such that a large subset of its image K' is contained in V^G. Then, we may project K' onto M^s. If π denotes this projection (which is a bijection), it remains to check that $V^{\pi(G)} \cap \pi(K')$ is large in $\pi(K')$. Since π is piecewise linear, it suffices to show that for every $A \subseteq K'$ on which π is linear, $V^{\pi(G)} \cap \pi(A)$ is large in $\pi(A)$. We show that $\pi(V^G) \cap \pi(A) \subseteq V^{\pi(G)} \cap \pi(A)$, which is clearly enough. Assume $\pi(G) = \langle \pi(G), *, 0\rangle$ is the induced group via π, and let $c \in \pi(V^G) \cap \pi(A)$. Since $\pi^{-1}(c) \in V^G$, there is a neighborhood V_c of c in $\pi(A)$ such that for all $x, y \in V_c$,

$$x -_* c * y = \pi\big(\pi^{-1}(x) \ominus_{K'} \pi^{-1}(a) \oplus_{K'} \pi^{-1}(y)\big) = \pi\big(\pi^{-1}(x) - \pi^{-1}(a) + \pi^{-1}(y)\big).$$

But since π is linear on A, the latter is equal to $\pi\big(\pi^{-1}(x)\big) - \pi\big(\pi^{-1}(a)\big) + \pi\big(\pi^{-1}(y)\big) = x - a + y$.

Until Section 5, we fix an n-dimensional, t-connected definable group G, and its compact part K, such that conditions (1)-(6) above hold. Let $s = \dim(K) = n - r$.

As mentioned in the Introduction, G is abelian.

Observe that the t-topology of K is the same as the quotient topology induced by the canonical surjection $q : G \to G/M^r$, by [ElSt, Fact 2.1].

Lemma 3.4. *For all $x_1, x_2 \in K$, there is $a \in M^r$, such that*

$$x_1 \oplus_K x_2 = x_1 \oplus x_2 \oplus a.$$

Proof. Since $q(x_1 \oplus_K x_2) = q\big(\sigma q(x_1 \oplus x_2)\big) = q\sigma\big(q(x_1 \oplus x_2)\big) = q(x_1 \oplus x_2)$, we have $(x_1 \oplus_K x_2) \ominus (x_1 \oplus x_2) \in \ker(q) = M^r$. □

We now proceed to define a suitable generic open s-parallelogram H in K. By Condition (6) above, $V^K \cap V^G \cap K$ is large in K. By Fact 2.7, there is an open s-parallelogram $H \subseteq K$, generic in K, contained in V^G and in V^K. We are going to show that H may be assumed to have center 0.

Lemma 3.5. *The group G is definably isomorphic to a group $G' = \langle G', +_{G'}, 0\rangle$, and there is a definably compact group $K' = \langle K', +_{K'}, 0\rangle$, such that:*

(1) $G' = M^r \times K' \subseteq M^n$
(2) $M^r \leqslant G'$
(3) (a) *$K' \subseteq G'$ is a definable complete set of representatives for G'/M^r, and*
(b) *$+_{K'}$ coincides with the group operation induced by the canonical surjection $q : G' \to G'/M^r$*
(4) *for every $a \in M^r$ and $u \in K'$, $a +_{G'} u = (a, u)$*
(5) *there is an open s-parallelogram $H' \subseteq K'$, generic in K', such that:*
(a) *H' has center 0,*
(b) *H' is contained in $V^{G'}$ and in $V^{K'}$.*

Proof. Let c be the center of H. Consider the following two definable bijections:

$$f_G : G \ni x \mapsto (x \oplus c) - c \in f_G(G) \subseteq M^n,$$

$$f_K : K \ni x \mapsto (x \oplus_K c) - c \in f_K(K) \subseteq M^n.$$

Now, let $G' = f_G(G)$, $K' = f_K(K)$, $H' = f_K(H \ominus_K c) = H - c$. Let $G' = \langle G', +_{G'}, 0\rangle$ and $K' = \langle K', +_{K'}, 0\rangle$ be the induced topological group structures induced by f_G and f_K, respectively. By [ElSt, Remark 2.2], f_G and f_K are definable isomorphisms, for all $x, y \in G'$,

$$x +_{G'} y = [(x + c) \ominus c \oplus (y + c)] - c,$$

and for all $x, y \in K'$,

$$x +_{K'} y = [(x + c) \ominus_K c \oplus_K (y + c)] - c.$$

(1) For every $x = (a, u) = a \oplus u \in G = M^r \times K$, we have:

$$\begin{aligned} f_G(x) &= (a \oplus u \oplus c) - c = [a \oplus u \oplus c \ominus (u \oplus_K c) \oplus (u \oplus_K c)] - c \\ &= \big(a \oplus u \oplus c \ominus (u \oplus_K c), u \oplus_K c\big) - (0, c) \\ &= \big(a \oplus u \oplus c \ominus (u \oplus_K c), (u \oplus_K c) - c\big) \in M^r \times K', \end{aligned}$$

by Lemma 3.4. Hence $G' \subseteq M^r \times K'$. On the other hand, if $y = (b, v) \in M^r \times K'$, let $x = (a, u)$ where $u = (v + c) \ominus_K c \in K$ and $a = b \ominus (u \oplus c) \oplus (u \oplus_K c) \in M^r$. It can be checked that $f_G(x) = y$. Hence $G' \supseteq M^r \times K'$.

(2) Observe that $f_G(M^r) = M^r$. Indeed, for every $x \in M^r$, $(x \oplus c) - c = (x, c) - (0, c) = (x, 0) = x$.

(3)(a) We first show that $K' \subseteq G'$. Let $(g \oplus_K c) - c \in K'$, for some $g \in K$. Then $g_1 = (g \oplus_K c) \ominus c \in G$ and $(g \oplus_K c) - c = (g_1 \oplus c) - c \in G'$.

We next show that K' is a definable set of representatives for G'/M^r. Let $g' = f_G(g) = (g \oplus c) - c \in G'$, for some $g \in G$. Since $G = \{a \oplus k : a \in M^r,\, k \in K\}$, there are $a \in M^r$ and $k \in K$ such that $g \oplus c = a \oplus k$. Then $f_G(a) +_{G'} f_K(k \ominus_K c) = [(a \oplus c) - c] +_{G'} (k - c) = (a \oplus c \ominus c \oplus k) - c = (a \oplus k) - c = (g \oplus c) - c = g'$.

Finally, K' is complete: assume $f_K(k_1) = f_G(a) +_{G'} f_K(k_2)$, for some $k_1, k_2 \in K$ and $a \in M^r$. We show $k_1 = k_2$ and, thus, $f_K(k_1) = f_K(k_2)$. We have, $f_K(k_1) = (k_1 \oplus_K c) - c$ and $f_G(a) +_{G'} f_K(k_2) = [(a \oplus c) - c] +_{G'} [(k_2 \oplus_K c) - c] = [a \oplus (k_2 \oplus_K c)] - c$. Thus, $k_1 \oplus_K c = a \oplus (k_2 \oplus_K c)$. Since K is a complete set of representatives for G/M^r, $k_1 \oplus_K c = k_2 \oplus_K c$ and, thus, $k_1 = k_2$.

(3)(b) We show that for every $x, y \in K'$, there is $a \in M^r$ such that $x +_{K'} y = x +_{G'} y +_{G'} a$. We have $x +_{K'} y = [(x+c) \ominus_K c \oplus_K (y+c)] - c = [(x+c) \ominus c \oplus (y+c) \oplus b] - c$, for some $b \in M^r$. Let $a = f_G(b) = (b \oplus c) - c \in M^r$. Then we have $x +_{G'} y +_{G'} a = \big([(x+c) \ominus c \oplus (y+c)] - c\big) +_{G'} [(b \oplus c) - c] = [(x+c) \ominus c \oplus (y+c) \ominus c \oplus b \oplus c] - c = x +_{K'} y$.

(4) We have

$$\begin{aligned} a +_{G'} u &= [(a + c) \ominus c \oplus (u + c)] - c = [(a, c) \ominus c \oplus (u + c)] - c \\ &= [(a \oplus c) \ominus c \oplus (u + c)] - c = (a, u + c) - (0, c) = (a, u). \end{aligned}$$

(5)(a) It is clear that $H' = H - c$ is an open s-parallelogram with center 0. Since H is generic in K, $H \ominus_K c$ is generic in K, and, thus, $H' = f_K(H \ominus_K c)$ is generic in K'. (b) By Fact 2.5 applied to f_G and f_K separately. $\square$

We may thus assume that: $H \subseteq K$ is an open s-parallelogram, generic in K, such that:

- H has center 0,
- H is contained in V^G and in V^K.

We let

$$H^G = \{a \oplus u : a \in M^r,\, u \in H\} = M^r \times H.$$

Since H is generic in K, it is easy to see that H^G is generic in G.

Lemma 3.6. *There is $\Xi \in \mathbb{N}$, such that $G = \underbrace{H^G \oplus \cdots \oplus H^G}_{\Xi-times}$.*

Proof. By [ElSt, Lemma 4.29], there is $\Xi \in \mathbb{N}$, such that $K = \underbrace{H \oplus \cdots \oplus H}_{\Xi-\text{times}}$. Since $G = \{a \oplus u : a \in M^r, u \in K\}$, Lemma 3.4 gives $G = \underbrace{H^G \oplus \cdots \oplus H^G}_{\Xi-\text{times}}$. $\square$

Definition 3.7. Let U_H be the subgroup of $\langle M^s, +, 0\rangle$ generated by H; that is, $U_H =< H >= \bigcup_{k<\omega} H^k$, where $H^k = \underbrace{H + \cdots + H}_{k-\text{times}}$. Let U be the subgroup of $M^n = \langle M^n, +, 0\rangle$ generated by H^G; that is,

$$U =< H^G >= \bigcup_{k<\omega} \left(H^G\right)^k.$$

Equivalently, $U = M^r \times U_H$. By Corollary 3.3, the following function $\phi : U \to G$ is well-defined. For all $x_1 = (a_1, u_1), \ldots, x_k = (a_k, u_k) \in H^G = M^r \times H$, if $x = x_1 + \cdots + x_n$, then

$$\phi(x) = x_1 \oplus \cdots \oplus x_k = (a_1 + \cdots + a_k) \oplus u_1 \oplus \cdots \oplus u_k.$$

Since M^r and $U_H = \langle U_H, +_{\restriction U_H}, 0\rangle$ are subgroups of M^n, so is their direct product $U = M^r \times U_H$. Easily, U is a $\bigvee$-definable group, and convexity of H implies convexity of U.

Proposition 3.8. *ϕ is a t-continuous group homomorphism from U onto G.*

Proof. ϕ is a group homomorphism, because if $x = x_1 + \cdots + x_l$ and $y = y_1 + \cdots + y_m$, with $x_i, y_i \in H$, then $\phi(x + y) = \phi(x_1 + \cdots + x_l + y_1 + \cdots + y_m) = x_1 \oplus \cdots \oplus x_l \oplus y_1 \oplus \cdots \oplus y_m = \phi(x) \oplus \phi(y)$. It is onto, by Lemma 3.6. Since $\oplus$ is t-continuous, so is ϕ. $\square$

Thus, if we let $L = \ker(\phi)$, we know that $U/L \cong G$ as abstract groups.

Step III. L is a lattice of rank s.

We begin with an easy lemma.

Lemma 3.9. *(i)* $\ker(\phi) \cap H^G = \{0\}$.

(ii) Let Ξ *be as in Lemma 3.6. Then* $\forall x \in U, \exists y \in (H^G)^\Xi$, $y - x \in ker(\phi)$.

Proof. (i) For every $x \in H^G, \phi(x) = x$.

(ii) For $x \in U$, since $\phi(x) \in G$, there are $x_1, \ldots, x_\Xi \in H^G$, such that $\phi(x) = x_1 \oplus \cdots \oplus x_\Xi$. Clearly, if $y = x_1 + \cdots + x_\Xi \in (H^G)^\Xi$, then $\phi(x) = \phi(y)$. □

Lemma 3.10. *For every* $x_1 = (a_1, u_1), \ldots, x_k = (a_k, u_k) \in G = M^r \times K$,

$$x_1 \oplus \cdots \oplus x_k = \big(a_1 + \cdots + a_k + h(u_1, \ldots, u_k), u_1 \oplus_K \cdots \oplus_K u_k\big),$$

where $h(u_1, \ldots, u_k) = u_1 \oplus \cdots \oplus u_k \ominus (u_1 \oplus_K \cdots \oplus_K u_k) \in M^r$.

Proof. The proof is quite standard, but we include it for completeness. Assume that $(a_1, u_1) \oplus \cdots \oplus (a_k, u_k) = (a, u)$. By taking $\sigma \circ q$ on both sides, we have $u_1 \oplus_K \cdots \oplus_K u_k = u$. On the other hand,

$$a_1 \oplus u_1 \oplus \cdots \oplus a_k \oplus u_k = (a_1, u_1) \oplus \cdots \oplus (a_k, u_k) = (a, u) = a \oplus u,$$

and hence $a = a_1 \oplus \cdots \oplus a_k \oplus h(u_1, \ldots, u_k)$. Finally, by Lemma 3.4, $h(u_1, \ldots, u_k) \in M^r$, allowing us to replace $\oplus$ by $+$ in the last equation. □

By Fact 2.6, $\phi^K : U_H \to K$ is well-defined and $L^K = \ker(\phi)$ has rank s. Let $\{w_1, \ldots, w_s\}$ be a fixed set of generators for L^K. For every $i \in \{1, \ldots, s\}$, define

$$v_i = \big(- h(w_i^1, \ldots, w_i^k), w_i\big) \in M^r \times U_H = U, \text{ where } w_i = w_i^1 + \cdots + w_i^k,\ w_i^j \in H.$$

Claim 3.11. $\{v_1, \ldots, v_s\}$ *is a* $\mathbb{Z}$*-independent set of generators for* L.

Proof. We first show that each v_i belongs to $L = \ker(\phi)$. By Lemma 3.10, if $x = (a, u) \in M^r \times U_H$, where $u = u_1 + \cdots + u_k$, $u_i \in H$, we have

$$\phi(x) = \big(a + h(u_1, \ldots, u_k), \phi^K(u)\big). \tag{1}$$

It follows that $\phi(v_i) = 0$.

Next we show that for every $x \in L$, there are $l_1, \ldots, l_s \in \mathbb{Z}$ such that $x = l_1 v_1 + \cdots + l_s v_s$. Denote by $\tau : U \to U_H$ the group homomorphism $(a, u) \mapsto u$. Observe then, by (1), that $\phi(x) = 0$ implies $\phi^K\big(\tau(x)\big) = 0$. Hence, there are $l_1, \ldots, l_k \in \mathbb{Z}$ such that

$$\tau(x) = l_1 w_1 + \cdots + l_s w_s = l_1 \tau(v_1) + \cdots + l_s \tau(v_s) = \tau(l_1 v_1 + \cdots + l_s v_s).$$

That is, $\tau\big(x - (l_1 v_1 + \cdots + l_s v_s)\big) = 0 \in H$. Hence $x - (l_1 v_1 + \cdots + l_s v_s) \in H^G = M^r \times H$. Since $L \cap H^G = \{0\}$, we have $x - (l_1 v_1 + \cdots + l_s v_s) = 0$.

Finally, if $v_1, \ldots, v_s$ were not $\mathbb{Z}$-independent, then $l_1 v_1 + \cdots + l_s v_s = 0$, for some $l_i \in \mathbb{Z}$. Hence $l_1 w_1 + \cdots + l_s w_s = \tau(l_1 v_1 + \cdots + l_1 v_s) = 0$, a contradiction. □

Proof of Theorem 1.4. In Definition 3.7, we defined a convex $\bigvee$-definable subgroup $U \leqslant M^n$, and an onto group homomorphism $\phi : U \to G$ (Proposition 3.8). In Claim 3.11 we showed that $L = \ker(\phi) \leqslant U$ is a lattice of rank s.

Let $\Sigma = (H^G)^\Xi$, where Ξ is as in Lemma 3.6. Then Σ and $\phi_{\restriction\Sigma}$ are definable. Moreover, the coset equivalence relation induced by U/L on Σ is definable, since, for all $x, y \in \Sigma$, we have $x - y \in L \Leftrightarrow \phi_{\restriction\Sigma}(x) = \phi_{\restriction\Sigma}(y)$. By Lemma 3.9(ii), Σ contains a complete set S of representatives for U/L, and thus, by definable choice, there is a definable such set S. By [ElSt, Claim 2.7], $U/L = \langle S, +_S\rangle$ is a definable quotient group. The restriction of ϕ on S is a definable group isomorphism between $\langle S, +_S\rangle$ and G. By [ElSt, Remark 2.2(ii)], we are done. □

Corollary 3.12. *For every $k \in \mathbb{N}$, the k-torsion subgroup $G[k]$ of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^s$.*

Proof. By Theorem 1.4, we may assume that G is a definable set of representatives for U/L. For every $x \in G$ then, we have

$$\underbrace{x \oplus \cdots \oplus x}_{k-\text{times}} = 0 \Leftrightarrow \phi(kx) = 0 \Leftrightarrow kx \in L \cong \mathbb{Z}^s.$$

Hence, $x \in G[k]$ if and only if there are unique $l_1, \dots, l_n \in \mathbb{Z}$ such that $x = l_1 \frac{v_1}{k} + \cdots + l_s \frac{v_s}{k}$. Equivalently, since $x \in G$,

$$x = \phi(x) = \underbrace{\phi\left(\frac{v_1}{k}\right) \oplus \cdots \oplus \phi\left(\frac{v_1}{k}\right)}_{l_1-\text{times}} \oplus \cdots \oplus \underbrace{\phi\left(\frac{v_s}{k}\right) \oplus \cdots \oplus \phi\left(\frac{v_s}{k}\right)}_{l_s-\text{times}}.$$

Clearly then, the map $f : G[k] \to (\mathbb{Z}/k\mathbb{Z})^s$, defined by

$$f(x) = (l_1 \, modk, \dots, l_s \, modk),$$

is a well-defined, surjective group homomorphism. To see that it is injective, check that $f(x) = f(y)$ implies $x - y \in L$ and, hence, $x = \phi(x) = \phi(y) = y$. $\square$

4. On Pillay's Conjecture

In [BOPP], the existence of G^{00} was established for a group G definable in any o-minimal structure. Here, we compute the dimension of the compact Lie group G/G^{00}, for our fixed G and $\mathcal{M}$. The special case where G is definably compact constitutes Pillay's Conjecture for $\mathcal{M}$, proved separately in [ElSt, Proposition 5.1] and [Ons]. The reader is referred to [Pi2] for any terminology.

Proposition 4.1. *There is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s. Namely, assuming Conditions (1)-(6) from page 6, $G^{00} = M^r \times K^{00}$.*

Proof. For $i < \omega$, we define H_i inductively as follows: $H_0 = H$, and $H_{i+1} = \frac{1}{2} H_i$. Let also for every $i < \omega$, $(H^G)_i = M^r \times H_i$. Denote

$$B = \bigcap_{i<\omega} (H^G)_i = \bigcap_{i<\omega} (M^r \times H_i) = M^r \times \left(\bigcap_{i<\omega} H_i \right).$$

Note that, by [ElSt, Proof of Proposition 5.1], $\bigcap_{i<\omega} H_i = K^{00}$. Now, by Lemma 3.2, it is easy to see that B is a subgroup of G. By induction and [ElSt, Lemma 4.28], each H_i is generic in K. It follows that each $(H^G)_i$ is generic in G, and, thus, B has bounded index in G. Moreover, it is not hard to see that B is torsion-free, and, thus, by [BOPP], it must be the smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} with the logic topology is a connected compact abelian Lie group. Hence $G^{00} = B$ is torsion-free. By [BOPP], G^{00} is also divisible. It follows that for all k, the k-torsion subgroup of G/G^{00} is isomorphic to the k-torsion subgroup of G, which is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^s$, by Corollary 3.12. Thus, G/G^{00} is isomorphic to the real s-torus and has dimension s. $\square$

5. Linear o-minimal expansions of ordered groups

Here we show that Theorem 1.4 and Proposition 4.1 hold for a group G definable in a sufficiently saturated linear o-minimal expansion of an ordered group (see Propositions 5.7 and 5.5, respectively). The relation with the context of the previous sections is the following.

Fact 5.1 ([LP]). *Let $\mathcal{M} = \langle M, +, <, 0, S\rangle$ be a linear o-minimal expansion of an ordered group. Then $\mathcal{M}$ can be elementarily embedded into a reduct of an ordered vector space $\mathcal{N} = \langle N, +, <, 0, \{\lambda\}_{\lambda \in D}\rangle$ over an ordered division ring D.*

Let $\mathcal{M}$ and $\mathcal{N}$ be as above, sufficiently saturated, and G a t-connected, $\mathcal{M}$-definable group of dimension n. We may assume that $\mathcal{M}$ is a reduct of $\mathcal{N}$, and, thus, G is also $\mathcal{N}$-definable. Then Theorem 1.4 and Proposition 4.1 are true but with all definability stated with respect to $\mathcal{N}$. Namely, since H is $\mathcal{N}$-definable, $U =< M^r \times H >$ is $\bigvee$-definable in $\mathcal{N}$, and $G^{00} = \bigcap_{i<\omega} (M^r \times H_i)$ is type-definable in $\mathcal{N}$. We show however in Proposition 5.5 below that G^{00} is 'absolute'.

For a group G definable in a sufficiently saturated o-minimal structure $\mathcal{M}$, we denote by $G^{00}_{\mathcal{M}}$ the smallest type-definable in $\mathcal{M}$ subgroup of G of bounded index (which exists by [BOPP, Theorem 1.1]). The following fact was pointed out by Pillay. (See [HPP] for any terminology.)

Fact 5.2 ([HPP]). *Let T be an o-minimal theory, $\mathcal{M}$ a sufficiently saturated model of T, and G a group definable in $\mathcal{M}$. Assume:*

(1) For all definable $X \subseteq G$, either X or $G \setminus X$ is generic.

(2) There is a left-invariant Keisler measure on G.

Then (G^{00} exists and) G^{00} is torsion-free.

Fact 5.3 ([BOPP], Corollary 1.2). *Let G be a group definable in some sufficiently saturated o-minimal structure $\mathcal{M}$. Assume that X is a torsion-free, type-definable in $\mathcal{M}$, subgroup of G of bounded index. Then $X = G^{00}_{\mathcal{M}}$.*

Corollary 5.4. *Let K be an abelian, definably compact group, definable in a sufficiently saturated o-minimal expansion $\mathcal{M}$ of an ordered group. Let $\mathcal{N}$ be a sufficiently saturated o-minimal expansion of $\mathcal{M}$. Then $K^{00}_{\mathcal{M}}$ is torsion-free and $K^{00}_{\mathcal{M}} = K^{00}_{\mathcal{N}}$.*

Proof. We first verify that the assumptions of Fact 5.2 hold for K: (1) holds by [ElSt, Lemma 3.10], and (2) holds because K is abelian. It follows that $K^{00}_{\mathcal{M}}$ is torsion-free. By Fact 5.3, $K^{00}_{\mathcal{M}} = K^{00}_{\mathcal{N}}$. □

In what follows, let $\mathcal{M} = \langle M, +, <, 0, S\rangle$ be a sufficiently saturated linear o-minimal expansion of an ordered group, G a t-connected, $\mathcal{M}$-definable group of dimension n, and $\mathcal{N}$ a sufficiently saturated ordered vector space over an ordered division ring expanding $\mathcal{M}$ as in Fact 5.1.

We may assume that there is a $\mathcal{M}$-definable group K of dimension s such that Conditions (1)-(6) from page 6 hold. Indeed, those conditions were established directly using the general Fact 1.3 (and not the assumption that $\mathcal{M}$ were a vector space.)

Proposition 5.5. $G^{00}_{\mathcal{M}} = G^{00}_{\mathcal{N}}$. *Therefore, $G/G^{00}_{\mathcal{M}}$ equipped with the logic topology is a compact Lie group of dimension s.*

Proof. Since G and K are also $\mathcal{N}$-definable, we can find $H \subseteq K$ as in Step II of Section 3, which is $\mathcal{N}$-definable. By Proposition 4.1, $G^{00}_{\mathcal{N}} = M^r \times K^{00}_{\mathcal{N}}$. Since K is abelian, $K^{00}_{\mathcal{M}}$ is torsion-free, by Corollary 5.4. Therefore, $M^r \times K^{00}_{\mathcal{M}}$ is torsion-free. Since $K^{00}_{\mathcal{M}}$ has bounded index in K, easily $M^r \times K^{00}_{\mathcal{M}}$ has bounded index in G. By Fact 5.3, $G^{00}_{\mathcal{M}} = M^r \times K^{00}_{\mathcal{M}}$. But, by Corollary 5.4, $K^{00}_{\mathcal{M}} = K^{00}_{\mathcal{N}}$. It follows that $G^{00}_{\mathcal{M}} = M^r \times K^{00}_{\mathcal{M}} = M^r \times K^{00}_{\mathcal{N}} = G^{00}_{\mathcal{N}}$.

The rest follows from Proposition 4.1. □

In case G is definably compact, we obtain Pillay's Conjecture in the linear setting.

Corollary 5.6 (Pillay's Conjecture)**.** *Assume G is a t-connected, definably compact, $\mathcal{M}$-definable group of dimension s. Then there is a smallest type-definable in $\mathcal{M}$ subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s.*

Proposition 5.7. *$U =< M^r \times H >$ is $\bigvee$-definable in $\mathcal{M}$. Therefore, G is definably isomorphic to a definable quotient group U/L, where U is a $\bigvee$-definable in $\mathcal{M}$ subgroup of M^n and L is a lattice of rank s.*

Proof. Since K^{00} is type-definable in $\mathcal{M}$ and it is contained in the $\mathcal{N}$-definable H, by compactness, there exists some $\mathcal{M}$-definable subset X of H that contains K^{00}. On the other hand, since $K^{00} = \bigcap_{i<\omega} H_k$ is contained in X, by compactness again, there exists some H_k contained in X. We have $H_k \subseteq X \subseteq H$, and therefore $U_H =< X >$ is a $\bigvee$-definable in $\mathcal{M}$ subgroup of M^s. We have that $U =< M^r \times X >$ is a $\bigvee$-definable in $\mathcal{M}$ subgroup of M^n.

The rest follows from Theorem 1.4. □

5.1. **O-minimal fundamental group.** The *o-minimal fundamental group* $\pi_1(G)$ of G can be defined as in the classical case except that all paths and homotopies are taken to be definable. We refer the reader to [ElSt, Section 6] for precise definitions. An adapted argument from that reference would show that $\pi_1(G) \cong L$, but the result in fact follows directly from [EdEl].

Proposition 5.8. *$\pi_1(G) \cong L$.*

Proof. By [ElSt, Corollary 1.5], there is $l \in \mathbb{N}$ such that $\pi_1(G) \cong \mathbb{Z}^l$ and $G[k] \cong (\mathbb{Z}/k\mathbb{Z})^l$. By Corollary 3.12, $l = s$. □

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