# A SEMI-LINEAR GROUP WHICH IS NOT AFFINE

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Abstract. In this short note we provide an example of a semi-linear group $G$ which does not admit a semi-linear affine embedding; in other words, there is no semi-linear isomorphism between topological groups $f : G \to G' \subseteq M^m$, such that the group topology on $G'$ coincides with the subspace topology induced by $M^m$.

Let $\mathcal{M}$ be an o-minimal structure. By "definable" we mean "definable in $\mathcal{M}$" possibly with parameters. A group $G = \langle G, \oplus, e_G \rangle$ is said to be definable if both its domain and its group operation are definable.

By [Pi], we know that every definable group $G \subseteq M^n$ can be equipped with a unique definable manifold topology that makes it into a topological group. We refer to this topology as the *group topology of* $G$. It is shown in [Pi] that the group topology of $G$ coincides with the subspace topology induced by $M^n$ on a large subset $V$ of $G$ $\big(\dim(G \setminus V) < \dim(G)\big)$. We call $G$ *affine* if the group topology of $G$ coincides with the subspace topology on (the whole of) $G$.

**Question.** *Is every definable group affine (up to definable isomorphism)?*

*Remark* 0.1. (i) An isomorphism between two topological groups is a group isomorphism which is also a topological homeomorphism.

(ii) By [ElSt, Remark 2.2], the Question can be restated as follows: *Given a definable group $G \subseteq M^n$, is there a definable injective map $\tau : G \to M^m$, $m \in \mathbb{N}$, such that the topology on $\tau(G)$ induced by the group topology of $G$ via $\tau$ coincides with the subspace topology on $\tau(G)$ induced by $M^m$?* If yes, then such a $\tau$ is called an *affine embedding of* $G$.

The Question admits an affirmative answer in case $\mathcal{M}$ expands a real closed field, by [BO, Proof of Lemma 10.4] and [vdD, Chapter 10, Theorem (1.8)]. In fact, these references concern affine embeddings of "abstract-definable manifolds", and the work in [BO] also yields affine embeddings which are moreover diffeomorphisms. The original proof of embedding semi-algebraic manifolds was given in [Ro].

We present here an example of a semi-linear group which is not affine. A *semi-linear* group is a group definable in an ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{d\}_{d \in D} \rangle$ over an ordered division ring $D$. Semi-linear groups were studied in [ElSt] and [El]. The main property of such an $\mathcal{M}$ that we use below is that every definable function $f : A \subseteq M^n \to M^m$ is piecewise-linear (PL); that is, there is a partition of $A$ into finitely many definable sets $A_i$, $i = 1, \dots, k$, such that for

*Date*: June 21, 2008.

2000 *Mathematics Subject Classification.* 03C64, 57Q35.

*Key words and phrases.* O-minimal structures, PL-embeddings.

Research supported by European Commission's Research Training Network MRTN-CT-2004-512234 (MODNET).

each $i = 1, \ldots, k$, the following holds: there is an $n \times m$ matrix $\lambda$ with entries from $D$, and an element $a \in M^m$, such that for every $x \in A_i$, $f(x) = \lambda x + a$. The group we present in our example below is definable in an ordered divisible abelian group $\langle M, +, <, 0\rangle$, which is naturally an ordered vector space over $\mathbb{Q}$.

For $x, y \in M$, we define:

$$x \prec_D y \Leftrightarrow \forall d \in D, d|x| < |y|.$$

**Example 0.2.** Let $\mathcal{M} = \langle M, +, <, 0\rangle$ be an ordered divisible abelian group with the following property: there are $a, b, c > 0$ in $M$ such that $b \prec_{\mathbb{Q}} c \prec_{\mathbb{Q}} a$. In particular, there is no definable function from $[0, b)$ onto $[0, c)$, and $\forall n \in \mathbb{N}$, $nc < a$.

Let $S = [0, a) \times [0, b)$ and $L = \mathbb{Z}(a, 0) + \mathbb{Z}(a - c, b)$ be the lattice in $M^2$ generated by the elements $(a, 0)$ and $(a - c, b)$. Define $G = \langle S, \oplus, 0\rangle$, where

$$x \oplus y = z \Leftrightarrow x + y - z \in L.$$

By [ElSt, Claim 2.7(ii)], $G$ is definable.

**Notation.** By $\lim^G$ we denote a limit with respect to the group topology of $G$. A path is a definable continuous map with respect to the subspace topology, and a $G$-path is a definable continuous map with respect to the group topology of $G$.

**Claim.** *There is no definable injective map $\tau : G \to M^m$, $m \in \mathbb{N}$, such that the induced topology on $\tau(G)$ coincides with the subspace topology.*

*Proof.* Assume, towards a contradiction, that there is such a $\tau$. For every element $t \in [0, a)$, consider the one-to-one $G$-path

$$\phi_t : [0, b) \to \{t\} \times [0, b), \text{ with } \phi_t(x) = (t, x).$$

By definition of $G$, we see that for every $t \in [0, a - c]$, $\lim^G_{x \to b} \phi_t(x) = (t + c, 0)$. Therefore, for every $t \in [0, a - c]$, $\tau(\phi_t)$ is a path in $M^m$ with the property:

$$\lim_{x \to b} \tau\big(\phi_t(x)\big) = \tau\big((t + c, 0)\big).$$

Consider now the image $\tau\left([0, a - c] \times \{0\}\right)$. By assumption, it contains an infinite number of elements of the form $\tau\big((nc, 0)\big)$, $n \in \mathbb{N}$. Since $\tau$ is piecewise-linear, there must exist some $n \in \mathbb{N}$ such that $\tau$ is linear on $\big[nc, (n+1)c\big) \times \{0\}$. Hence the image $\tau\Big(\big[nc, (n + 1)c\big) \times \{0\}\Big)$ is in bijection with an interval $J \subseteq M$ via some projection map $\pi_i : M^m \to M$ onto one of the $m$ coordinates. Since $\tau(\phi_{nc}) : [0, b) \to M^m$ is a path with

$$\tau\big(\phi_{nc}(0)\big) = \tau\big((nc, 0)\big) \text{ and } \lim_{x \to b} \tau\big(\phi_{nc}(x)\big) = \tau\Big(\big((n + 1)c, 0\big)\Big),$$

the image of $\tau\Big(\phi_{nc}\big([0, b)\big)\Big)$ under $\pi_i$ covers $J$. Clearly, then, there is a definable map from $\tau\Big(\phi_{nc}\big([0, b)\big)\Big)$ onto $J$, and, therefore, there is a definable map from $[0, b)$ onto $J$. It follows that there is a definable map from $[0, b)$ onto $\big[nc, (n + 1)c\big)$, and, therefore, a definable map from $[0, b)$ onto $[0, c)$, a contradiction. $\square$

Note that the group $G$ given in our example is definably compact ([PeS]). If a definable group $G$ is not definably compact, then by [PeS], $G$ contains a torsion-free one-dimensional definable subgroup. If in particular $G$ is torsion-free, as well as semi-linear, then by [EdEl], $G$ is definably isomorphic to $\langle M^n, +\rangle$, where $n$ is the dimension of $G$. That is, a torsion-free semi-linear group admits an affine embedding.

We conclude with some remarks on the classical literature for real PL-manifolds. These are abstract-definable manifolds (in the sense of [BO]) definable in $\langle \mathbb{R}, <, +, 0, \{r\}_{r\in\mathbb{R}}\rangle$. It is well-known that every real PL-manifold of dimension $n$ admits an affine embedding into $\mathbb{R}^{2n}$ (see [ReSk, Theorem 3.1], or [Wh] for the original proof). The dimension $2n$ is the best possible in general, but it can be dropped under further topological assumptions (see, for example, [PWZ]). On the other hand, stronger notions of embeddings have also been investigated, such as "isometric" embeddings. In [BuZa], it is shown that every orientable real PL-manifold of dimension 2 admits an isometric affine embedding into $\mathbb{R}^3$. The generalization of this statement to manifolds of higher dimension is open. The reader is referred to [ReSk] for a survey on embeddings of real PL-manifolds, whereas a classical textbook for piecewise-linear topology over the reals is [RouSa].

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