GROUPS DEFINABLE IN LINEAR O-MINIMAL STRUCTURES

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Abstract

by

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Let $\mathcal{M} = \langle M, +, <, 0, \dots \rangle$ be a linear o-minimal expansion of an ordered group, and $G = \langle G, \oplus, e_G \rangle$ an n-dimensional group definable in $\mathcal{M}$. We show that if G is definably connected with respect to the t-topology, then it is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup U of $\langle M^n, + \rangle$ and a lattice L of rank equal to the dimension of the 'compact part' of G. This is suggested as a structure theorem analogous to the classical theorem that every connected abelian Lie group is Lie isomorphic to a direct sum of copies of the additive group $\langle \mathbb{R}, + \rangle$ of the reals and the circle topological group S^1. We then apply our analysis and prove Pillay's Conjecture and the Compact Domination Conjecture for a saturated $\mathcal{M}$ as above. En route, we show that the o-minimal fundamental group of G is isomorphic to L. Finally, we state some restrictions on L.

To my parents, Eleftherios and Maria,

and my brother, Dimitris

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CHAPTER 1

INTRODUCTION

The main motivation for this work arises from the intuition that a group G definable in an o-minimal structure $\mathcal{M}$ should 'resemble' a real Lie group. The foremost supporting evidence for this intuition was given in [Pi1], where it was shown that every such group G can be equipped with a (unique) 'definable manifold' structure over $\mathcal{M}$ that makes it into a topological group. A series of definable analogues of classical theorems in the o-minimal context followed and the intuition was recently formalized in Pillay's Conjecture, stated below.

Let $\mathcal{M}$ be a big saturated o-minimal structure. Let G be a group definable in $\mathcal{M}$ equipped with its definable manifold topology, henceforth called 't-topology'. The group G is *definably compact* ([PeS]) if for every definable continuous map $\sigma : (a, b) \subseteq M \to G$, the limit $lim_{x \to a} \sigma(x)$ exists in G, taken with respect to the t-topology of G. A set $X \subseteq M^n$ is *type-definable* if it is the intersection of $< |M|$ many definable sets. A subgroup H of G has *bounded index* if $|G/H| < |M|$. If H has bounded index and $\pi : G \to G/H$ denotes the canonical homomorphism, then a set $A \subseteq G/H$ is *closed in the logic topology* if $\pi^{-1}(A) \subseteq G$ is type-definable ([LaPi]).

Pillay's Conjecture ([Pi2]). *Assume G is a definably compact group definable in a saturated o-minimal structure $\mathcal{M}$, and $\dim(G) = n$. Then there is a smallest* *type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension n.*

In [HPP] Pillay's Conjecture was proved for $\mathcal{M}$ an o-minimal expansion of an ordered field.

The next conjecture is intended to formalize the further intuition that $\pi : G \to G/G^{00}$ is a kind of 'standard part map'. Let **Haar** denote the unique normalized Haar measure on G/G^{00}.

Compact Domination Conjecture ([HPP]). *Assume G satisfies the assumptions and the conclusion of Pillay's Conjecture. Then for all definable subsets $X \subseteq G$,*

$$\dim(X) < n \Rightarrow \mathbf{Haar}\big(\pi(X)\big) = 0.$$

In this case, we say that G is *compactly dominated.*

In [HPP] it was shown that compact domination holds if $\dim(G) = 1$ or if G has 'very good reduction'.

In this dissertation we analyze groups definable in 'linear' o-minimal structures. For these groups we first prove a structure theorem analogous to the classical theorem that every connected abelian Lie group is Lie isomorphic to a direct sum of copies of the additive group $\langle \mathbb{R}, + \rangle$ of the reals and the circle topological group S^1. We then apply our analysis and settle positively the above two conjectures in this context.

We briefly introduce the terminology required to state our results. The terminology is further explained in the following chapters and the references given therein.

An o-minimal expansion $\mathcal{M} = \langle M, <, +, \dots \rangle$ of an ordered group is called *linear* ([LP]) if for every definable function $f : A \subseteq M^n \to M$, there is a partition of A into finitely many definable A_i, such that for each i, if $x, y, x+t, y+t \in A_i$, then

$$f(x+t) - f(x) = f(y+t) - f(y).$$

The main example of a linear o-minimal structure is that of an ordered vector space $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$ over an ordered division ring D.

Until the end of this Introduction, $\mathcal{M} = \langle M, <, +, 0, \dots \rangle$ denotes a linear o-minimal expansion of an ordered group, and 'definable' means 'definable in $\mathcal{M}$ with parameters'.

Let $G = \langle G, \oplus, e_G \rangle$ be a definable group of dimension n equipped with the t-topology. Assume G is definably connected, that is, G is not the disjoint union of any two open definable proper subsets. It is known that G is abelian. (See for example [PeSt, Corollary 5.1].) By [PeS], if G is not definably compact, then it has a 1-dimensional torsion-free definable subgroup. By induction on the dimension of G, there are definable subgroups $\{e_G\} = G_0 < G_1 < \ldots < G_r \leqslant G$, such that G/G_r is definably compact and G_{i+1}/G_i is a 1-dimensional torsion-free group, for $i = 0, \dots, r-1$. By [EdEl1], the torsion-free subgroup G_r of G is definably isomorphic to $M^r = \langle M^r, +, 0 \rangle$. We call G/M^r the *compact part* of G. The dimension of G/M^r is then $s = n - r$.

A *lattice* L of rank $m \leq n$ is a subgroup of $M^n = \langle M^n, + \rangle$ generated by m $\mathbb{Z}$-linearly independent elements of M^n. If $U \leqslant M^n$ is a subgroup of M^n and $L \leqslant U$ is a lattice, then U/L is called a *definable quotient group* if there is a definable complete set $S \subseteq U$ of representatives for U/L, such that the induced group structure $\langle S, +_S \rangle$ is definable. In this case, we identify U/L with $\langle S, +_S \rangle$.

Let $\{X_k : k < \omega\}$ be a collection of definable subsets of M^n. Assume that $U = \bigcup_{k<\omega} X_k$ is equipped with a binary map $\cdot$ so that $\langle U, \cdot \rangle$ is a group. Then U is called *a* $\bigvee$*-definable group* ([PeSt]) if, for all $i, j < \omega$, there is $k < \omega$, such that $X_i \cup X_j \subseteq X_k$ and the restriction of $\cdot$ to $X_i \times X_j$ is a definable function into M^n.

For $l, m \in \mathbb{Z} \setminus \{0\}$, and $x = (x_1, \ldots, x_n) \in M^n$, let $\frac{l}{m}x$ be the unique $y = (y_1, \ldots, y_n) \in M^n$ such that for all i, $lx_i = my_i$. A set $X \subseteq M^n$ is called *convex* if for all $x, y \in X$ and $q \in \mathbb{Q} \cap [0, 1]$, $qx + (1 - q)y \in X$.

We show:

Theorem 1 - The Structure Theorem. *Let G be a definably connected group definable in a linear o-minimal expansion $\mathcal{M}$ of an ordered group. Assume that the dimension of G is n, and that the dimension of the compact part of G is s. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant \langle M^n, + \rangle$ and a lattice $L \leqslant U$ of rank s.*

Theorem 1 has the following corollary.

Theorem 2. *Let G be a group definable in a saturated linear o-minimal expansion $\mathcal{M}$ of an ordered group. Assume that the dimension of the compact part of G is s. Then there is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s.*

If, in addition, G is definably compact, then $s = n$ and Theorem 2 is Pillay's Conjecture for $\mathcal{M}$ a saturated linear o-minimal expansion of an ordered group.

Note that regarding Theorems 1 and 2 for G definably compact and $\mathcal{M}$ archimedean, an independent work appears also in [Ons].

Using the analysis from the proof of Theorem 1, we show the following stronger version of compact domination.

Theorem 3. *Let G be a definably compact group definable in a saturated linear o-minimal expansion $\mathcal{M}$ of an ordered group. Then for all definable subsets $X \subseteq G$ defined in* any *o-minimal expansion of $\mathcal{M}$,*

$$\dim(X) < n \Rightarrow \mathbf{Haar}\big(\pi(X)\big) = 0.$$

Note that the assumption of definable connectedness in Theorems 2 and 3 above is at no loss of generality, easily, by [Pi1].

The *o-minimal fundamental group* $\pi_1(G)$ of G can be defined as in the classical case except that all paths and homotopies are taken to be definable.

Theorem 4. *Let G be a definably connected group definable in a linear o-minimal expansion $\mathcal{M}$ of an ordered group. Assume that the dimension of the compact part of G is s. Then $\pi_1(G) \cong L \cong \mathbb{Z}^s$, where L is as in the Structure Theorem.*

The Structure Theorem can be seen as a procedure for recovering a lattice L, given the definable group G. We investigate a partial 'converse' of this procedure; namely, we provide necessary and sufficient conditions that a lattice L must satisfy so that the following question admits a positive answer.

Question 5. *Given a lattice $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n \leqslant M^n$ of rank n, is there a convex $\bigvee$-definable subgroup U of $\langle M^n, +\rangle$ such that U/L is a definably connected definably compact definable quotient group of dimension n?*

The structure of this dissertation is as follows.

In Chapter 2, we recall some basic facts about groups definable in o-minimal structures. We also introduce our terminology and set the scene for our results.

In Chapter 3, we prove Theorems 1, 2, and 4 for G definably compact and $\mathcal{M}$ an ordered vector space over an ordered division ring. See Theorems 3.1.2, 3.3.1 and 3.4.13, respectively. The material of this chapter appears in [ElSt].

In Chapter 4, we extend our proofs to the case where G is not necessarily definably compact (Theorems 4.1.5, 4.2.25 and 4.2.28), and $\mathcal{M}$ is any linear o-minimal expansion of an ordered group (Theorems 4.3.8, 4.3.6 and 4.3.11). The material of this chapter appears in [El1].

In Chapter 5, we prove Theorem 3. The material of this chapter appears in [El2].

In Chapter 6, we address Question 5.

Throughout this dissertation, unless stated otherwise, we denote by $\mathcal{M} = \langle M, <, +, 0, \ldots \rangle$ a big saturated o-minimal expansion of an ordered group, and 'definable' means 'definable in $\mathcal{M}$ with parameters'.

We omit bars from tuples in M^n, for $n \in \mathbb{N}$.

We assume that the reader has some familiarity with the basics of o-minimality. A standard reference is [vdD].

CHAPTER 2

PRELIMINARIES

In this chapter we fix our terminology and prove a few lemmas to be used in the sequel. Our discussion centers around the following four topics. In Section 2.1, we recall some basic facts about groups definable in o-minimal structures. In Section 2.2, we introduce the notion of a definable quotient group. In Section 2.3, we discuss definability in an ordered vector space over an ordered division ring. In Section 2.4, we present a construction of a standard part map.

2.1 Definable groups

We begin this section by collecting some facts about groups definable in o-minimal structures from [Pi1]. A group $G = \langle G, \oplus, e_G \rangle$ is said to be definable if both its domain G and the graph of its group operation are definable subsets of M^n and M^{3n}, for some n, respectively. A topological group is a group equipped with a topology in a way that makes its multiplication and inverse operations continuous. An isomorphism between two topological groups G and G' is at the same time a group isomorphism and a topological homeomorphism between G and G'.

For the rest of Section 2.1, let $G = \langle G, \oplus, e_G \rangle$ be a definable group with $G \subseteq M^n$ and $\dim(G) = m \leq n$.

A definable manifold topology on G is a Hausdorff topology on G satisfying the following: there is a finite set $\mathcal{A} = \{\langle S_i, \phi_i\rangle : i \in J\}$ such that

(i) for each $i \in J$, S_i is a definable open subset of G and $\phi : S_i \to M^m$ is a definable homeomorphism between S_i and $K_i := \phi(S_i) \subseteq M^m$,

(ii) $G = \cup_{i\in J} S_i$, and

(iii) for all $i, j \in J$, if $S_i \cap S_j \neq \emptyset$, then $S_{ij} := \phi_i(S_i \cap S_j)$ is a definable open set and $\phi_j \circ \phi_i^{-1} \restriction_{S_{ij}}$ is a definable homeomorphism onto its image.

We fix our notation for a definable manifold topology on G as above. Moreover, we refer to each ϕ_i as *a chart map*, to each $\langle S_i, \phi_i\rangle$ as *a definable chart on* G, and to $\mathcal{A}$ as *a definable atlas on* G for this topology. If all of G, S_i and ϕ_i, $i \in J$, are A-definable, for some $A \subseteq M$, we say that G admits an A-definable manifold structure.

The main result in [Pi1] is the following.

Fact 2.1.1. *There is a unique definable manifold topology that makes G into a topological group. We refer to this topology as the* t-topology (on G)*, or as the* t_G-topology *if more than one definable group is present.*

Remark 2.1.2. (i) Whenever $f : K \to K'$ is a definable bijection between two definable subsets of cartesian powers of M, and $K = \langle K, \star, e\rangle$ is a definable group, f induces on K' a definable group structure $\langle K', \circ, f(e)\rangle$, where $\circ$ is defined as follows: $x \circ y = f\big(f^{-1}(x) \star f^{-1}(y)\big)$. Clearly, f is a definable group isomorphism between K and K'. Moreover, if K is a topological group, f induces on K' a group topology that makes f a definable isomorphism between topological groups.

(ii) By uniqueness of the t-topology, a definable group isomorphism between two definable groups also preserves their associated t-topologies, and, thus, it is a definable isomorphism between the corresponding topological groups.

Let $X \subseteq M^n$ be an A-definable set, for some set of parameters $A \subseteq M$. Then $a \in X$ is called a *dim-generic element of X over A* if $\dim(a/A) = \dim(X)$. If $A = \emptyset$, a is called a *dim-generic element of* X. A definable set $V \subseteq X$ is called *large in* X if $\dim(X \setminus V) < \dim(X)$. Equivalently, V contains all dim-generic elements of X over A, for any A over which X and V are defined. We freely use any properties of dim-generic elements of definable groups from [Pi1].

We make a few comments about the two topologies on G, the t-topology on the one hand, and the subspace topology induced by M^n, henceforth called the *$\mathcal{M}$-topology*, on the other. First, $\oplus$ is continuous with respect to the t-topology, and $+\restriction_A$ with respect to the $\mathcal{M}$-topology, for $A = \{(x, y) \in G \times G : x + y \in G\}$. Moreover, by [Pi1], there is a large subset W^G of G which is open in both the t- and $\mathcal{M}$- topologies, such that, for all $Z \subseteq W$, Z is open in the t-topology if and only if Z is open in the $\mathcal{M}$-topology. For $a \in M^n$ and $r > 0$ in M, we denote by $\mathcal{B}^n_a(r)$ the open n-box centered at a of size r,

$$\mathcal{B}^n_a(r) := a + (-r, r)^n = \{a + \varepsilon : \varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in M^n,\ \varepsilon_i \in (-r, r)\},$$

whereas for $a \in G$, by a *t-neighborhood* V_a of a (in G) we mean a definable open neighborhood of a in G with respect to the t-topology. We omit the index 'n' from $\mathcal{B}^n_a(r)$ when it is clear that $a \in M^n$. Note that if $\dim(G) = n$ and $a \in W^G$, then for sufficiently small r, $\mathcal{B}_a(r)$ is also a t-neighborhood of a in G.

In general, we distinguish between topological notions when taken with respect to the product topology of M^n and when taken with respect to the t-topology on G, by adding an index 't' in the latter case. For example, we write $\overline{A}^t$, $\mathrm{Int}(A)^t$, $\mathrm{bd}(A)^t = \overline{A}^t \setminus \mathrm{Int}(A)^t$ to denote, respectively, the closure, interior and boundary of a set $A \subseteq G$ with respect to the t-topology. Similarly, $A \subseteq G$ is called *t-open*, *t-closed*, or *t-connected*, if it is definable and, respectively, open, closed, or definably connected with respect to the t-topology. We call a function $f : M^n \to G$ *t-continuous* if it is continuous with respect to the t-topology in the range. Accordingly, $\lim^t_{x \to x_0} f(x)$ denotes the limit of f with respect to the t-topology in the range. Definable compactness of a definable group G is always meant with respect to the t-topology, that is ([PeS]): for every definable t-continuous map $\sigma : (a, b) \subseteq M \to G$, $-\infty \leq a < b \leq \infty$, there are $c, d \in G$ such that $\lim^t_{x \to a^+} \sigma(x) = c$ and $\lim^t_{x \to b^-} \sigma(x) = d$. By a *t-path* we mean a definable t-continuous function $\gamma : [p, q] \to G$, $p, q \in M$, $p \leq q$, and by a *path (in M^n)*, just a definable continuous function $\gamma : [p, q] \to M^n$, $p, q \in M$, $p \leq q$. A *(t-)loop* is then a (t-)path γ with $\gamma(p) = \gamma(q)$. The concatenation of two (t-)paths $\gamma : [0, p] \to M^n$ (G) and $\delta : [0, q] \to M^n$ (G) with $\gamma(p) = \delta(0)$ is a (t-)path $\gamma \vee \delta : [0, p + q] \to M^n$ (G) with:

$$(\gamma \vee \delta)(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, p], \\ \delta(t - p) & \text{if } t \in [p, p + q]. \end{cases}$$

The image of a (t-)path γ is denoted by $\mathrm{Im}(\gamma)$. Finally, a definable subset of M^n (G) is called *(t-)path-connected* if any two points of it can be connected by a (t-)path.

Notice the systematic omitting of the words ‘definable’ or ‘definably’ in our terminology.

Remark 2.1.3. *If G is t-connected, then it is t-path-connected.* In o-minimal expansions of ordered groups, definable connectedness is equivalent to definable path-connectedness. Recall, G can be covered by finitely many t-open sets S_i, that can be taken to be t-connected, each of which is homeomorphic to a definably connected and, thus, path-connected subset of M^m. The homeomorphisms imply that the S_i's are t-path-connected, and, thus, so is G.

Of course, G, as a definable subset of M^n, has finitely many path-connected components.

The following sets are going to be important for our work.

Definition 2.1.4. Let W^G be a fixed definable large subset of G which is both t-open and open in the $\mathcal{M}$-topology, such that, for all $Z \subseteq W$, Z is t-open if and only if Z is open in the $\mathcal{M}$-topology. Let

$$V^G := \{a \in W^G : \text{there is a } t\text{-neighborhood}\, V_a \text{ of } a \text{ in}\, G,$$

$$\text{such that}\, \forall x, y \in V_a,\ x \ominus a \oplus y = x - a + y\}.$$

Lemma 2.1.5. *(i) V^G is definable.*

(ii) V^G is t-open and, thus, also open in the $\mathcal{M}$-topology of G.

Proof. (i) Recall that G admits a definable atlas $\mathcal{A} = \{\langle S_i, \phi_i\rangle : i \in J\}$. Thus, for every element $a \in S_i \subseteq G$, the existence of a t-neighborhood V_a of a in G amounts to the existence of some $r \in M$ such that the image of a under $\phi_i : S_i \to K_i$ belongs to the open m-box $\mathcal{B}_{\phi_i(a)}(r) \subseteq K_i$ in M^m.

(ii) Let $v \in V^G$ and a t-neighborhood $V_v \subseteq G$ of v be such that $\forall x, y \in V_v$, $x \ominus v \oplus y = x - v + y$. By the definable manifold structure of G and Remark 2.1.2, we may assume that $V_v = \mathcal{B}^m_v(r)$ for some $r > 0$ in M. We claim that $\forall u \in \mathcal{B}^m_v(r)$, $u \in V^G$. To see that, let $u \in \mathcal{B}^m_v(r)$ and pick $\delta > 0$ in M such that $\mathcal{B}^m_u(\delta) \subseteq \mathcal{B}^m_v(r)$. Let $x, y \in \mathcal{B}^m_u(\delta)$. Then $v + x - u \in \mathcal{B}^m_v(r)$ and

$$(v + x - u) \ominus v \oplus u = v + x - u - v + u = x.$$

Therefore, $x \ominus u = (v + x - u) \ominus v$. It follows that

$$x \ominus u \oplus y = (v + x - u) \ominus v \oplus y = v + x - u - v + y = x - u + y.$$

$\square$

2.2 Definable quotient groups

In this section we introduce and discuss the particular kind of a definable group that we are interested in.

Recall, M is equipped with the order topology. $M^n = \langle M^n, + \rangle$ is then the topological group whose group operation is defined point-wise, that has $0 = (0, \ldots, 0)$ as its unit element, and whose topology is the product topology. If L is a subgroup of M^n, we denote by E_L the equivalence relation on M^n induced by L; namely, $xE_Ly \Leftrightarrow x - y \in L$. For $U \subseteq M^n$, we let $E_L^U := E_L \restriction_{U \times U}$ and $U/L := U/E_L^U$. The elements of U/L are denoted by $[x]_L^U$, $x \in U$. If $U \leqslant M^n$ is a subgroup of M^n, then it is a topological group equipped with the subspace topology. If, moreover, $L \leqslant U$ is a subgroup of U, then $U/L = \langle U/L, +_{U/L}, [0]_L^U \rangle$ is the *quotient topological group*, whose topological and group structure are both induced by the canonical surjection $q : U \to U/L$. If $S \subseteq U$ is a complete set of representatives for E_L^U (that is, it contains exactly one representative for each equivalence class), then the bijection $U/L \ni [x]_L^U \mapsto x \in S$ induces on S a topological group structure $\langle S, +_S \rangle$:

(i) $x +_S y = z \Leftrightarrow [x]_L^U +_{U/L} [y]_L^U = [z]_L^U \Leftrightarrow (x + y) \, E_L^U \, z$, and

(ii) $A \subseteq S$ is open in the *quotient topology on* S if and only if $q^{-1}(A)$ is open in U.

Definition 2.2.1. Let $U \subseteq M^n$ and $L \leqslant M^n$. Then U/L is said to be *a definable quotient* if there is a definable complete set $S \subseteq U$ of representatives for E_L^U. If, in addition, $L \leqslant U \leqslant M^n$ and for some S as above $+_S$ is definable, then the topological group U/L is called *a definable quotient group*.

Convention. We identify a definable quotient group U/L with $\langle S, +_S \rangle$, for some fixed, definable complete set of representatives S for E_L^U, via the bijection $U/L \ni [x]_L^U \mapsto x \in S$.

That is, a definable quotient group U/L is a definable group and, thus, it can be equipped with the t-topology. As it is shown in Claim 2.2.4 below, the t-topology on U/L coincides with the quotient topology on it in the case where L is a 'lattice'. Let us define the notion of a lattice. The abelian subgroup of M^n generated by the elements $v_1, \dots, v_m \in M^n$ is denoted by $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$. If $v_1, \dots, v_m$ are $\mathbb{Z}$-linearly independent, then the free abelian subgroup $\mathbb{Z}v_1 + \ldots + \mathbb{Z}v_m$ of M^n is called *a lattice of rank m*.

Moreover, it is shown in Claim 2.2.4 that, if L is a lattice, $L \leqslant U \leqslant M^n$, and U/L is a definable quotient, then U can be generated by some definable subset H of it, that is, it has form $U = \bigcup_{k<\omega} H^k$, where $H^k := \underbrace{H + \ldots + H}_{k-\text{times}}$. Such a group U is called '$\bigvee$-definable' in [PeSt], 'locally definable' in [Ed3], and 'Ind-definable' in [HPP].

Definition 2.2.2 ([PeSt]). Let $\{X_k : k < \omega\}$ be a collection of definable subsets of M^n. Assume that $U = \bigcup_{k<\omega} X_k$ is equipped with a binary map $\cdot$ so that $\langle U, \cdot \rangle$ is a group. U is called *a $\bigvee$-definable group* if, for all $i, j < \omega$, there is $k < \omega$, such that $X_i \cup X_j \subseteq X_k$ and the restriction of $\cdot$ to $X_i \times X_j$ is a definable function into M^n.

The reader is referred to [PeSt] for a more detailed discussion of $\bigvee$-definable groups. The main fact about a $\bigvee$-definable group $U = \bigcup_{k<\omega} X_k$ that we use here is that every definable subset of U is contained in some X_k, $k < \omega$, by use of compactness.

2.2.1 Definable quotients and $\bigvee$-definable groups

We first prove a general statement about quotient topological groups:

Lemma 2.2.3. *Let $L \leqslant U \leqslant M^n$, and $S \subseteq U$ a complete set of representatives for E_L^U. Let $R \subseteq S$ be open in U. Then, for any $D \subseteq R$, D is open in U if and only if D is open in the quotient topology on S.*

Proof. First, we claim that every $A \subseteq S$ open in U is open in the quotient topology on S. Let $A \subseteq S$ be open in U. We need to show that $q^{-1}(A)$ is open in U. But $q^{-1}(A) = \bigcup_{x\in L}(x + A)$. Since $\langle U, +, 0\rangle$ is a topological group, we have that for all $x \in L$, $x + A$ is open in U. Thus, $\bigcup_{x\in L}(x + A)$ is open in U.

Now let $R \subseteq S$ be open in U, and $D \subseteq R$. The left-to-right direction is given by the previous paragraph. For the right-to-left one, assume D is open in the quotient topology on S, that is, $q^{-1}(D) = \bigcup_{x\in L}(x + D)$ is open in U. Since R is also open in U, it suffices to show

$$D = q^{-1}(D) \cap R.$$

$D \subseteq q^{-1}(D) \cap R$ is clear. Now, let $a \in q^{-1}(D) \cap R$. We have $a = x + d = r$, for some $x \in L$, $d \in D$ and $r \in R$. Thus, $d - r \in L$. Since S is a complete set of representatives for E_L^U, and $d, r \in S$, we have $d = r$. Thus, $x = 0$ and $a = d \in D$. □

Claim 2.2.4. *Let $L \leqslant U \leqslant M^n$, with L a lattice of rank $m \leq n$. Suppose U/L is a definable quotient, $S \subseteq U$ is a definable complete set of representatives for E_L^U, and* $\dim(S) = n$. *Then:*

(i) U is a $\bigvee$-definable group,

(ii) U/L is a definable quotient group, and

(iii) the quotient topology on S coincides with the t-topology on S.

Proof. (i) We have, $\forall x \in U, \exists y \in S, x - y \in L$. Let $L = \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_n$, and for each $k < \omega$,

$$L_k := \{l_1 v_1 + \ldots + l_n v_n \in L : -k \leq l_i \leq k\}$$

and

$$U_k := \{x \in M^n : \exists y \in S,\, x - y \in L_k\} = S + L_k.$$

Clearly, all L_k and U_k are definable. Moreover, $U = \bigcup_{k<\omega} U_k$. Since $\forall k$, $U_k \subseteq U_{k+1}$, it is easy to see that U is $\bigvee$-definable.

(ii) Since $U = \bigcup_{k<\omega} U_k$ is $\bigvee$-definable and $S + S \subseteq U$, there must be some $K < \omega$ such that $S + S \subseteq U_K$. It follows that $+_S$ is definable, since $\forall x, y, z \in S$, $x +_S y = z \Leftrightarrow (x + y)\, E_L^{U_K}\, z \Leftrightarrow x + y - z \in L_{2K}$.

(iii) Since $\langle S, +_S \rangle$ is a topological group with respect to the quotient topology as well as with respect to the t-topology, it suffices to show that the two topologies coincide on a large subset Y of S. Let W^S be as in Definition 2.1.4, that is, W^S is a large t-open and open subset of S where the t- and $\mathcal{M}$- topologies coincide.

Let $R := \mathrm{Int}(S)$ denote the interior of S in M^n. Then R is a large definable subset of S which is also open in U. It follows that $Y := R \cap W^S \subseteq M^n$ is a large definable subset of S, and that for every $D \subseteq Y$, we have: $D \subseteq R$ is open in the quotient topology on S if and only if (by Lemma 2.2.3) D is open in U if and only if $D \subseteq W^S$ is open in the t-topology on S. $\square$

2.3 Definability in $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$

In Section 2.3, we fix $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$ to be a saturated ordered vector space over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1 \rangle$.

By [vdD, Chapter 1, (7.6)], $\mathcal{M}$ is o-minimal. Following [vdD, Chapter 1, §7], a *linear (affine) function* on $A \subseteq M^n$ is a function $f : A \to M$ of the form $f(x_1, \ldots, x_n) = \lambda_1 x_1 + \ldots + \lambda_n x_n + a$, for some fixed $\lambda_i \in D$ and $a \in M$.[1] A *basic semilinear set* in M^n is a set of the form

$$\{x \in M^n : f_1(x) = \ldots = f_p(x) = 0, g_1(x) > 0, \ldots, g_q(x) > 0\},$$

where f_i and g_j are linear functions on M^n. Then (7.6), (7.8) and (7.10) of the above reference say that:

(1) $Th(\mathcal{M})$ admits quantifier elimination and, in particular, the definable subsets of M^n are the *semilinear sets* in M^n, that is, finite unions of basic semilinear sets in M^n.

(2) Every definable function $f : A \subseteq M^n \to M$ is *piecewise linear*, that is, there is a finite partition of A into basic semilinear sets A_i ($i \in \{1, \ldots, k\}$), such that $f \restriction_{A_i}$ is linear, for each $i \in \{1, \ldots, k\}$.

In fact, the above can be subsumed in a refinement of the classical Cell Decomposition Theorem ([vdD, Chapter 3, (2.11)]) stated below. First, the notion of a 'linear cell' can be defined similarly to the one of a usual cell

[1] We keep the term 'linear' and mean it in the 'affine' sense, conforming to the literature such as [Hud] or [LP].

([vdD, Chapter 3, (2.2)-(2.4)]) by using linear functions in place of definable continuous ones. Namely, for a definable set $X \subseteq M^n$, we let

$$L(X) := \{f : X \to M : f \text{ is linear}\}.$$

If $f \in L(X)$, we denote by $\Gamma(f)$ the graph of f. If $f, g \in L(X) \cup \{\pm\infty\}$ with $f(x) < g(x)$ for all $x \in X$, we write $f < g$ and denote by $(f, g)_X$ the 'generalized cylinder' $(f, g)_X = \{(x, y) \in X \times M : f(x) < y < g(x)\}$ between f and g. Then

- a *linear cell in* M is either a singleton subset of M, or an open interval with endpoinds in $M \cup \{\pm\infty\}$,
- a *linear cell in* M^{n+1} is a set of the form $\Gamma(f)$, for some $f \in L(X)$, or $(f, g)_X$, for some $f, g \in L(X) \cup \{\pm\infty\}$, $f < g$, where X is a linear cell in M^n.

One can then adapt the classical proof of the Cell Decomposition Theorem and inductively show:

Linear Cell Decomposition Theorem. *Let $A \subseteq M^n$ and $f : A \to M$ be definable. Then there is a decomposition of M^n that partitions A into finitely many linear cells A_i, such that each $f\restriction_{A_i}$ is linear.* (See [vdD, Chapter 3, (2.10)] for a definition of decomposition of M^n.)

Since $D = \langle D, +, \cdot, <, 0, 1\rangle$ is a division ring, $\langle \mathbb{Q}, +, \cdot, <, 0, 1\rangle$ naturally embeds into D. If $a \in M$ and $0 < m \in \mathbb{N}$, we write $\frac{a}{m}$ for $\frac{1}{m}a$, which is also the unique $b \in M$ such that $a = mb = \underbrace{b + \ldots + b}_{m\text{-times}}$, since $\mathcal{M}$ is divisible and torsion-free..

We write $0 := (0, \ldots, 0)$. If $\lambda \in D$, $x = (x_1, \ldots, x_n) \in M^n$ and $X \subseteq M^n$, then $\lambda x := (\lambda x_1, \ldots, \lambda x_n)$ and $\lambda X := \{\lambda x : x \in X\}$, whereas if $\lambda = (\lambda_1, \ldots, \lambda_n) \in D^n$ and $x \in M$, $\lambda x := (\lambda_1 x, \ldots, \lambda_n x)$. If $\lambda \in \mathbb{M}(n, D)$ is an $n \times n$ matrix over D and

$x \in M^n$, then λx denotes the resulting n-tuple of the matrix multiplication of λ with x. The unit element of $\mathbb{M}(n, D)$ is denoted by $\mathbb{I}_n$. Again, if $a \in M^n$ and $0 < m \in \mathbb{N}$, then $\frac{a}{m} := \frac{1}{m}a$.

Let $m, n \in \mathbb{N}$. The elements $a_1, \ldots, a_m \in M^n$ are called *linearly independent over* $\mathbb{Z}$ or just $\mathbb{Z}$-*independent* if for all $\lambda_1, \ldots, \lambda_m$ in $\mathbb{Z}$, $\lambda_1 a_1 + \ldots + \lambda_m a_m = 0$ implies $\lambda_1 = \ldots = \lambda_m = 0$. The elements $\lambda_1, \ldots, \lambda_m \in D^n$ are called M-*independent* if for all $t_1, \ldots, t_m \in M$, $\lambda_1 t_1 + \ldots + \lambda_m t_m = 0$ implies $t_1 = \ldots = t_m = 0$.

For $\lambda \in D$, $|\lambda| := \max\{-\lambda, \lambda\}$. For $x \in M$, $|x| := \max\{-x, x\}$, and for $x = (x_1, \ldots, x_n) \in M^n$, $|x| := |x_1| + \ldots + |x_n|$.

Definition 2.3.1. Let $A \subseteq M^n$.

(i) A is called *convex* if $\forall x, y \in A, \forall q \in \mathbb{Q} \cap [0, 1]$, $qa + (1 - q)b \in A$.

(ii) A is called *bounded* if $\exists r \in M, \forall x \in A$, $|x| \leq r$, that is, $\exists r' \in M$, $A \subseteq \mathcal{B}_0(r')$.

For example, a linear cell is a convex basic semilinear set, and it is bounded if no endpoints or functions involved in its construction are equal to $\pm\infty$. Below we define a special kind of bounded definable convex sets, the 'parallelograms' (Definition 2.3.5), and make explicit their relation to bounded linear cells (Lemma 2.3.6).

We consider throughout definable functions $f = (f_1, \ldots, f_n) : M^m \to M^n$, $m, n \in \mathbb{N}$. All definitions apply to f through its components, for example, f is called linear on M^m if every f_i is linear on M^m. Moreover, the Linear Cell Decomposition Theorem holds for definable functions of this form. In fact, a linear function $f : M^n \times M^n \to M^n$ can be written in the usual form, $f(x_1, x_2) = \lambda_1 x_1 + \lambda_2 x_2 + a$, for some fixed $\lambda_i \in \mathbb{M}(n, D)$ and $a \in M^n$.

Definition 2.3.2. Let $a \in M^n \setminus \{0\}$. We say *a has definable slope* if there are $\lambda \in D^n \setminus \{0\}$ and $e > 0$ in M, such that $a = \lambda e$. In this case, and if $x \in M^n$, we call

$$[0, e] \ni t \mapsto x + \lambda t \in M^n$$

a *linear path from x to $x + a$.*

Remark 2.3.3. (i) Any two linear paths from x to $x+a$ must have the same image. Indeed, let $a = \lambda d = \mu e$ with $\lambda = (\lambda_1, \ldots, \lambda_n), \mu = (\mu_1, \ldots, \mu_n) \in D^n \setminus \{0\}$, and $0 < d, e \in M$. It is then easy to see that for all $i = 1, \ldots, n$, it is either $\lambda_i = \mu_i = 0$ or $\lambda_i, \mu_i \neq 0$. Moreover, for all $i, j = 1, \ldots, n$ such that $\mu_i, \mu_j \neq 0$, we have $\mu_i^{-1}\lambda_i = \mu_j^{-1}\lambda_j$. Assume $\mu_1 \neq 0$. Then the reader can check that for every $t \in [0, d]$, if $t' = \mu_1^{-1}\lambda_1 t$, then $t' \in [0, e]$ and $\mu t' = \lambda t$.

(ii) By the Linear Cell Decomposition Theorem, every definable path is, piecewise, a linear path, that is, it is the concatenation of finitely many linear paths.

Lemma 2.3.4. *Let $A \subseteq M^n$ be definable and convex, and $x, y \in A$. If γ is a linear path from x to y, then* $\mathrm{Im}(\gamma) \subseteq A$.

Proof. Let $\gamma(t) : [0, e] \ni t \mapsto x + \lambda t \in M^n$. Assume, towards a contradiction, that $P := \{t \in [0, e] : x + \lambda t \notin A\} \neq \emptyset$. By o-minimality, P is a finite union of points and open intervals. If it is a finite union of points and t_0 is one of them, then there must be some small $z > 0$ in M such that $t_0 - z, t_0 + z \in [0, e] \setminus P$. But since A is convex, $x + \lambda t_0 = \frac{x+\lambda(t_0-z)+x+\lambda(t_0+z)}{2}$ has to be in A, a contradiction. Similarly, if P contains some intervals, it is possible to find one such with endpoints $t_1 < t_2$, and some $z_1, z_2 \geq 0$ in M, such that $t_1 - z_1, t_2 + z_2 \in [0, e] \setminus P$ and $t_1 < \frac{t_1-z_1+t_2+z_2}{2} < t_2$. Then $x + \lambda\frac{t_1-z_1+t_2+z_2}{2} = \frac{x+\lambda(t_1-z_1)+x+\lambda(t_2+z_2)}{2} \in A$, again a contradiction. $\square$

Definition 2.3.5. Let $a_1, \ldots, a_m \in M^n \setminus \{0\}$, $0 < m \leq n$, have definable slopes, and $c \in M^n$. Then the *open m-parallelogram with center c and generated by* $a_1, \ldots, a_m$ is the definable set

$$c + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_1 < t_i < e_i\},$$

where $a_i = \lambda_i e_i$, $e_i > 0$, $1 \leq i \leq m$. The *closed m-parallelogram with center a and generated by* $a_1, \ldots, a_m$ is the closed definable set

$$c + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i \leq t_i \leq e_i\}.$$

We say just *open (* or *closed) m-parallelogram* if c and $a_1, \ldots, a_m$ are not specified. The 2^m elements $c + \lambda_1 t_1 + \ldots + \lambda_m t_m$, $t_i = -e_i, e_i$, are called the *corners* of the open or closed m-parallelogram.

Remark 2.3.3(i) guarantees that the definition of an open (or closed) m-parallelogram does not depend on the choice of λ_i and e_i, $1 \leq i \leq m$. Clearly, an open or closed m-parallelogram is a definable bounded convex set. Notice also that an open m-parallelogram is an open subset of M^n only if $m = n$.

In the next lemma we use the following notation for a closed m-parallelogram. Let $a_1, \ldots, a_m \in M^n \setminus \{0\}$, $0 < m \leq n$, with $a_i = \lambda_i e_i$, $e_i > 0$, for $0 < i \leq m$, and $a \in M^n$. We let

$$\overline{P}_a(a_1, \ldots, a_m) := a + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : t_i \in [0, e_i]\}.$$

It is clear that if $c = a + \frac{1}{2}\sum_{i=1}^{m} a_i$, then

$$\overline{P}_a(a_1, \ldots, a_m) = c + \left\{\lambda_1 t_1 + \ldots + \lambda_m t_m : -\frac{1}{2}e_i \le t_i \le \frac{1}{2}e_i\right\}$$

is a closed m-parallelogram with center c and generated by $\frac{1}{2}a_i$, $1 \le i \le m$. Conversely, if $c + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i \le t_i \le e_i\}$ is a closed parallelogram with center c and generated by $a_i = \lambda_i e_i$, then if $a = c - \sum_{i=1}^{m} a_i$, we have

$$c + \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i \le t_i \le e_i\} = \overline{P}_a(2a_1, \ldots, 2a_m).$$

Lemma 2.3.6. *The closure of every bounded n-dimensional linear cell $Y \subset M^n$, $n > 0$, is a finite union of closed n-parallelograms.*

Proof. By induction on n.

$\mathbf{n = 1}$. $Y = (a, b) \subset M$, $a, b \in M$. Then $\overline{Y} = \overline{P}_a(b - a)$.

$\mathbf{n > 1}$. A bounded n-dimensional linear cell Y must have the form $Y = (f, g)_X$, for some $(n-1)$-dimensional linear cell X in M^{n-1} and $f < g \in L(X)$. By Inductive Hypothesis, $\overline{X}$ is a finite union of closed $(n-1)$-parallelograms, and, thus, it suffices to show that for any closed $(n-1)$-parallelogram $Q \subset M^{n-1}$ and $f < g \in L(Q)$, $\overline{(f, g)_Q}$ is a finite union of closed n-parallelograms. Let $Q = \overline{P}_{q_0}(q_1, \ldots, q_{n-1})$ in M^{n-1}, $a_0 = \big(q_0, f(q_0)\big)$, $b_0 = \big(q_0, g(q_0)\big)$, and $\forall i \in \{0, \ldots, n-1\}$,

$$a_i = \big(q_0 + q_i, f(q_0 + q_i)\big) - a_0 = \big(q_i, f(q_0 + q_i) - f(q_0)\big) \in M^n$$

and

$$b_i = \big(q_0 + q_i, g(q_0 + q_i)\big) - b_0 = \big(q_i, g(q_0 + q_i) - g(q_0)\big) \in M^n.$$

Then $\Gamma(f) = \overline{P}_{a_0}(a_1, \ldots, a_{n-1})$ and $\Gamma(g) = \overline{P}_{b_0}(b_1, \ldots, b_{n-1})$. Indeed, it is not very hard to see that for $0 < i \leq n-1$, if $[0, e_i] \ni t_i \mapsto q_i(t_i) \in M^{n-1}$ is a linear path from 0 to q_i, then

$$[0, e_i] \ni t_i \mapsto a_i(t_i) := \Big(q_i(t_i), f\big(q_0 + q_i(t_i)\big) - f(q_0)\Big) \in M^n$$

is a linear path from 0 to a_i, and

$$[0, e_i] \ni t_i \mapsto b_i(t_i) := \Big(q_i(t_i), g\big(q_0 + q_i(t_i)\big) - g(q_0)\Big) \in M^n$$

is a linear path from 0 to b_i. Moreover, for any $x = q_0 + \sum_{i=1}^{n-1} q_i(t_i) \in Q$, we have $f(x) = \sum_{i=1}^{n-1} f\big(q_0 + q_i(t_i)\big) - (n-2)f(q_0)$, since by linearity of f, for any $j \in \{2, \ldots, n-1\}$, $f\left(q_0 + \sum_{i=1}^{j} q_i(t_i)\right) - f\left(q_0 + \sum_{i=1}^{j-1} q_i(t_i)\right) = f\big(q_0 + q_j(t_j)\big) - f(q_0)$. Thus,

$$\begin{aligned} a_0 + \sum_{i=1}^{n-1} a_i(t_i) &= \big(q_0, f(q_0)\big) + \sum_{i=1}^{n-1} \Big(q_i(t_i), f\big(q_0 + q_i(t_i)\big) - f(q_0)\Big) \\ &= \left(q_0 + \sum_{i=1}^{n-1} q_i(t_i), \sum_{i=1}^{n-1} f\big(q_0 + q_i(t_i)\big) - (n-2)f(q_0)\right) = \big(x, f(x)\big). \end{aligned}$$

It follows that $\Gamma(f) = \overline{P}_{a_0}(a_1, \ldots, a_{n-1})$. Similarly, $\Gamma(g) = \overline{P}_{b_0}(b_1, \ldots, b_{n-1})$.

Now, if $\exists c \in M^n, \forall i \in \{0, \ldots, n-1\}$, $b_i - a_i = c$, then for all $i > 0$, $a_i - a_0 = b_i - b_0$ and $\overline{(f,g)_Q}$ is the closed n-parallelogram $\overline{P}_{a_0}(a_1, \ldots, a_{n-1}, b_0 - a_0)$. Indeed, one first can see that $\forall x \in Q$, $g(x) - f(x) = b_0 - a_0 = c$ and, thus, $\overline{(f,g)_Q} = \{(x, y) \in M^{n-1} \times M : x \in Q, y \in f(x) + [0, (b_0)_n - (a_0)_n]\}$. On the other hand, consider the linear path $[0, b_0 - a_0] \ni t \mapsto (b_0 - a_0)(t) := (0, t) \in M^{n-1} \times M$ from 0 to $(0, b_0 - a_0)$ in M^n. Then every element in $\overline{P}_{a_0}(a_1, \ldots, a_{n-1}, b_0 - a_0)$ has the form $a_0 + \sum_{i=1}^{n-1} a_i(t_i) + (b_0 - a_0)(t) = \big(x, f(x)\big) + (0,t) = \big(x, f(x) + t\big)$, for $x \in Q$ and $t \in [0, (b_0)_n - (a_0)_n]$.

Otherwise, we may assume that $\overline{(f,g)_Q}$ is such that for some $i \in \{0, \dots, n-1\}$, $a_i = b_i$. Indeed, let $C = \{|b_i - a_i| : 0 \le i \le n-1\}$, and let $j \in \{0, \dots, n-1\}$ be such that $|b_j - a_j| = (b_j)_n - (a_j)_n$ is minimum in C. If, say, $j = 0$, and $a_0 \neq b_0$, it is easy to see as before that $\overline{(f,g)_Q} = \overline{(f,f')_Q} \cup \overline{P}_{a_0}(b_1, \dots, b_{n-1}, b_0 - a_0)$, where $\forall x \in Q, f'(x) = g(x) - (b_0 - a_0)$, that is, $\overline{(f,g)_Q}$ is the union of the closure of a cell of the desired form and of a closed n-parallelogram.

We may further assume that all generators of $\Gamma(f)$ and $\Gamma(g)$ but one coincide. For, if $\Gamma(f) = \overline{P}_{a_0}(a_1, \dots, a_{n-1})$ and $\Gamma(g) = \overline{P}_{a_0}(b_1, \dots, b_{n-1})$, with say $a_1 \neq b_1$ and $a_2 \neq b_2$, then $\overline{(f,g)_Q} = \overline{(f,f')_Q} \cup \overline{(f',g)_Q}$, where $f' \in L(Q)$ such that $\Gamma(f') = \overline{P}_{a_0}(b_1, a_2, \dots, a_{n-1})$. Clearly, on the one hand, all generators of $\Gamma(f)$ and $\Gamma(f')$ but one coincide. On the other hand, the number of generators of $\Gamma(f')$ and $\Gamma(g)$ which do not coincide is by one smaller than the number of generators of $\Gamma(f)$ and $\Gamma(g)$ which do not coincide. Thus, repeating this process, we see that $\overline{(f,g)_Q}$ is a union of closures of cells of the desired form.

Now let $\Gamma(f) = \overline{P}_{a_0}(a_1, a_2, \dots, a_{n-1})$ and $\Gamma(g) = \overline{P}_{a_0}(b_1, a_2, \dots, a_{n-1})$. Let $\bar{a} := (a_2, \dots, a_{n-1})$. Then $\overline{(f,g)_Q} = P_1 \cup P_2 \cup P_3$, where

$P_1 = \overline{P}_{a_0}\left(\frac{a_1}{2}, \frac{b_1}{2}, \bar{a}\right)$,

$P_2 = \overline{P}_{a_0 + \frac{a_1}{2}}\left(\frac{a_1}{2}, \frac{b_1 - a_1}{2}, \bar{a}\right)$, and

$P_3 = \overline{P}_{a_0 + \frac{b_1}{2}}\left(\frac{b_1}{2}, \frac{a_1 - b_1}{2}, \bar{a}\right)$.

Indeed, let $x = q_0 + \sum_{i=1}^{n-1} q_i(t_i) \in Q$, and $\big(x, f(x) + t\big) \in \overline{(f,g)_Q}$, $t \in [0, g(x) - f(x)]$. Then the following are easy to check. If $t_1 \le \frac{e_1}{2}$, then $(x, f(x) + t) \in P_1$. If $t_1 \ge \frac{e_1}{2}$, then if $t \le \frac{(b_1)_n - (a_1)_n}{2}$, $\big(x, f(x) + t\big) \in P_2$, whereas if $t \ge \frac{(b_1)_n - (a_1)_n}{2}$, $\big(x, f(x) + t\big) \in P_3$. □

For the rest of Section 2.3, let $G = \langle G, \oplus, e_G \rangle$ be a definable group with $G \subseteq M^n$ and $\dim(G) = m \leq n$.

Note that if a definable set $A \subseteq M^n$ is unbounded, then there is a definable continuous injective map $\gamma : [0, \infty) \to A$.

Lemma 2.3.7. *If G is definably compact, then G is definably bijective to a bounded subset of M^m. Thus, in this case, we may assume $m = n$ (see Remark 2.1.2).*

Proof. Recall, G admits a finite t-open covering $\{S_i\}_{i \in J}$, such that each S_i is definably homeomorphic to an open subset K_i of M^m via $\phi_i : S_i \to K_i$. It is not hard to see that it suffices to show that each K_i is bounded in M^m. If, say, K_1 is not, then there must be a definable continuous injective map $\gamma : [0, \infty) \to K_1$. Since G is definably compact, there is some $g \in G$ with $\lim^t_{x \to \infty} \phi_1^{-1}\big(\gamma(x)\big) = g$. If $g \in S_l$, $l \in J$, take a bounded open subset B of K_l in M^m containing $\phi_l(g)$. Then the restriction of the map $\phi_l \circ \phi_1^{-1} \circ \gamma$ on some $[a, \infty)$ such that $\phi_l \circ \phi_1^{-1} \circ \gamma\big([a, \infty)\big) \subseteq B$ is a piecewise linear bijection between a bounded and an unbounded set in M^m, a contradiction. □

Definition 2.3.8. Assume G is abelian. Let $X \subseteq G \subseteq M^n$. A *$\oplus$-translate of X* is a set of the form $a \oplus X$, for $a \in G$. We say that X *is generic (in G)* if finitely many $\oplus$-translates of X cover G.

Fact 2.3.9. *Assume G is abelian. Then:*

(i) every large definable subset of G is generic.

Assume, further, that X is a definable subset of G. Then:

(ii) if $X \subseteq G$ is generic, then $\dim(X) = \dim(G)$.

(iii) $X \subseteq G$ is generic if and only if $\overline{X}^t$ is generic.

Proof. (i) is by [Pi1], whereas (ii) and (iii) constitute [PePi, Lemma 3.4]. □

Let us note here that, although in [PePi] the authors work over an o-minimal expansion $\mathcal{M}$ of a real closed field, their proofs of several facts about generic sets, such as [PePi, Lemma 3.4], that is, Fact 2.3.9 above, go through in the present context as well. More significantly, their Corollary 3.9 holds. To spell out a few more details, their use of the field structure of $\mathcal{M}$ is to ensure that G is affine ([vdD, Chapter 10, (1.8)]), and, therefore, that a definably compact subset X of G is closed and bounded ([PeS]). Theorem 2.1 from [PePi] (which is extracted from Dolich's work, and is shown in their Appendix to be true if $\mathcal{M}$ expands an ordered group), then applies and shows their Lemma 3.6 and, following, Corollary 3.9. Although in our context G may not be affine, [PePi, Theorem 2.1] can be restated for any $X \subseteq G$, which is definably compact, instead of closed and bounded, assuming G is definably compact, as below. The rest of the proof of [PePi, Corollary 3.9] then works identically.

Lemma 2.3.10. *Let both G and $X \subseteq G$ be definably compact, and $\mathcal{M}_0$ a small elementary substructure of $\mathcal{M}$ (that is, $|M_0| < |M|$), such that the manifold structure of G is $\mathcal{M}_0$-definable. Then the following are equivalent:*

(i) The set of $\mathcal{M}_0$-conjugates of X is finitely consistent.

(ii) X has a point in $\mathcal{M}_0$.

Therefore ([PePi, Corollary 3.9]), if G is abelian, the union of any two non-generic definable subsets of G is also non-generic.

Proof. Recall that G is Hausdorff. We use the notation for the definable manifold topology on G from Section 2.1. One can then show that there are $\mathcal{M}_0$-definable t-open subsets $O_i \subseteq G$, $i \in J$, such that $G = \bigcup_{i \in J} O_i$ and $\overline{O_i}^t \subset S_i$ (see [BO1, Lemmas 10.4, 10.5], for example, where the authors work over a real closed field but their arguments go word-by-word through in the present context, as well).

Now, for the non-trivial direction $(i) \Rightarrow (ii)$, let $X \subseteq G$, $X = \bigcup_{i \in J} X_i$, with $X_i := X \cap \overline{O_i}^t$, and assume that the set of $\mathcal{M}_0$-conjugates of X is finitely consistent. Since O_i and the chart maps $\phi_i : S_i \to M^m$ are $\mathcal{M}_0$-definable, if $f \in \mathrm{Aut}_{\mathcal{M}_0}(M)$, then $f(X_i) \subseteq \overline{O_i}^t$, and, thus, the set $\big\{ \bigcup_{i \in J} \phi_i\big(f(X_i)\big)\big\}_{f \in \mathrm{Aut}_{\mathcal{M}_0}(M)}$ is finitely consistent. Moreover, it is not hard to see that $f\big(\bigcup_{i \in J} \phi_i(X_i)\big) = \bigcup_{i \in J} \phi_i\big(f(X_i)\big)$, which gives that the set of $\mathcal{M}_0$-conjugates of $\bigcup_{i \in J} \phi_i(X_i)$ is finitely consistent. Since each X_i is definably compact, $\bigcup_{i \in J} \phi_i(X_i)$ is closed and bounded in M^m. By [PePi, Theorem 2.1], $\bigcup_{i \in J} \phi_i(X_i)$ has a point in $\mathcal{M}_0$, say $a \in \phi_1(X_1)$, and, thus, X_1 has a point b in $\mathcal{M}_0$ $\big($since $\mathcal{M}_0 \prec \mathcal{M} \models \exists y \in X_1 \phi_1(y) = a\big)$. $\square$

Remark 2.3.11. The proof (and the result) of Lemma 2.3.10 are valid in any o-minimal expansion $\mathcal{M}$ of an ordered group. Moreover, the proof of Lemma 2.3.10 shows that Lemma 2.3.7 is also valid in any o-minimal expansion $\mathcal{M}$ of an ordered group. Indeed, with the above notation, each $\overline{O_i}^t$ is definably compact (as a t-closed subset of the definably compact G), hence $\phi_i(\overline{O_i}^t) \subseteq M^m$ is definably compact in M^m and, thus, (closed and) bounded.

2.4 Standard part maps

In Section 2.4, we fix $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D}\rangle$ to be a saturated ordered vector space over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1\rangle$.

2.4.1 Non-commutative linear algebra

In addition to the notation fixed in Section 2.3 we introduce the following. If $\lambda = (\lambda^1, \ldots, \lambda^n) \in D^n$ and $\mu \in D$, we denote $\mu\lambda := (\mu\lambda^1, \ldots, \mu\lambda^n)$ and $\lambda\mu := (\lambda^1\mu, \ldots, \lambda^n\mu)$.

The elements $\lambda_1, \ldots, \lambda_m \in D^n$ are called *left (right) D-independent* if for all $\mu_1, \ldots, \mu_m$ in D, $\mu_1\lambda_1 + \ldots + \mu_m\lambda_m = 0$ ($\lambda_1\mu_1 + \ldots + \lambda_m\mu_m = 0$ implies $\mu_1 = \ldots = \mu_m = 0$).

Many definitions and facts from linear algebra over a field go through over D. Let $A \in \mathbb{M}(n, D)$ be an $n \times n$ matrix with entries from D. The *row rank* of A is the cardinality of a maximal left D-independent set of rows from A, and the *column rank* of A is the cardinality of a maximal right D-independent set of columns from A.

Fact 2.4.1. *(i) The row rank and the column rank of A are equal. We refer to either of them as the* rank *of A.*

(ii) The row rank of A is n if and only if A has a right inverse.

The column rank of A is n if and only if A has a left inverse.

(iii) A left inverse of A is also its right inverse, and vice versa. We refer to either of them as the inverse A^{-1} *of A.*

Proof. (i) See [Jac, Chapter II, Theorem 9].

(ii) Similar to [Lang, Chapter IV, Theorem 2.2].

(iii) By (i) and (ii). Also, see [Jac, Chapter I, Theorem 6]. □

Note that there is an analogue of Cramer's rule for computing the entries of A^{-1}, using the notion of a 'quasi-determinant'. See [GGRW] for a general reference.

Now, for $n \in \mathbb{N}$ and $i = 1, \ldots, n$, let

$$\lambda_i = \begin{pmatrix} \lambda_i^1 \\ \vdots \\ \lambda_i^n \end{pmatrix} \in D^n,$$

and $e_i \in M$ positive, such that the open n-parallelogram

$$H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$$

has dimension n. Consider the following matrix with entries from D.

$$A = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \ldots & \lambda_n^1 \\ \vdots & \ldots & \vdots \\ \lambda_1^n & \ldots & \lambda_n^n \end{pmatrix}.$$

We show that A has rank n:

Claim 2.4.2. *A has column rank equal to n.*

Proof. Assume not. Without loss of generality, we may then assume that there are $\mu_1, \ldots, \mu_{n-1} \in D$ such that

$$\lambda_n = \lambda_1 \mu_1 + \ldots + \lambda_{n-1} \mu_{n-1}.$$

But then

$$\begin{aligned} H &= \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\} \\ &= \{\lambda_1 t_1 + \ldots + \lambda_{n-1} t_{n-1} + (\lambda_1 \mu_1 + \ldots + \lambda_{n-1} \mu_{n-1}) t_n : -e_i < t_i < e_i\} \\ &= \{\lambda_1 (t_1 + \mu_1 t_n) \ldots + \lambda_{n-1} (t_{n-1} + \mu_{n-1} t_n) : -e_i < t_i < e_i\}, \end{aligned}$$

which clearly has dimension less than n, a contradiction. □

Corollary 2.4.3. *A is invertible.*

Corollary 2.4.4. *$\lambda_1, \ldots, \lambda_n$ are M-independent.*

Proof. For any $e_1, \ldots, e_n \in M$, if $A\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$, then $\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} = A^{-1}\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$. □

2.4.2 Construction of a standard part map

Here we present the construction of a standard part map which will be very useful in the chapters following. Let again, for $n \in \mathbb{N}$ and $i = 1, \ldots, n$,

$$\lambda_i = \begin{pmatrix} \lambda_i^1 \\ \vdots \\ \lambda_i^n \end{pmatrix} \in D^n,$$

and $e_i \in M$ positive, such that the open n-parallelogram

$$H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$$

has dimension n. We define U_H to be the subgroup of M^n generated by H. That is,

$$U_H =< H >= \bigcup_{k<\omega} H^k,$$

where $H^k := \underbrace{H + \ldots + H}_{k-\text{times}}$. Then, for all $x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \in U_H$, there are $\chi^i \in M$ with $\forall i \exists q \in \mathbb{Z}$, $-qe_i < \chi^i < qe_i$, such that

$$x = \lambda_1 \chi^1 + \ldots + \lambda_n \chi^n = A\begin{pmatrix} \chi^1 \\ \vdots \\ \chi^n \end{pmatrix}. \tag{2.1}$$

That is, for $i = 1, \ldots, n$,

$$x^i = \lambda_1^i \chi^1 + \ldots + \lambda_n^i \chi^n.$$

Clearly, for all $k \in \mathbb{N}$,

$$x \in H^k \Leftrightarrow \forall i, -ke_i < \chi^i < ke_i. \tag{2.2}$$

By Corollary 2.4.4, for every $x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \in U_H$, the χ^i's in equation (2.1) are unique. We can, thus, define a *standard part map* $st_H : U_H \to \mathbb{R}^n$ as follows: for every $x \in U_H$ with form (2.1), let

$$st_H(x) = \big(st_1(\chi^1), \ldots, st_n(\chi^n)\big),$$

where for every i,

$$st_i(\chi^i) := \sup\{q \in \mathbb{Q} : qe_i < \chi^i\}.$$

It can be checked that each of st_H and st_i is a surjective group homomorphism.

An easy but useful lemma is given next. Notice that if $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_k \leqslant U_H$ is a lattice of rank $k \leq n$, then, clearly, $st_H(L) = \mathbb{Z}st_H(a_1) + \ldots + \mathbb{Z}st_H(a_k)$ is a lattice in $\mathbb{R}^n$ of rank $\leq k$.

Lemma 2.4.5. *Assume that $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_k \leqslant U_H$ is a lattice of rank k, such that $L \cap H = \{0\}$. Then $st_H(L) \leqslant \mathbb{R}^n$ is a lattice of rank k.*

Proof. Clearly, $st_H(L)$ has rank at most k. If $st_H(L)$ has rank strictly less than k, then for some $l_1, \ldots, l_k \in \mathbb{Z}$, not all zero, $l_1 st_H(a_1) + \ldots + l_k st_H(a_k) = 0$. Since $st_H : U \to \mathbb{R}^n$ is a group homomorphism, $st_H(l_1a_1 + \ldots + l_ka_k) = 0$. Thus, $l_1a_1 + \ldots + l_ka_k \in H$. Hence, since $L \cap H = \{0\}$, we have $l_1a_1 + \ldots + l_ka_k = 0$, contradicting the fact that L has rank k. $\square$

2.4.3 Properties of the standard part map

Here we prove some properties of $st_H : U_H \to \mathbb{R}^n$ that will be particularly useful in Chapter 5. We fix an open n-parallelogram

$$H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$$

of dimension n, as in the previous subsection. For simplicity, we write U for the subgroup $U_H =< H >$ generated by H, and st for the standard part map $st_H : U_H \to \mathbb{R}^n$.

The definition of a $\mathbb{Q}$-box in U given below is in analogy with the definition from [BO3] of a box in cartesian powers of an o-minimal expansion of an ordered field.

Definition 2.4.6. Let $n \in \mathbb{N}$, $n > 0$. A *$\mathbb{Q}$-box in U* (of dimension n) is a subset of U of the form

$$B = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : p_i e_i \leq t_i \leq q_i e_i\},$$

for some $p_i, q_i \in \mathbb{Q}$.

A *real $\mathbb{Q}$-box* (of dimension n) is a subset of $\mathbb{R}^n$ of the form

$$C = [k_1, l_1] \times \cdots \times [k_n, l_n],$$

for some $k_i, l_i \in \mathbb{Q}$.

If B is a $\mathbb{Q}$-box in U as above, then $B^{\mathbb{R}}$ denotes the real $\mathbb{Q}$-box defined by $k_i = p_i$ and $l_i = q_i$, for $i = 1, \ldots, n$.

Remark 2.4.7. It is easy to see that, for all $x \in U$ and $y \in \mathbb{R}^n$,

$$st(x) = y \Leftrightarrow \text{for every } \mathbb{Q}\text{-box } B \text{ in } U,\ x \in B \text{ implies } y \in B^{\mathbb{R}}.$$

The proof of the following lemma is almost word-by-word the one of [BO3, Proposition 4.2].

Lemma 2.4.8. *For every $\mathbb{Q}$-box B in U, $st(B) = B^{\mathbb{R}}$.*

Proof. The inclusion $st(B) \subseteq B^{\mathbb{R}}$ is by Remark 2.4.7. For the equality, let $y \in B^{\mathbb{R}}$. We write $\{y\} = \bigcap_{i \in \mathbb{N}} B_i^{\mathbb{R}}$, where $\{B_i^{\mathbb{R}} : i \in \mathbb{N}\}$ is an enumeration of all real $\mathbb{Q}$-boxes containing y. The set of all formulas $x \in B_i$ is a type in M which must be realized by some element $x \in U$. For this x, we have $st(x) = y$ and $x \in B$. □

The first five of the properties listed below will be used in the sequel. The sixth is recorded in the interests of completeness.

Lemma 2.4.9. *(i) For all $X_1, X_2 \subseteq U$, $st(X_1 \cup X_2) = st(X_1) \cup st(X_2)$.*

(ii) For all $X \subseteq U$, $st^{-1}\big(st(X)\big) = X + \ker(st)$.

(iii) For all $X \subseteq U$, $st(\overline{X}) = st(X)$.

(iv) For all $X \subseteq U$, $X + \ker(st) = \overline{X} + \ker(st)$.

(v) A bounded set $A \subseteq \mathbb{R}^n$ is closed if and only if $st^{-1}(A)$ is type-definable.

(vi) For all definable $X \subseteq U$, $st(X)$ is closed.

Proof. (ii) $\forall y \in U$,

$$y \in st^{-1}\big(st(X)\big) \Leftrightarrow \exists x \in X, st(x) = st(y) \Leftrightarrow y \in X + \ker(st).$$

(iii) For the non-trivial inclusion ($\subseteq$), let $x \in \overline{X}$. We need to find $x' \in X$, such that $st(x') = st(x)$. Since $\ker(st)$ is open and $x \in \overline{X}$, $\big(x + \ker(st)\big) \cap X \neq \emptyset$. We can take x' to be any element in $\big(x + \ker(st)\big) \cap X$.

(iv) By (ii) and (iii), $X + \ker(st) = st^{-1}\big(st(X)\big) = st^{-1}\big(st(\overline{X})\big) = \overline{X} + \ker(st)$.

(v) The proof is almost word-by-word the one of [BO3, Proposition 5.4]. Note that by (ii) and Lemma 2.4.8, for every $\mathbb{Q}$-box B in U, $st^{-1}(B^{\mathbb{R}}) = B + \ker(st)$.

Let $A \subseteq \mathbb{R}^n$ be bounded.

For the left-to-right direction, if A is closed, then A is the intersection of a countable family of real $\mathbb{Q}$-boxes $\{B_i^{\mathbb{R}} : i \in \mathbb{N}\}$. Thus, $st^{-1}(A) = \{x : \bigwedge_{i\in\mathbb{N}} \big(x \in B_i + \ker(st)\big)\}$, so $st^{-1}(A)$ is type-definable.

For the right-to-left direction, let $st^{-1}(A)$ be type-definable, say $st^{-1}(A) = \{x : \bigwedge_{i\in I}(x \in X_i)\}$. To show that A is closed, let $y \in \overline{A}$. We show that $y \in A = st\big(st^{-1}(A)\big)$. We need to find an $x \in \bigcap_{i\in I} X_i$ such that $st(x) = y$. It suffices to show that the type $\bigwedge_{i\in I}(x \in X_i) \wedge \bigwedge_{j\in\mathbb{N}}(x \in B_j)$ is consistent, where $\{B_j : j \in \mathbb{N}\}$ is an enumeration of all $\mathbb{Q}$-boxes B in U with $y \in \mathrm{Int}(B^{\mathbb{R}})$. By compactness, it suffices to show the consistency of the type $p(x) = \bigwedge_{i\in I}(x \in X_i) \wedge (x \in B)$, where B is any $\mathbb{Q}$-box in U with $y \in \mathrm{Int}(B^{\mathbb{R}})$. But, since $y \in \overline{A}$, there is $y' \in \mathrm{Int}(B^{\mathbb{R}}) \cap A \neq \emptyset$. Thus, there is $x' \in B$, such that $st(x') = y \in B^{\mathbb{R}} \cap A$. This x' realizes the type p.

(vi) Observe that the boundedness of $A \subseteq \mathbb{R}^n$ in (v) was only used in the left-to-right direction. Now, let $X \subseteq U$. By (ii), $st^{-1}\big(st(X)\big) = X + \ker(st)$, so if X is definable, then $st^{-1}\big(st(X)\big)$ is type-definable and we can apply (v), right-to-left. $\square$

CHAPTER 3

THE COMPACT CASE

3.1 Introduction

In this chapter we prove Theorems 1, 2, and 4 from Chapter 1 for G definably compact and $\mathcal{M}$ an ordered vector space over an ordered division ring. See Theorems 3.1.2, 3.3.1 and 3.4.13 below, respectively.

In Chapter 3, we fix $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$ **to be a saturated ordered vector space over an ordered division ring** $D = \langle D, +, \cdot, <, 0, 1 \rangle$.

In this compact case, Theorem 1 is suggested as a structure theorem analogous to the following classical result from the theory of topological groups (see [Pon, Theorem 42], for example):

Fact 3.1.1. *Any compact connected abelian locally Euclidean group is (as a topological group) isomorphic to a direct sum of copies of the circle topological group* $S^1 = \langle \mathbb{R}, + \rangle / \mathbb{Z} = \langle [0, 1), \oplus, 0 \rangle$, *where*

$$x \oplus y = \begin{cases} x + y & \text{if } x + y < 1, \\ x + y - 1 & \text{if } x + y \geq 1. \end{cases}$$

Let us argue next why a model theoretic analogue of Fact 3.1.1 would have to take the form of Theorem 3.1.2 below. First, it is clear that the assumptions should be weakened (to their definable versions), since in the non-archimedean $\mathcal{M}$ compactness and connectedness almost always fail. Also, caution is needed in order to state a *definable* version of the conclusion, since: i) $\mathbb{Z}$ is not definable in any o-minimal structure and therefore $M/\mathbb{Z}$ is not a priori a definable object, ii) no $[0, a)$, $a \in M$, can serve as a fundamental domain for $M/\mathbb{Z}$, as it cannot contain a representative for the $\mathbb{Z}$-class of infinitely large elements, and iii) we cannot always expect G to be a direct product of 1-dimensional definable subgroups of it, known by examples in [Str] (see also [PeS]).

We state:

Theorem 3.1.2. *Let G be an n-dimensional definable group which is t-connected and definably compact. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant M^n$ and a lattice $L \leqslant U$ of rank n.*

Theorem 3.1.2 has two corollaries.

Theorem 3.3.1 (Pillay's Conjecture). *Let G be as in Theorem 3.1.2. Then there is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension n.*

Theorem 3.4.13. *Let G be as in Theorem 3.1.2. Then the o-minimal fundamental group of G is isomorphic to L.*

As it was mentioned in Chapter 1, Pillay's Conjecture was stated in [Pi2] for groups definable in any o-minimal structure, and it was shown in [HPP] to be true for an o-minimal expansion of an ordered field. Earlier, it was shown in [EdOt] (and was used in [HPP]) that for a group G satisfying the assumptions of the conjecture and definable over an o-minimal expansion of an ordered field, the following hold: (i) the o-minimal fundamental group of G is equal to $\mathbb{Z}^n$, and (ii) the k-torsion subgroup of G is equal to $(\mathbb{Z}/k\mathbb{Z})^n$. Theorems 3.1.2 and 3.4.13 show that (i) and (ii) are true in the present context as well.

The structure of this chapter is as follows. Section 3.2 contains the proof of Theorem 3.1.2. En route, we show that *any* m-dimensional group definable in $\mathcal{M}$ is locally isomorphic to M^m. In Section 3.3, we apply our analysis to define G^{00} and prove Theorem 3.3.1. In Section 3.4, we prove Theorem 3.4.13.

3.2 The Structure Theorem

Outline. We split our proof into three steps. We let $G = \langle G, \oplus, e_G \rangle$ be a $\emptyset$-definable group with $G \subseteq M^n$.

In Step I, we begin with a local analysis on G and show that the set V^G (from Definition 2.1.4) is large in G. We then let G be n-dimensional, definably compact and t-connected, and, based on the set V^G, we compare the two group operations $\oplus$ and $+$. A key notion is that of a 'jump' of a t-path (Definition 3.2.16), and the main results of this first step are Lemma 3.2.23 and Proposition 3.2.24.

In Step II, we invoke [PePi, Corollary 3.9] (see Lemma 2.3.10 here) in order to establish the existence of a generic open n-parallelogram H in G, which is used to generate a subgroup $U \leqslant M^n$. Using Lemma 3.2.23(i) from Step I, we can define a group homomorphism ϕ from U onto G, and we let $L := \ker(\phi)$.

In Step III, we use Proposition 3.2.24 to prove that L is a lattice generated by some elements of M^n recovered in Step I, namely, by some $\mathbb{Z}$-linear combinations of 'jump vectors'. We then use H to obtain a standard part map $st_H : U \to \mathbb{R}^n$ as in Section 2.4. This allows us to compute the rank of L and finish the proof.

STEP I. Comparing $\oplus$ with $+$.

We start with a $\emptyset$-definable group $G = \langle G, \oplus, e_G \rangle$ with $G \subseteq M^n$ and $\dim(G) = m \leq n$**.** (We do not yet assume that G is definably compact or t-connected.)

Our first goal is to show that V^G is a large subset of G, which, among other things, implies that G is locally isomorphic to $M^m = \langle M^m, +, 0 \rangle$.

A consequence of the Linear Cell Decomposition Theorem is that for any two independent dim-generic elements a and b of G, there are t-neighborhoods V_a of a and V_b of b in G, such that for all $x \in V_a$ and $y \in V_b$, $x \oplus y = \lambda x + \mu y + d$, for some fixed $\lambda, \mu \in \mathbb{M}(n, D)$, and $d \in M^n$. Moreover, λ and μ have to be invertible matrices (for example, setting $y = b$, $x \oplus b = \lambda x + \mu b + d$ is invertible, showing that λ is invertible).

Proposition 3.2.1. *For every dim-generic element a of G, there exists a t-neighborhood V_a of a in G, such that for all $x, y \in V_a$,*

$$x \ominus a \oplus y = x - a + y.$$

Proof. We proceed through several lemmas.

Lemma 3.2.2. *For every two independent dim-generics $a, b \in G$, there exist t-neighborhoods V_a of a and V_b of b in G, invertible λ, $\lambda' \in \mathbb{M}(n, D)$, and $c = b - \lambda a$, $c' = b - \lambda' a \in M^n$, such that for all $x \in V_a$,*

$$x \ominus a \oplus b = \lambda x + c \in V_b \quad \textit{and} \quad \ominus a \oplus b \oplus x = \lambda' x + c' \in V_b.$$

Proof. Since a and b are independent dim-generics of G, a and $\ominus a \oplus b$ are independent dim-generics of G as well. Therefore, there are t-neighborhoods V_a of a and $V_{\ominus a \oplus b}$ of $\ominus a \oplus b$ in G, as well as invertible $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that $\forall x \in V_a$, $\forall y \in V_{\ominus a \oplus b}$, $x \oplus y = \lambda x + \mu y + d$. In particular, for all $x \in V_a$, $x \ominus a \oplus b = \lambda x + \mu(\ominus a \oplus b) + d$. Letting $c = \mu(\ominus a \oplus b) + d$ and $V_b = \{x \ominus a \oplus b : x \in V_a\}$ shows the first equality. That $c = b - \lambda a$, it can be verified by setting $x = a$. The second equality can be shown similarly. $\square$

Lemma 3.2.3. *Let a be a dim-generic element of G. Then there exist a t-neighborhood V_a of a in G, $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that for all $x, y \in V_a$,*

$$x \ominus a \oplus y = \lambda x + \mu y + d.$$

Proof. Take a dim-generic element a_1 of G independent from a. Then $a_2 = a \ominus a_1$ is also a dim-generic element of G independent from a. By Lemma 3.2.2, we can find t-neighborhoods V_{a_1}, V_{a_2}, V_a of a_1, a_2, a, respectively, in G, as well as $\lambda_1, \lambda_2 \in \mathbb{M}(n, D)$ and $c_1, c_2 \in M^n$, such that $\forall x \in V_a$, $x \ominus a \oplus a_1 = \lambda_1 x + c_1 \in V_{a_1}$ and $\forall y \in V_a$, $\ominus a \oplus a_2 \oplus y = \lambda_2 y + c_2 \in V_{a_2}$. Moreover, since a_1 and $a_2 = a \ominus a_1$ are independent dim-generics of G, we could choose V_{a_1}, V_{a_2} and V_a to be such that for some fixed $\nu, \xi \in \mathbb{M}(n, D)$ and $o \in M^n$, we have: $\forall x \in V_{a_1}$, $\forall y \in V_{a_2}$, $x \oplus y = \nu x + \xi y + \varepsilon$. Now for all $x, y \in V_a$, we have:

$$\begin{aligned} x \ominus a \oplus y &= x \ominus a \oplus a_1 \ominus a_1 \oplus y \\ &= (x \ominus a \oplus a_1) \oplus (\ominus a \oplus a_2 \oplus y) \\ &= \nu(\lambda_1 x + c_1) + \xi(\lambda_2 y + c_2) + o \\ &= \nu\lambda_1 x + \xi\lambda_2 y + \nu c_1 + \xi c_2 + o \end{aligned}$$

Setting $\lambda = \nu\lambda_1, \mu = \xi\lambda_2$, and $d = \nu c_1 + \xi c_2 + o$ finishes the proof of the lemma. $\square$

We can now finish the proof of Proposition 3.2.1. By Lemma 3.2.3, there exists a t-neighborhood V_a of a in G, $\lambda, \mu \in \mathbb{M}(n, D)$ and $d \in M^n$, such that for all $x, y \in V_a$, $x \ominus a \oplus y = \lambda x + \mu y + d$. In particular, for all $x, y \in V_a$,

$$y = a \ominus a \oplus y = \lambda a + \mu y + d$$

$$x = x \ominus a \oplus a = \lambda x + \mu a + d$$

and, therefore, $x + y = (\lambda x + \mu y + d) + (\lambda a + \mu a + d)$. But, $\lambda x + \mu y + d = x \ominus a \oplus y$, and

$$a = a \ominus a \oplus a = \lambda a + \mu a + d.$$

Hence, $x + y = (x \ominus a \oplus y) + a$, or, $x \ominus a \oplus y = x - a + y$. $\square$

Corollary 3.2.4. *G is 'definably locally isomorphic' to M^m. That is, there is a definable homeomorphism f from some t-neighborhood V_{e_G} of e_G in G to a definable open neighborhood W_0 of 0 in M^m, such that:*

(i) for all $x, y \in V_{e_G}$, if $x \oplus y \in V_{e_G}$, then $f(x \oplus y) = f(x) + f(y)$, and

(ii) for all $x, y \in W_0$, if $x + y \in W_0$, then $f^{-1}(x + y) = f^{-1}(x) \oplus f^{-1}(y)$.

(See [Pon, Definition 30] for more on the definition of a local isomorphism.)

Proof. Let a be a dim-generic element of G. The function $G \ni x \mapsto x \oplus a \in G$ witnesses that the topological group $(G, \oplus, e_G)$ is definably isomorphic to $(G, *, a)$, where $x * y = x \ominus a \oplus y$ (Remark 2.1.2). Now, since a is dim-generic, some t-neighborhood V_a of a in G can be projected homeomorphically onto an open subset W_a of M^m, inducing on W_a the group structure from V_a. We may thus assume that $V_a \subseteq M^m$. By Proposition 3.2.1, the definable function $f : G \ni x \mapsto$

$x - a \in M^m$ witnesses, easily, that $(G, *, a)$ is definably locally isomorphic to M^m. Thus, $(G, \oplus, e_G)$ is (definably isomorphic to a group which is) definably locally isomorphic to M^m. $\square$

The following corollary is already known; for example, see [Ed1, Corollary 6.3] or [PeSt, Corollary 5.1]. It can also be extracted from [LP].

Corollary 3.2.5. *G is abelian-by-finite.*

Proof. Let V_{e_G} and f be as in Corollary 3.2.4. Since $\oplus$ is t-continuous, there is a t-open $U' \subseteq G$ containing e_G with $\forall x, y \in U'$, $x \oplus y \in V_{e_G}$. Thus, if we let $U := U' \cap V_{e_G}$, then $\forall x, y \in U$, $x \oplus y = f^{-1}\big(f(x) + f(y)\big) = f^{-1}\big(f(y) + f(x)\big) = y \oplus x$.

Now let G^0 be the t-connected component of e_G in G. Then for every element $a \in U$, its centralizer $C(a) = \{x \in G : a \oplus x = x \oplus a\}$ contains the t-open (m-dimensional) subset $U \subseteq G$, and, thus, $G^0 \subseteq C(a)$. It follows that the center $Z(G^0) = \{x \in G^0 : \forall y \in G^0, x \oplus y = y \oplus x\}$ of G^0 contains U, thus, $Z(G^0)$ must have dimension m and be equal to G^0. That is, G^0 is abelian. $\square$

For the rest of Chapter 3, we fix a definable group $G = \langle G, \oplus, e_G \rangle$, definably compact and t-connected, with $G \subseteq M^n$.

By Lemma 2.3.7, we may assume $\dim(G) = n$. By Corollary 3.2.5, G is abelian.

Proposition 3.2.1 says that the set V^G is large in G. We omit the index 'G' and write just V. Then V is t-open as well as open, and, by cell decomposition, it is the disjoint union of finitely many open definably connected components $V_0, \dots, V_N$, that is, $V = \bigsqcup_{i \in I} V_i$, for a fixed index set $I := \{0, \dots, N\}$.

The next goal is to show that the property

$$(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v$$

may be assumed to be true for any $u, v \in V$ and 'small' $\varepsilon \in M^n$ (Corollary 3.2.12). In what follows, whenever we write a property that includes an expression of the form '$x \oplus y$', it is meant that $x, y \in G$ (and the property holds).

Corollary 3.2.6. *For all $u \in V$, there is $r > 0$ in M, such that for all $v \in \mathcal{B}_u(r)$ and $\varepsilon \in (-r, r)^n$,*

$$(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v.$$

Proof. By definition of V, there is $r > 0$ in M, such that for all $v \in \mathcal{B}_u(r)$ and $\varepsilon \in (-r, r)^n$,

$$(u + \varepsilon) \ominus u \oplus v = u + \varepsilon - u + v = v + \varepsilon.$$

□

Lemma 3.2.7. *For all u, v in the same definably connected component of V, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, +r)^n$,*

$$(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v.$$

Proof. Let V_i be a definably connected component of V and u some element in V_i. We show that the set

$$\Gamma = \{v \in V_i : \exists r > 0 \in M \, \forall \varepsilon \in (-r, +r)^n \, [(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v]\}$$

is a nonempty clopen subset of V_i. First, Γ is nonempty since it contains u. To show that Γ is open, consider an element $v \in \Gamma$. Let $r_v \in M$ be such that $\forall \varepsilon \in (-r_v, r_v)^n$, $(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v$. By Corollary 3.2.6, there is $s_v > 0$ in M such that for all $v' \in \mathcal{B}_v(s_v)$ and $\varepsilon \in (-s_v, s_v)^n$, $(v + \varepsilon) \ominus v = (v' + \varepsilon) \ominus v'$. By letting $r := \min\{r_v, s_v\}$, we obtain that for all $v' \in \mathcal{B}_v(r)$, for all $\varepsilon \in (-r, r)^n$,

$$(v' + \varepsilon) \ominus v' = (v + \varepsilon) \ominus v = (u + \varepsilon) \ominus u,$$

that is, $\mathcal{B}_r(v) \subseteq \Gamma$, and therefore Γ is open.

To show that Γ is closed in V_i, pick some v in $V_i \setminus \Gamma$. It should satisfy

$$\forall r > 0\, \exists \varepsilon_v \in (-r, r)^n\, [(u + \varepsilon) \ominus u \neq (v + \varepsilon) \ominus v]. \tag{3.1}$$

Now, let as before $s_v > 0$ be so that for all $v' \in \mathcal{B}_v(s_v)$ and $\varepsilon \in (-s_v, s_v)^n$, $(v + \varepsilon) \ominus v = (v' + \varepsilon) \ominus v'$. We want to show $v' \in V \setminus \Gamma$, that is, $\forall r_{v'} > 0$,

$$\exists \varepsilon_{v'} \in (-r_{v'}, r_{v'})^n [(u + \varepsilon_{v'}) \ominus u \neq (v' + \varepsilon_{v'}) \ominus v']. \tag{3.2}$$

It suffices to show (3.2) for all $r_{v'}$ with $\mathcal{B}_{v'}(r_{v'}) \subseteq \mathcal{B}_v(s_v)$. Let $r_{v'}$ be one such. Apply (3.1) for $r_{v'}$ to get an $\varepsilon_{v'} \in (-r_{v'}, r_{v'})^n \subseteq (-s_v, s_v)^n$ satisfying $(u+\varepsilon_{v'}) \ominus u \neq (v+\varepsilon_{v'}) \ominus v$. But since $\varepsilon_{v'} \in (-s_v, s_v)^n$, we also have $(v+\varepsilon_{v'}) \ominus v = (v'+\varepsilon_{v'}) \ominus v'$. It follows that $(u + \varepsilon_{v'}) \ominus u \neq (v' + \varepsilon_{v'}) \ominus v'$. □

More generally, the following is true.

Lemma 3.2.8. *There are invertible $\lambda_0, \ldots, \lambda_N \in \mathbb{M}(n, D)$ such that for any $i, j \in I = \{0, \ldots, N\}$, $u \in V_i$ and $v \in V_j$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u + \lambda_i \varepsilon) \ominus u = (v + \lambda_j \varepsilon) \ominus v.$$

In particular, $\lambda_0 = \mathbb{I}_n$.

Proof. By Lemma 3.2.2, for any two independent dim-generics $u \in V_0$ and $v \in V_j$, $j \in I$, there is invertible $\lambda_j \in \mathbb{M}(n, D)$ such that for all x in some small t-neighborhood of u in G, $x \ominus u \oplus v = \lambda_j x + v - \lambda_j u$, or, equivalently, for sufficiently small ε, $(u+\varepsilon) \ominus u \oplus v = \lambda_j(u+\varepsilon) + v - \lambda_j u = v + \lambda_j \varepsilon$, that is, $(u+\varepsilon) \ominus u = (v + \lambda_j \varepsilon) \ominus v$. By Lemma 3.2.7, the last equation holds for any $u \in V_0$ and $v \in V_j$, perhaps for some smaller epsilon's. Clearly, $\lambda_0 = \mathbb{I}_n$. Now, pick any $i, j \in I$, and any $v_0 \in V_0$, $u \in V_i$, $v \in V_j$. We derive that for sufficiently small ε:

$$(u + \lambda_i \varepsilon) \ominus u = (v_0 + \varepsilon) \ominus v_0 = (v + \lambda_j \varepsilon) \ominus v.$$

$\square$

We next show (Lemma 3.2.11) that all λ_i's in Lemma 3.2.8 may be assumed to be equal to $\mathbb{I}_n$. First, let us notice that it is harmless to assume $0 = e_G \in V$, which in particular means that in a t-neighborhood of 0 the $\mathcal{M}$- and t- topologies coincide.

Lemma 3.2.9. *$(G, \oplus, e_G)$ is definably isomorphic to a topological group $(G', +_1, 0)$ with $0 \in V^{G'}$.*

Proof. Pick a dim-generic point $b \in G$. Consider the definable bijection

$$f : G \ni x \mapsto (x \oplus b) - b \in f(G) \subseteq M^n.$$

Let $G' := f(G)$ and let $\langle G', +_1, 0 = f(e_G)\rangle$ be the topological group structure induced on G' by f. Then f is a definable isomorphism between $\langle G, \oplus, e_G\rangle$ and $\langle G', +_1, 0\rangle$ (Remark 2.1.2). We show that

$$V^{G'} = V - b,$$

and, therefore, since $b \in V$, we have $0 \in V^{G'}$.

For all $x, y, c \in G'$, we have that $x+b, y+b, c+b \in G$ and the following holds:

$$\begin{aligned} x -_1 c +_1 y &= f\big(f^{-1}(x) \ominus f^{-1}(c) \oplus f^{-1}(y)\big) \\ &= \big([(x+b) \ominus b \ominus (c+b) \oplus b \oplus (y+b) \ominus b] \oplus b\big) - b \\ &= [(x+b) \ominus (c+b) \oplus (y+b)] - b. \end{aligned}$$

Now, assume that $c + b \in V$. We claim that $c \in V^{G'}$. Indeed, if x, y are sufficiently close to c, then $x+b$, $y+b$ will be close to $c+b \in V$, hence

$$[(x+b) \ominus (c+b) \oplus (y+b)] - b = x + b - c - b + y + b - b = x - c + y.$$

This shows $V - b \subseteq V^{G'}$ (which is what we need). The inverse inclusion can be shown similarly. □

Remark 3.2.10. The above proof can be split into two parts: (i) for every element b in G, the definable bijection $f_1 : G \ni x \mapsto x \oplus b \in G$ preserves V, and (ii) for every element b in G, the definable bijection $f_2 : G \ni x \mapsto x - b \in G'$ maps V to $V^{G'}$, that is, $V^{G'} = V - b$. Later, we use the property that a bijection such as f_2 maps open m-parallelograms to open m-paralellograms.

We let V_0 be the component of V that contains $0 = e_G$.

Lemma 3.2.11. *G is definably isomorphic to a group $G' = \langle G', +_1, 0\rangle$ whose corresponding $\lambda_i^{G'}$'s (as in Lemma 3.2.8) are all equal to $\mathbb{I}_n$.*

Proof. For any $i \in I$, let a_i be some element in V_i. Consider the definable function $f : G \to M^n$, such that

$$f(x) = \begin{cases} \lambda_i^{-1}(x - a_i) + a_i & \text{if } x \in V_i,\ i \in I, \\ x & \text{if } x \in G \setminus V. \end{cases}$$

We may assume that f is one-to-one, by definably moving the definably connected components of G sufficiently 'far away' from each other if needed, which is possible, by Lemma 2.3.7. We show that in the induced group $G' = \langle f(G) = G', +_1, f(0) = 0\rangle$ the corresponding set $V^{G'}$ is exactly the set $f(V) = f(V_0) \bigsqcup \ldots \bigsqcup f(V_N)$, with $f(V_0), \ldots, f(V_N)$ as its definably connected components. First, notice that for $x \in V_i \subseteq G$ and ε 'small', $\lambda_i \varepsilon$ is also small, and $f(x + \lambda_i \varepsilon) = \lambda_i^{-1}(x + \lambda_i \varepsilon - a_i) + a_i = {\lambda_i}^{-1}(x - a_i) + a_i + \varepsilon = f(x) + \varepsilon$. Thus, for all $x, y, c \in G'$, with x, y close to c, $f^{-1}(x), f^{-1}(y)$ must be close to $f^{-1}(c)$. Moreover, if $f^{-1}(c) \in V_i$, then $x, y, c \in f(V_i)$ and

$$\begin{aligned} x -_1 c +_1 y = {} & f\big(f^{-1}(x) \ominus f^{-1}(c) \oplus f^{-1}(y)\big) = f\big(f^{-1}(x) - f^{-1}(c) + f^{-1}(y)\big) \\ = {} & {\lambda_i}^{-1}\Big(\big[\big(\lambda_i(x - a_i) + a_i\big) - \big(\lambda_i(c - a_i) + a_i\big) + \big(\lambda_i(y - a_i) + a_i\big)\big] - a_i\Big) + a_i \\ = {} & x - c + y, \end{aligned}$$

This shows that $f(V_i) \subseteq V_i^{G'}$. The inverse inclusion can be shown similarly.

It then suffices to show that for any $i \in \{0, \ldots, N\}$, for all $u = f(u) \in V_0^{G'} = V_0$, $f(v) \in V_i^{G'}$, and sufficiently small ε,

$$(u + \varepsilon) -_1 u = \big(f(v) + \varepsilon\big) -_1 f(v).$$

We have

$$\big(f(v)+\varepsilon\big)-_1 f(v) = f(v+\lambda_i\varepsilon)-_1 f(v) = f\big((v+\lambda_i\varepsilon)\ominus v\big) = f\big((u+\varepsilon)\ominus u\big) = (u+\varepsilon)-_1 u,$$

by Lemma 3.2.8 and since f is the identity on V_0. □

By Lemma 3.2.11, we may assume that for any $i \in I = \{0, \dots, N\}$, $\lambda_i = \mathbb{I}_n$. Therefore, Lemma 3.2.8 becomes the following:

Corollary 3.2.12. *For all $u, v \in V$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u + \varepsilon) \ominus u = (v + \varepsilon) \ominus v.$$

Corollary 3.2.13. *For all $u \in V$, $v \in G$, such that $u \oplus v \in V$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u + \varepsilon) \oplus v = (u \oplus v) + \varepsilon. \tag{3.3}$$

Proof. By Corollary 3.2.12, there is $r > 0$ in M, such that $\forall \varepsilon \in (-r, r)^n$,

$$(u + \varepsilon) \ominus u = [(u \oplus v) + \varepsilon] \ominus (u \oplus v).$$

□

The final goal in this first step (Lemma 3.2.23 and Proposition 3.2.24) is to obtain suitable versions of the equation (3.3), where i) u, v and $u \oplus v$ are in G, and ii) ε is arbitrary in M^n.

Definition 3.2.14. We let $\sim_G$ be the following definable equivalence relation on $\overline{G}$:

$$a \sim_G b \Leftrightarrow \forall t > 0 \text{ in } M, \exists a_t, b_t \in G, \text{ such that}$$
$$a_t \in \mathcal{B}_a(t),\ b_t \in \mathcal{B}_b(t) \text{ and } a_t \ominus b_t \in \mathcal{B}_0(t).$$

Clearly, $\forall a, b \in G$, $a \sim_G b \Leftrightarrow a = b$.

We may assume that $G \subseteq \overline{V}$:

Lemma 3.2.15. *G is definably isomorphic to a group G' with $G' \subseteq \overline{V^{G'}}$.*

Proof. Since V is large in G, it is everywhere dense, so $G \subseteq \overline{V}^t$. This implies that $\forall a \in G, \exists b \in \overline{V}$, such that $a \sim_G b$. Indeed, for any $a \in G$ and any $t > 0$ in M, there is $b_t \in V$ so that $a \ominus b_t \in \mathcal{B}_0(t)$. Since $V \subseteq G$ is bounded (Remark 2.3.7), $\overline{V}$ is closed and bounded. Thus, $b := \lim_{t \to 0} b_t \in \overline{V}$, by [PeS]. We have $a \sim_G b$. Now, by definable choice, there is a definable subset Y of $\overline{V}$ of representatives for $\sim_G$ (by considering the restriction of $\sim_G$ on $\overline{V} \times \overline{V}$). Since each class can contain only one element of G, the definable function:

$$f : G \ni x \mapsto \text{the unique element } a \text{ with } x \sim_G a \in Y \subseteq \overline{V},$$

is a definable bijection between G and Y. We can let G' be the topological group with domain Y and structure induced by f, according to Remark 2.1.2. □

Note that now $\text{bd}(V) = \text{bd}(G)$. Indeed, since $V \subseteq G \subseteq \overline{V}$, we have $\overline{V} \subseteq \overline{G} \subseteq \overline{V}$ and $\text{Int}(V) \subseteq \text{Int}(G) \subseteq \text{Int}(\overline{V}) = \text{Int}(V)$, that is, $\overline{G} = \overline{V}$ and $\text{Int}(G) = \text{Int}(V)$.

Definition 3.2.16. Let $\gamma : [0,p] \subseteq M \to G$ be a t-path. An element $w \in M^n$, $w \neq 0$, is said to be a *jump (vector) of* γ if there is some $t_0 \in [0,p]$ such that

$$w = \gamma(t_0) - \lim_{t \to t_0^-} \gamma(t) \text{ or } w = \lim_{t \to t_0^+} \gamma(t) - \gamma(t_0). \tag{3.4}$$

We say that γ *jumps at* t_0.

An element $w \in M^n$ is called a *jump vector (for G)* if it is the jump of some t-path.

Remark 3.2.17. (i) One can see that: w is a jump of some t-path $\Leftrightarrow \exists$ distinct $a, b \in \mathrm{bd}(V)$, such that $a \sim_G b$ and $w = b - a$. Thus, *the set of all jump vectors is a definable subset of* M^n.

(ii) Since γ is a t-path, $\lim^t_{t \to t_0^-} \gamma(t) = \gamma(t_0) = \lim^t_{t \to t_0^+} \gamma(t)$ $\big($contrasting (3.4)$\big)$, or, equivalently, $\lim_{z \to 0} \big[\gamma(t_0 - z) \ominus \gamma(t_0 + z)\big] = 0$.

(iii) In case $\gamma : [0,p] \to G$ is a t-path with no jumps, then it is a path in M^n as well and it has the form $u + \varepsilon(t)$, where $u = \gamma(0)$, and $\varepsilon(t) = \gamma(t) - u$ is a path in M^n with $\varepsilon(0) = 0$. Conversely, if a t-path has the form $u + \varepsilon(t)$ for some path $\varepsilon(t)$ in M^n, then it has no jumps. For example, every t-path in V is of this form, as the $\mathcal{M}$- and t- topologies coincide on V.

Lemma 3.2.18. *Let* $u, v \in V$ *such that* $u \oplus v \in V$, *and* $u + \varepsilon(t) : [0,p] \to V$, $\varepsilon(0) = 0$, *a* t*-path. Then* $\exists t_0 \in (0,p]$, *such that* $\forall t \in [0, t_0]$,

$$\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + \varepsilon(t).$$

Proof. Let $r > 0$ be as in Corollary 3.2.13 and choose $t_0 \in (0,p]$ such that $\forall t \in [0, t_0]$, $u + \varepsilon(t) \in \mathcal{B}_u(r)$. $\square$

Lemma 3.2.19. *Let $\gamma(t) = u + \varepsilon(t) : [0,p] \to V$, $\varepsilon(0) = 0$, be a t-path, such that $\forall t \in [0,p]$, $\varepsilon(t) \in V$. Then:*

$$\big(u + \varepsilon(p)\big) \ominus u = \varepsilon(p).$$

Proof. Consider the function $f : G \ni x \mapsto x - (x \ominus u) \in M^n$. By Lemma 3.2.18, f is locally constant on $\mathrm{Im}(\gamma)$. Indeed, first observe that $\forall s \in [0,p]$, $\exists z > 0$, such that $\forall t \in [s-z, s+z] \cap [0,p]$,

$$\big(u + \varepsilon(t)\big) \ominus u = \big(u + \varepsilon(s) + \varepsilon(t) - \varepsilon(s)\big) \ominus u = \big[\big(u + \varepsilon(s)\big) \ominus u\big] + \varepsilon(t) - \varepsilon(s).$$

Then, $\forall t \in [s-z, s+z]$, $f\big(u + \varepsilon(t)\big) = u + \varepsilon(t) - \big[\big(u + \varepsilon(t)\big) \ominus u\big] = u + \varepsilon(s) - \big[\big(u + \varepsilon(s)\big) \ominus u\big] = f\big(u + \varepsilon(s)\big)$.

It follows that f is constant on $\mathrm{Im}(\gamma)$ and equal to $u - (u \ominus u) = u$. Thus, $\forall t \in [0,p]$, $u + \varepsilon(t) - \big[\big(u + \varepsilon(t)\big) \ominus u\big] = u$, that is, $\big(u + \varepsilon(t)\big) \ominus u = \varepsilon(t)$. □

We can replace V from the previous lemma by G, as follows:

Lemma 3.2.20. *Let $u + \varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, be a t-path that does not jump at $t = 0$, such that $\forall t \in (0,p]$, $u + \varepsilon(t) \in V$, and $\forall s,t \in [0,p]$, $\varepsilon(s) - \varepsilon(t) \in V$. Then:*

$$\big(u + \varepsilon(p)\big) \ominus u = \varepsilon(p).$$

Proof. By Lemma 3.2.19, we have $\forall t \in (0,p]$, $\big(u + \varepsilon(t) + \varepsilon(p) - \varepsilon(t)\big) \ominus \big(u + \varepsilon(t)\big) = \varepsilon(p) - \varepsilon(t)$; that is,

$$\big(u + \varepsilon(p)\big) \ominus \big(u + \varepsilon(t)\big) = \varepsilon(p) - \varepsilon(t).$$

On the other hand, since for all (small) $t \in [0,p]$, $\varepsilon(p) - \varepsilon(t) \in V$, the limits of the expression above, with respect to the t- and $\mathcal{M}$- topologies as $t \to 0$, must coincide and be equal to $\varepsilon(p)$:

$$\lim_{t\to 0}{}^{t}\big[\big(u+\varepsilon(p)\big) \ominus \big(u+\varepsilon(t)\big)\big] = \lim_{t\to 0}\big(\varepsilon(p)-\varepsilon(t)\big) = \varepsilon(p).$$

Since $u+\varepsilon(t) : [0,p] \to G$ does not jump at $t=0$, we also have $\lim^{t}_{t\to 0}\big(u+\varepsilon(t)\big) = u$. It follows, $\big(u+\varepsilon(p)\big) \ominus u = \lim^{t}_{t\to 0}\big[\big(u+\varepsilon(p)\big) \ominus \big(u+\varepsilon(t)\big)\big] = \varepsilon(p)$. □

Lemma 3.2.21. *Let $u+\varepsilon(t) : [0,p] \to G$, $\varepsilon(0) = 0$, be a t-path that does not jump at $t = 0$. Then: $\exists t_0 \in (0,p]$, such that $\forall t \in [0,t_0]$,*

$$\big(u+\varepsilon(t)\big) \ominus u = \varepsilon(t).$$

Proof. By curve selection, since $G \subseteq \overline{V}$ and $u+\varepsilon(t)$ does not jump at $t=0$, it is not hard to see that there is some $t_0 \in (0,p]$ and, for all $s \in [0,t_0]$, a t-path $u+\delta_s(t) : [0,s] \to G$, with no jumps, such that:

(i) $\delta_s(0) = 0$, $\delta_s(s) = \varepsilon(s)$, and $\forall t \in (0,s)$, $u+\delta_s(t) \in V$, and

(ii) $\forall t_1, t_2 \in [0,s]$, $\delta_s(t_1) - \delta_s(t_2) \in V$.

Now, by Lemma 3.2.20, $\forall s \in [0,t_0]$, $\forall t \in [0,s)$,

$$\big(u+\delta_s(t)\big) \ominus u = \delta_s(t) \in V_0.$$

Since $u+\delta_s(t)$ does not jump at $t=s$,

$$\big(u+\delta_s(s)\big) \ominus u = \lim_{t\to s}{}^{t}\big[\big(u+\delta_s(t)\big) \ominus u\big] = \lim_{t\to s}{}^{t}\delta_s(t) = \delta_s(s).$$

We have shown: $\forall s \in [0,t_0]$, $\varepsilon(s) = \delta_s(s) = \big(u+\delta_s(s)\big) \ominus u = \big(u+\varepsilon(s)\big) \ominus u$. □

Lemma 3.2.22. *Let $u, v \in G$ and $u + \varepsilon(t) : [0, p] \to G$, $\varepsilon(0) = 0$, a t-path that does not jump at $t = 0$, such that*

(i) $(u \oplus v) + \varepsilon(t)$ is a t-path,

or

(ii) $\big(u + \varepsilon(t)\big) \oplus v$ is a t-path that does not jump at $t = 0$.

Then: $\exists t_0 \in (0, p]$, such that $\forall t \in [0, t_0]$,

$$\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + \varepsilon(t).$$

Proof. (i) Notice, by Remark 3.2.17(iii), $(u \oplus v) + \varepsilon(t)$ does not jump at $t = 0$. Applying Lemma 3.2.21 both to $u + \varepsilon(t)$ and to $(u \oplus v) + \varepsilon(t)$, we obtain: $\exists t_0 \in (0, p]\, \forall t \in [0, t_0]$,

$$\big(u + \varepsilon(t)\big) \ominus u = \varepsilon(t) = [(u \oplus v) + \varepsilon(t)] \ominus (u \oplus v).$$

(ii). Since $\big(u + \varepsilon(t)\big) \oplus v$ does not jump at $t = 0$, there exists some $s \in (0, p]$, such that $\forall t \in [0, s]$, $\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + d_\varepsilon(t)$ for some path $d_\varepsilon(t)$ in M^n, that is,

$$[(u \oplus v) + d_\varepsilon(t)] \ominus (u \oplus v) = \big(u + \varepsilon(t)\big) \ominus u.$$

On the other hand, by Lemma 3.2.21, there is $t_0 \in (0, s]$, such that $\forall t \in [0, t_0]$,

$$[(u \oplus v) + d_\varepsilon(t)] \ominus (u \oplus v) = d_\varepsilon(t) \text{ and } \big(u + \varepsilon(t)\big) \ominus u = \varepsilon(t).$$

It follows that $\forall t \in [0, t_0]$, $d_\varepsilon(t) = \varepsilon(t)$. $\square$

Lemma 3.2.23. *Let $u, v \in G$, and $\gamma(t) = u + \varepsilon(t) : [0, p] \to G$, $\varepsilon(0) = 0$, be a t-path with no jumps, such that*

(i) $(u \oplus v) + \varepsilon(t)$ is a t-path,

or

(ii) $\big(u + \varepsilon(t)\big) \oplus v$ is a t-path with no jumps.

Then:

$$\big(u + \varepsilon(p)\big) \oplus v = (u \oplus v) + \varepsilon(p).$$

Proof. Notice, by Remark 3.2.17(iii), $(u \oplus v) + \varepsilon(t)$ has no jumps. Consider the function $f : G \ni x \mapsto x + v - (x \oplus v) \in G$. By Lemma 3.2.22, it follows that f is locally constant on $\mathrm{Im}(\gamma)$. Thus, it is constant on $\mathrm{Im}(\gamma)$ and equal to $u+v-(u\oplus v)$. Hence for all $t \in [0, p]$, $u + \varepsilon(t) + v - \big[\big(u + \varepsilon(t)\big) \oplus v\big] = u + v - (u \oplus v)$, that is, $\big(u + \varepsilon(t)\big) \oplus v = (u \oplus v) + \varepsilon(t)$. $\square$

By o-minimality, a t-path γ jumps at only finitely many points $t_1, \ldots, t_r$ of its domain. If $w_1, \ldots, w_r$ are the jumps, their sum is denoted by

$$J_\gamma := \sum_{i=1}^{r} w_i.$$

Proposition 3.2.24. *Let $u, v \in G$, and $\gamma(t) = u + \varepsilon(t) : [0, p] \to G$, $\varepsilon(0) = 0$, be a t-path with no jumps. Then:*

$$\big(u + \varepsilon(p)\big) \oplus v = (u \oplus v) + \varepsilon(p) + J_{\gamma \oplus v}.$$

Proof. Assume that $\gamma(t) \oplus v$ has a jump w_i at each t_i, for $1 \le i \le r$ and $0 = t_0 \le t_1 \le \ldots \le t_r \le t_{r+1} = 1$. Let $w_0, w_{r+1} := 0$, and for all $i \in \{0, \ldots, r+1\}$,

$J_i := \sum_{k=0}^{i} w_k$, and $\gamma^i := \gamma \restriction_{[0,t_i]}$. By induction on i, we show that for all $i \in \{0, \ldots, r+1\}$ the proposition is true for γ^i, that is,

$$\gamma(t_i) \oplus v = (u \oplus v) + \varepsilon(t_i) + J_{\gamma^i \oplus v}. \tag{3.5}$$

(3.5) is clearly true for $i = 0$. Now, assume that (3.5) holds for γ^i, for some $i \in \{0, \ldots, r\}$. We show that (3.5) holds for γ^{i+1}. If $t_i = t_{i+1}$ there is nothing to show, so assume $t_i < t_{i+1}$.

Claim. For all $s \in (t_i, t_{i+1})$,

$$\gamma(s) \oplus v = (u \oplus v) + \varepsilon(s) + J_i. \tag{3.6}$$

Proof of Claim. We first show

$$\lim_{t \to t_i^+} \big(\gamma(t) \oplus v\big) = (u \oplus v) + \varepsilon(t_i) + J_i. \tag{3.7}$$

Case 1: $w_i = \big(\gamma(t_i) \oplus v\big) - \lim_{t \to t_i^-} \big(\gamma(t) \oplus v\big)$. Then $\gamma(t_i) \oplus v = \lim_{t \to t_i^+} \big(\gamma(t) \oplus v\big)$, and $J_{\gamma^i \oplus v} = J_i$. By the Inductive Hypothesis, (3.7) follows.

Case 2: $w_i = \lim_{t \to t_i^+} \big(\gamma(t) \oplus u\big) - \big(\gamma(t_i) \oplus v\big)$. Then $J_{\gamma^i \oplus v} + w_i = J_i$, and by the Inductive Hypothesis, (3.7) follows.

Now, for any t with $t_i < t < s$, Lemma 3.2.23(ii) gives $\big(u + \varepsilon(s)\big) \oplus v = \big(u + \varepsilon(t) + \varepsilon(s) - \varepsilon(t)\big) \oplus v = \big[\big(u + \varepsilon(t)\big) \oplus v\big] + \varepsilon(s) - \varepsilon(t)$. Therefore, $\big(u + \varepsilon(s)\big) \oplus v = \lim_{t \to t_i^+} \big[\big(u + \varepsilon(s)\big) \oplus v\big] = \lim_{t \to t_i^+} \big[\big(u + \varepsilon(t)\big) \oplus v\big] + \varepsilon(s) - \varepsilon(t_i)$. By (3.7), we obtain $\big(u + \varepsilon(s)\big) \oplus v = (u \oplus v) + \varepsilon(s) + J_i$, that is, (3.6) holds. This proves the Claim. □

We now show that (3.5) is true for γ^{i+1}. Taking limits from the left of t_{i+1} in equation (3.6) we obtain:

$$\lim_{s\to t_{i+1}^-}\big(\gamma(s)\oplus v\big) = (u\oplus v)+\varepsilon(t_{i+1})+J_i. \tag{3.8}$$

Case 1: $w_{i+1}=\lim_{t\to t_{i+1}^+}\big(\gamma(t)\oplus v\big)-\big(\gamma(t_{i+1})\oplus v\big)$. Then $\gamma(t_{i+1})\oplus v =$ $=\lim_{t\to t_{i+1}^-}\big(\gamma(t)\oplus v\big)$ and $J_{\gamma^{i+1}\oplus v}=J_i$. By (3.8), equation (3.5) is true for γ^{i+1}.

Case 2: $w_{i+1}=\big(\gamma(t_{i+1})\oplus v\big)-\lim_{t\to t_{i+1}^-}\big(\gamma(t)\oplus v\big)$. Then $J_{\gamma^{i+1}\oplus v}=J_i+w_{i+1}$, and by (3.8), again, (3.5) is true for γ^{i+1}. $\square$

STEP II. A generic open n-parallelogram of G. Since V is large in G, it is also generic, by Fact 2.3.9(i). By the Linear Cell Decomposition Theorem, V is a finite union of linear cells, and by Lemma 2.3.10, one of them, call it Y, must be generic. By Fact 2.3.9(ii), Y has dimension n, and by Lemma 2.3.7, it is bounded. Therefore, by Lemma 2.3.6, $\overline{Y}$ is a finite union of closed n-parallelograms, say $W_1,\dots,W_l$. For $i\in\{1,\dots,l\}$, let $Y_i := Y\cap W_i$. Then $Y=Y_1\cup\ldots\cup Y_l$. By Lemma 2.3.10 again, one of the Y_i's must be generic, say Y_1. Let $H:=\mathrm{Int}(Y_1)$. Since on V the $\mathcal{M}$- and t- topologies coincide, $H=\mathrm{Int}(Y_1)^t$. By Fact 2.3.9(iii), H is generic. Since W_1 is a closed n-parallelogram and $\mathrm{Int}(W_1)=\mathrm{Int}(W_1\cap\overline{Y})=\mathrm{Int}(W_1\cap Y)=\mathrm{Int}(Y_1)=H$, we have that H is an open n-parallelogram.

Let c be the center of H. By translation in M^n, we may assume that $c=0$. Indeed, in Lemma 3.2.9 we could have let $f: G\ni x\mapsto (x\oplus c)-c\in M^n$. Since H is generic, $H\ominus c$ is generic, and, thus, $f(H\ominus c)=H-c$ is a generic open n-parallelogram of $f(G)$ centered at 0. To see that the $\mathcal{M}$- and t- topologies coincide on $H-c\subseteq f(G)$, consider the definable automorphism

$$\overline{f}: M^n\ni x\mapsto x-c\in M^n,$$

and notice moreover that $\overline{f}\restriction_G: G \to f(G)$ is in fact a homeomorphism, since for all $x \in G$, $\overline{f}(x) = f(x \ominus c)$.

Summarizing, we may assume that:

- *H is a generic, t-open, open n-parallelogram, with center 0, on which the $\mathcal{M}$- and t- topologies coincide.*

Since H is generic, it has dimension n and, by Section 2.4, it has the form:

$$H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\},$$

for some M-independent $\lambda_i \in D^n$ and positive $e_i \in M$.

Lemma 3.2.25. *Let $a, b \in H$, such that $a + b \in H$. Then there is a path $\varepsilon(t)$ in H from 0 to a, such that the path $\varepsilon(t) + b$ lies entirely in H, as well.*

Proof. We prove the statement for *any* open m-parallelogram $H = \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i < t_i < e_i\} \subset M^n$, $0 < m \leq n$, for M-independent $\lambda_i \in D^n$ and positive $e_i \in M$, by induction on m.

$\mathbf{m = 1}$. Let $H = \{\lambda_1 t_1 : -e_1 < t_1 < e_1\}$ containing a, b and $a + b$. Assume $a = \lambda_1 t_{a1}$, for some $t_{a1} \in (-e_1, e_1)$. It is then easy to see that the path $\varepsilon(t) : [0, t_{a1}] \ni t \mapsto \lambda_1 t \in H$ satisfies the conclusion, by convexity of H and Lemma 2.3.4.

$\mathbf{m > 1}$. Let $H = \{\lambda_1 t_1 + \ldots + \lambda_m t_m : -e_i < t_i < e_i\}$ containing a, b and $a + b$, and let $a = \lambda_1 t_{a1} + \ldots + \lambda_m t_{am}$, $b = \lambda_1 t_{b1} + \ldots + \lambda_m t_{bm}$, for some $t_{ai}, t_{bi} \in (-e_i, e_i)$. Consider the open $(m-1)$-parallelogram $H' = \{\lambda_2 t_2 + \ldots + \lambda_m t_m : -e_i < t_i < e_i\}$, and let $a' := \lambda_2 t_{a2} + \ldots + \lambda_m t_{am}$, $b' := \lambda_2 t_{b2} + \ldots + \lambda_m t_{bm}$. By the Inductive Hypothesis, there is a path ε' in H' from 0 to a', such that $b' + \varepsilon'(t)$ is a path in H' from b' to $a' + b'$. Let $\varepsilon(t)$ be the concatenation of ε' with the linear path

$a' + \lambda_1 t$, $t \in [0, t_{a1}]$, from a' to a. It is then easy to check, using the convexity of H and Lemma 2.3.4, that both $\varepsilon(t)$ and $b + \varepsilon(t)$ lie entirely in H. □

Since the two topologies coincide on H, the paths $\varepsilon(t)$ and $b+\varepsilon(t)$ from Lemma 3.2.25 are also t-paths.

Lemma 3.2.26. *Let $x_1, \ldots, x_l \in H$ be such that for any subset σ of $\{1, \ldots, l\}$, $\sum_{j\in\sigma} x_j \in H$. Then $x_1 + \ldots + x_l = x_1 \oplus \ldots \oplus x_l$.*

Proof. By induction on l.

$\mathbf{l = 2}$. Let $a = x_1$, $b = x_2$, and $\gamma(t) = \varepsilon(t)$ as in Lemma 3.2.25. Then, by Lemma 3.2.23(i), for $u = 0$ and $v = b = x_2$, we have: $x_1 \oplus x_2 = (0 \oplus x_2) + x_1 = x_1 + x_2$.

$\mathbf{l > 2}$. $x_1 + \ldots + x_l = x_1 + (x_2 + \ldots + x_l) = x_1 \oplus (x_2 \oplus \ldots \oplus x_l) = x_1 \oplus \ldots \oplus x_l$. □

Lemma 3.2.27. *For every $x_1, \ldots, x_l, y_1, \ldots, y_m \in H$, if $x_1 + \ldots + x_l = y_1 + \ldots + y_m$, then $x_1 \oplus \ldots \oplus x_l = y_1 \oplus \ldots \oplus y_m$.*

Proof. Assume $x_1 + \ldots + x_l = y_1 + \ldots + y_m$, $x_i, y_i \in H$. We want to show $x_1 \oplus \ldots \oplus x_l = y_1 \oplus \ldots \oplus y_m$. Clearly, by convexity of H, for any subset σ of $\{1, \ldots, l\}$, $\sum_{i\in\sigma} \frac{x_i}{l} \in H$, and therefore $\sum_{i\in\sigma} \frac{x_i}{lm} \in H$. Similarly, for any subset τ of $\{1, \ldots, m\}$, $\sum_{j\in\tau} \frac{y_j}{m} \in H$ and $\sum_{j\in\tau} \frac{y_j}{lm} \in H$. By Lemma 3.2.26, on the one hand we have $\frac{x_1}{lm} \oplus \ldots \oplus \frac{x_l}{lm} = \frac{x_1}{lm} + \ldots + \frac{x_l}{lm} = \frac{y_1}{lm} + \ldots + \frac{y_m}{lm} = \frac{y_1}{lm} \oplus \ldots \oplus \frac{y_m}{lm}$, and, on the other, $\underbrace{\frac{x_i}{lm} \oplus \ldots \oplus \frac{x_i}{lm}}_{lm\text{-times}} = x_i$ and $\underbrace{\frac{y_j}{lm} \oplus \ldots \oplus \frac{y_j}{lm}}_{lm\text{-times}} = y_j$, for every i, j. Thus, $x_1 \oplus \ldots \oplus x_l = \bigoplus_{1\le i\le l} \Big(\underbrace{\frac{x_i}{lm} \oplus \ldots \oplus \frac{x_i}{lm}}_{lm\text{-times}} \Big) = \bigoplus_{lm\text{-times}} \left(\frac{x_1}{lm} \oplus \ldots \oplus \frac{x_l}{lm} \right) = \bigoplus_{lm\text{-times}} \left(\frac{y_1}{lm} \oplus \ldots \oplus \frac{y_m}{lm} \right) = \bigoplus_{1\le i\le m} \Big(\underbrace{\frac{y_i}{lm} \oplus \ldots \oplus \frac{y_i}{lm}}_{lm\text{-times}} \Big) = y_1 \oplus \ldots \oplus y_m$. □

Lemma 3.2.28. *Let* $H_1 := \frac{1}{2}H = \{\frac{1}{2}x : x \in H\}$. *Then* H_1 *is generic.*

Proof. We show that finitely many $\oplus$-translates of H_1 cover H. By Lemma 3.2.26, it suffices to find finitely many $a_i \in H$, such that $H = \bigcup_i (a_i + H_1)$. Let $H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$. Then $H_1 = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -\frac{e_i}{2} < t_i < \frac{e_i}{2}\}$. It is a routine to check that $H = \bigcup_{i=1}^{2^n}(a_i + H_1)$, where the a_i's are the corners of H_1. □

Lemma 3.2.29. *There is* $\Xi \in \mathbb{N}$*, such that* $G = \underbrace{H \oplus \ldots \oplus H}_{\Xi-times}$.

Proof. Let H_1 as in Lemma 3.2.28. Assume that for $\Xi \in \mathbb{N}$, $\{a_i \oplus H_1\}_{\{1 \leq i \leq \Xi\}}$ covers G. Since G is t-connected (and H_1 is t-open), for any $x \in G$, one can find $0 = x_0, x_1, \ldots, x_l = x \in G$, $l \leq \Xi$, such that $\forall i \in \{1, \ldots, l-1\}$, after perhaps reordering $\{a_i \oplus H_1\}_{\{1 \leq i \leq \Xi\}}$, $x_i \in (a_i \oplus H_1) \cap (a_{i+1} \oplus H_1)$, $0 \in a_1 \oplus H_1$, and $x \in a_l \oplus H_1$. Then, for $h_i := x_i \ominus x_{i-1} \in H$, $1 \leq i \leq l$, we have: $x = h_1 \oplus \ldots \oplus h_l$. □

Definition 3.2.30. Let U denote the subgroup $U_H \leqslant M^n$ generated by H as in Section 2.4; that is,

$$U =< H >= \bigcup_{k<\omega} H^k,$$

where $H^k := \underbrace{H + \ldots + H}_{k-\text{times}}$. By Lemma 3.2.27, the following function $\phi : U \to G$ is well-defined. For all $x \in U$, if $x = x_1 + \ldots + x_k$, $x_i \in H$, then

$$\phi(x) = x_1 \oplus \ldots \oplus x_k.$$

$U = \langle U, +_{\restriction U}, 0\rangle$ is a $\bigvee$-definable group. Easily, convexity of H implies convexity of U. Moreover:

Proposition 3.2.31. *ϕ is a t-continuous group homomorphism from U onto G.*

Proof. ϕ is a group homomorphism, because if $x = x_1 + \ldots + x_l$ and $y = y_1 + \ldots + y_m$, with $x_i, y_i \in H$, then $\phi(x+y) = \phi(x_1 + \ldots + x_l + y_1 + \ldots + y_m) = x_1 \oplus \ldots \oplus x_l \oplus y_1 \oplus \ldots \oplus y_m = \phi(x) \oplus \phi(y)$. It is onto, by Lemma 3.2.29. Since $\oplus$ is t-continuous, so is ϕ. □

Thus, if we let $L := \ker(\phi)$, we know that $U/L \cong G$ as abstract groups.

STEP III. L is a lattice of rank n. We show that L is a lattice generated by n $\mathbb{Z}$-independent elements of M^n, namely, by some $\mathbb{Z}$-linear combinations of jump vectors for G. Recall that (Remark 3.2.17(i)) $w \in M^n$ is a jump vector if and only if there are distinct $a, b \in \text{bd}(V)$ such that $a \sim_G b$ and $w = b - a$. The following is a consequence of the local analysis from Step I.

Lemma 3.2.32. *There are only finitely many jump vectors.*

Proof. Since the set of all jump vectors is definable, if there were infinitely many jump vectors, by o-minimality, one of the following should be true:

(A) there exists a non-constant path γ on $\text{bd}(V)$, such that all points in $\text{Im}(\gamma)$ are $\sim_G$-equivalent,

(B) there exist two disjoint non-constant paths γ and δ on $\text{bd}(V)$, such that every element a in $\text{Im}(\gamma)$ is $\sim_G$-equivalent with a unique element b_a in $\text{Im}(\delta)$, and vice versa, and all jump vectors $w_a = b_a - a$, $a \in \text{Im}(\gamma)$, are distinct.

Assume (A) holds. By o-minimality again, we may assume that $\gamma(t) = a + \varepsilon(t) : [0, p] \to M^n$, for some path $\varepsilon(t)$ in H with $\varepsilon(0) = 0$ and $\varepsilon := \varepsilon(p) \neq 0$. Moreover, we may assume that there is a path $\rho(s) : [0, q] \to M^n$, with $\rho(0) = 0$, such that $\forall s > 0$, $a + \rho(s)$ and $a + \varepsilon + \rho(s)$ are in G, and $a + \rho(s) + \varepsilon(t) : [0, p] \to G$ is a t-path, with no jumps, from $a + \rho(s)$ to $a + \rho(s) + \varepsilon$. By Lemma 3.2.23(i), we have that for all $s \in (0, p]$,

$$\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big) = \varepsilon.$$

Thus, $\lim_{s\to 0}\big[\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big)\big] = \varepsilon \neq 0$, contradicting the fact that $a \sim_G a + \varepsilon$.

Now assume (B) holds and, without loss of generality, let $\gamma(t) = a + \varepsilon(t) : [0, p_\gamma] \to M^n$, for some path $\varepsilon(t)$ in H with $\varepsilon(0) = 0$ and $\varepsilon := \varepsilon(p_\gamma) \neq 0$. Let also $\delta(t) = b + \zeta(t) : [0, p_\delta] \to M^n$, for $b \sim_G a$ and some path $\zeta(t)$ in H with $\zeta(0) = 0$ and $\zeta := \zeta(p_\delta) \neq 0$. As before, we may assume that there is a path $\rho(s) : [0, q] \to M^n$, with $\rho(0) = 0$, such that $\forall s > 0$, $a + \rho(s)$ and $a + \varepsilon + \rho(s)$ are in G, and $a + \rho(s) + \varepsilon(t) : [0, p_\gamma] \to G$ is a t-path, with no jumps, from $a + \rho(s)$ to $a + \rho(s) + \varepsilon$. Similarly, we may assume that there is a path $\sigma(s) : [0, q] \to M^n$, with $\sigma(s) = 0$, such that $\forall s > 0$, $b + \sigma(s)$ and $b + \zeta + \sigma(s)$ are in G, and $b+\sigma(s)+\zeta(t) : [0, p_\delta] \to G$ is a t-path, with no jumps, from $b+\sigma(s)$ to $b+\sigma(s)+\zeta$. We show that if $a + \varepsilon \sim_G b + \zeta$, then $\varepsilon = \zeta$, which contradicts the fact that all jump vectors from $\mathrm{Im}(\gamma)$ to $\mathrm{Im}(\delta)$ are distinct. As before, we have that for any $s \in (0, p_\gamma] \cap (0, p_\delta]$,

$$\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big) = \varepsilon \text{ and } \big(b + \sigma(s) + \zeta\big) \ominus \big(b + \sigma(s)\big) = \zeta.$$

On the other hand, since $a \sim_G b$ and $a + \varepsilon \sim_G b + \zeta$,

$$\lim_{s\to 0}\big[\big(a + \rho(s)\big) \ominus \big(b + \sigma(s)\big)\big] = 0 \text{ and } \lim_{s\to 0}\big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big)\big] = 0.$$

Since in a t-neighborhood of 0 the $\mathcal{M}$- and t- topologies coincide,

$$\lim_{s\to 0}{}^{t}\big[\big(a+\rho(s)\big)\ominus\big(b+\sigma(s)\big)\big]=0 \text{ and } \lim_{s\to 0}{}^{t}\big[\big(a+\varepsilon+\rho(s)\big)\ominus\big(b+\zeta+\sigma(s)\big)\big]=0,$$

and, thus,

$$\begin{aligned}\varepsilon\ominus\zeta &= \lim_{s\to 0}{}^{t}(\varepsilon\ominus\zeta)\\ &= \lim_{s\to 0}{}^{t}\big[\big(a+\varepsilon+\rho(s)\big)\ominus\big(a+\rho(s)\big)\ominus\big(b+\zeta+\sigma(s)\big)\oplus\big(b+\sigma(s)\big)\big]\\ &= \lim_{s\to 0}{}^{t}\big[\big(a+\varepsilon+\rho(s)\big)\ominus\big(b+\zeta+\sigma(s)\big)\big]\ominus\lim_{s\to 0}{}^{t}\big[\big(a+\rho(s)\big)\ominus\big(b+\sigma(s)\big)\big]\\ &= 0,\end{aligned}$$

hence $\varepsilon=\zeta$. $\square$

Let $\{w_1,\ldots,w_l\}$ be the set of all jump vectors for G.

Lemma 3.2.33. $\ker(\phi)\subseteq\mathbb{Z}w_1+\ldots+\mathbb{Z}w_l$.

Proof. Let $x=x_1+\ldots+x_m\in\ker(\phi)\subseteq U$, with $x_i\in H$. For all $i\in\{1,\ldots,m\}$, let $x_i(t)$ be a path in H from 0 to x_i. By Proposition 3.2.24,

$$\phi(x)=x_1\oplus\ldots\oplus x_m=x_1+\ldots+x_m+J_\gamma,$$

where γ is the t-loop $\big(x_1(t)\big)\vee\big(x_1\oplus x_2(t)\big)\vee\ldots\vee\big(x_1\oplus\ldots\oplus x_{m-1}\oplus x_m(t)\big)$ from 0 to $x_1\oplus\ldots\oplus x_m=\phi(x_1+\ldots+x_m)=\phi(x)=0$. We have: $x=-J_\gamma\in\mathbb{Z}w_1+\ldots+\mathbb{Z}w_l$. $\square$

A subgroup of the torsion-free group M^n is torsion-free. Thus, $\mathbb{Z}w_1+\ldots+\mathbb{Z}w_l$ is a finitely generated torsion-free abelian subgroup of M^n, and therefore it is free. Since $\ker(\phi)\leqslant\mathbb{Z}w_1+\ldots+\mathbb{Z}w_l$, it follows that $\ker(\phi)$ is a free abelian subgroup of

U generated by k $\mathbb{Z}$-independent elements, for some $k \leq l$. (The reader is referred to [Lang, Chapter I] for any of the above assertions.) In Claims 3.2.36 and 3.2.37 below we show that $k = n$.

Recall that, since H is generic in G, it must have dimension n, and therefore we can obtain a standard part map $U \to \mathbb{R}^n$ as in Section 2.4. We let $st := st_H$. We define

$$\forall x \in U,\ ||x|| := |st(x)|_{\mathbb{R}},$$

where $|\cdot|_{\mathbb{R}}$ is the Euclidean norm in $\mathbb{R}^n$. It is easy to check that $||\cdot||$ is a 'seminorm on U over $\mathbb{Q}$', that is:

(i) $\forall x, y \in U$, $||x+y|| \leq ||x|| + ||y||$, and (ii) $\forall q \in \mathbb{Q}, \forall x \in U$, $||qx|| = |q|\,||x||$.

Lemma 3.2.34. *For all $x \in U$ and $m \in \mathbb{N}$,*

$$x \in H^m \Leftrightarrow ||x|| < m\sqrt{n}.$$

Proof. With the notation of equations (2.1) and (2.2) from Section 2.4, we have $x \in H^m \Leftrightarrow \forall i, -me_i < \chi^i < me_i \Leftrightarrow st(x) \in [-m,m]^n \subset \mathbb{R}^n \Leftrightarrow |st(x)|_{\mathbb{R}} < \sqrt{nm^2} = m\sqrt{n}$. □

Let us also gather together two easy but helpful facts about $\ker(\phi)$:

Lemma 3.2.35. *(i)* $\ker(\phi) \cap H = \{0\}$.

(ii) Let Ξ be as in Lemma 3.2.29. Then $\forall x \in U, \exists y \in H^{\Xi}$, $y - x \in \ker(\phi)$.

Proof. (i) For all $x \in H$, $\phi(x) = x$.

(ii) For $x \in U$, since $\phi(x) \in G$, there are $x_1, \ldots, x_{\Xi} \in H$, such that $\phi(x) = x_1 \oplus \ldots \oplus x_{\Xi}$. Clearly, if $y = x_1 + \ldots + x_{\Xi} \in H^{\Xi}$, then $\phi(x) = \phi(y)$. □

We are now ready to compute the rank of $L = \ker(\phi)$. Fix a set $\{v_1, \ldots, v_k\}$ of generators for L.

Claim 3.2.36. *$k \geq n$.*

Proof. Assume, towards a contradiction, that $k < n$. For any $a \in U$, let $S_a := a + \mathbb{Z}v_1 + \ldots + \mathbb{Z}v_k$. Let Ξ be as in Lemma 3.2.29.

Subclaim. *There is $a \in U$, such that $S_a \cap H^{\Xi} = \emptyset$.*

Proof of Subclaim. By Lemma 3.2.34, it suffices to show that there is $a \in U$, such that $\forall l_1, \ldots, l_k \in \mathbb{N}$, $||a + l_1 v_1 + \ldots + l_k v_k|| \geq \Xi\sqrt{n}$. But,

$$||a + l_1 v_1 + \ldots + l_k v_k|| = |st(a) + l_1 st(v_1) + \ldots + l_k st(v_k)|_{\mathbb{R}},$$

and, since $k < n$, there is $\bar{a} \in \mathbb{R}^n$ such that $\forall l_1, \ldots, l_k \in \mathbb{N}$,

$$|\bar{a} + l_1 st(v_1) + \ldots + l_k st(v_k)|_{\mathbb{R}} \geq \Xi\sqrt{n}.$$

(This is true for *any* number $\Xi\sqrt{n}$.) We can take a to be any element in $st^{-1}(\bar{a})$. □

This contradicts Lemma 3.2.35(ii). □

Claim 3.2.37. *$k \leq n$.*

Proof. Notice that $st(L)$ is a lattice in $\mathbb{R}^n$ contained in $\mathbb{Z}st(v_1) + \ldots + \mathbb{Z}st(v_k)$. By Lemmas 3.2.35(i) and 2.4.5, $st(L)$ has rank k. Lemma 3.2.35(i) also gives us that $st(L)$ is discrete: the interior of $st(H)$ is an open neighborhood of 0 that contains no other elements from $st(L)$. But it is a classical fact that every discrete subgroup of $\mathbb{R}^n$ is generated by $\leq n$ elements (see [BD, Chapter I, Lemma 3.8], for example). Thus, $k \leq n$. □

Proof of Theorem 3.1.2. For convenience, we recall the main definitions and facts. In Step II, Definition 3.2.30, we defined the convex $\bigvee$-definable subgroup $U = \langle U, +_{\restriction U}, 0\rangle$ of M^n, generated by a generic, t-open, open n-parallelogram $H \subseteq G$ centered at 0. We also let $\phi : U \to G$ be such that $(\forall k \in \mathbb{N})(\forall x = x_1 + \ldots + x_k, h_i \in H)[\phi(x) = x_1 \oplus \ldots \oplus x_k]$. In Proposition 3.2.31, we proved that ϕ is an onto homomorphism, and in Step III, Claims 3.2.36 and 3.2.37, that $L := \ker(\phi) \leqslant U$ is a lattice of rank n. We have $U/L \cong G$ as abstract groups. Notice, ϕ restricted to a definable subset of U is a definable map.

Let $\Sigma := H^{\Xi}$, where Ξ is as in Lemma 3.2.29. Clearly, Σ is definable, and, thus, $\phi_{\restriction\Sigma}$ is definable. Moreover, E_L^{Σ} is definable, since, for all $x, y \in \Sigma$, we have $xE_L^{\Sigma}y \Leftrightarrow x - y \in L \Leftrightarrow \phi_{\restriction\Sigma}(x) = \phi_{\restriction\Sigma}(y)$. By Lemma 3.2.35(ii), Σ contains a complete set S of representatives for E_L^U, and, by definable choice, there is a definable such set S. By Claim 2.2.4(ii), $U/L = \langle S, +_S\rangle$ is a definable quotient group. The restriction of ϕ on S is a definable group isomorphism between $\langle S, +_S\rangle$ and G. By Remark 2.1.2(ii), we are done. □

The following is immediate.

Corollary 3.2.38. *For every $k \in \mathbb{N}$, the k-torsion subgroup of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$.*

3.3 On Pillay's Conjecture

In this section we show Pillay's Conjecture in the present context, that is, for $\mathcal{M}$ a saturated ordered vector space over an ordered division ring. The terminology was introduced in Chapter 1, but the reader is referred to [Pi2] for further details.

Theorem 3.3.1 (Pillay's Conjecture). *There is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension n.*

Proof. Recall that H is an open n-parallelogram with center 0. For $i < \omega$, we define H_i inductively as follows: $H_0 = H$, and $H_{i+1} = \frac{1}{2}H_i$. By Lemma 3.2.26, $B = \bigcap_{i<\omega} H_i$ is then a type-definable subgroup of G. As in the proof of Lemma 3.2.28, one can show that for all i, finitely many $\oplus$-translates of H_{i+1} cover H_i, and, thus, inductively, finitely many $\oplus$-translates of H_{i+1} cover G. It follows that B has bounded index in G. Note also that B is torsion-free: if $m \in \mathbb{N}$ and $x \in B \setminus \{0\}$, then $x \in H_m$, and, thus, by Lemma 3.2.26, $\underbrace{x \oplus \ldots \oplus x}_{m-\text{times}} = mx \neq 0$.

By [BOPP], there is a smallest type-definable subgroup G^{00} of bounded index, which is divisible, and G/G^{00} with the logic topology is a connected compact abelian Lie group. By [BOPP, Corollary 1.2], a torsion-free type-definable subgroup of G of bounded index is equal to G^{00}, hence $B = G^{00}$. Since G^{00} is torsion-free and divisible, it follows that for all k, the k-torsion subgroup of G/G^{00} is isomorphic to the k-torsion subgroup of G, which is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, by Corollary 3.2.38. Thus, G/G^{00} is isomorphic to the real n-torus and has dimension n. □

3.4 O-minimal fundamental group

The o-minimal fundamental group is defined as in the classical case (see [Hat], for example) except that all paths and homotopies are definable. The following is a restatement of the definition given in [BO2], where $\mathcal{M}$ expanded an ordered field. A different definition of the o-minimal fundamental group $\pi(G)$ was given in [Ed2] for a $\bigvee$-definable group in any o-minimal structure $\mathcal{M}$, namely, $\pi(G)$ is the kernel of the 'o-minimal universal covering homomorphism' $\widetilde{p} : \widetilde{G} \to G$ of G. In [EdEl2] it is shown that the two o-minimal fundamental groups coincide, for $\mathcal{M}$ any o-minimal expansion of an ordered group.

The next two definitions run in parallel with respect to the product topology of M^n and the t-topology on G. Notice that until Lemma 3.4.8, $\mathcal{M}$ can be any o-minimal expansion of an ordered group and G any group definable in $\mathcal{M}$.

Definition 3.4.1 ([vdD], Chapter 8, (3.1)). Let $f, g : M^m \supseteq X \to M^n$ (G) be two definable (t-)continuous maps in M^n (in G). A *(t-)homotopy between f and g* is a definable (t-)continuous map $F(t,s) : X \times [0,q] \to M^n$ (G), for some $q > 0$ in M, such that $f = F_0$ and $g = F_q$, where $\forall s \in [0,q]$, $F_s := F(\cdot, s)$. We call f and g *(t-)homotopic*, denoted by $f \sim g$ $(f \sim_t g)$.

Definition 3.4.2. Two (t-)paths $\gamma : [0,p] \to M^n$ (G), $\delta : [0,q] \to M^n$ (G), with $\gamma(0) = \delta(0)$ and $\gamma(p) = \delta(q)$, are called *(t-)homotopic* if there is some $t_0 \in [0, \min\{p,q\}]$, and a (t-)homotopy $F(t,s) : [0, \max\{p,q\}] \times [0,r] \to M^n$ (G), for some $r > 0$ in M, between

$$\gamma_{\restriction[0,t_0]} \vee \mathbf{c} \vee \gamma_{\restriction[t_0,p]} \text{ and } \delta \text{ (if } p \leq q\text{), or}$$

$$\delta_{\restriction[0,t_0]} \vee \mathbf{d} \vee \delta_{\restriction[t_0,q]} \text{ and } \gamma \text{ (if } q \leq p\text{),}$$

where $\mathbf{c}(t) = \gamma(t_0)$ and $\mathbf{d}(t) = \delta(t_0)$ are the constant paths with domain $[0, |p-q|]$.

If $\mathbb{L}(G)$ denotes the set of all t-loops that start and end at 0, then the restriction $\sim_t\restriction_{\mathbb{L}(G)\times\mathbb{L}(G)}$ is an equivalence relation on $\mathbb{L}(G)$. Let $\pi_1(G) := \mathbb{L}(G)/\sim_t$ and $[\gamma] :=$ the class of $\gamma \in \mathbb{L}(G)$.

It is clear that any two constant (t-)loops with image $\{0\}$ (but perhaps different domains) are (t-)homotopic. We can thus write $\mathbf{0}$ for the constant (t-)loop at 0 without specifying its domain.

Proposition 3.4.3. *$\langle \pi_1(G), \cdot, [\mathbf{0}] \rangle$ is a group, with $[\gamma] \cdot [\delta] := [\gamma \vee \delta]$.*

Proof. Definition 3.4.2 provides that for all t-paths $\gamma, \gamma', \delta, \delta'$, if $\gamma \sim_t \gamma', \delta \sim_t \delta'$, then $(\gamma \vee \delta) \sim_t (\gamma' \vee \delta')$, and therefore $\cdot$ is well-defined. Associativity is trivial since for all t-paths γ, δ, σ, $(\gamma \vee \delta) \vee \sigma = \gamma \vee (\delta \vee \sigma)$. Clearly, $[\mathbf{0}]$ is a left and right unit element. Finally, for $\gamma : [0, p] \to G$ a t-path, the class of $\gamma^*(t) := \gamma(p - t)$ is the left and right inverse $[\gamma]^{-1}$ of $[\gamma]$. Indeed, $(\gamma \vee \gamma^*) \sim_t \mathbf{0} : [0, 2p] \to \{0\}$ is witnessed by the t-homotopy $F(t, s) : [0, 2p] \times [0, p] \to G$, $F_t = \gamma_t \vee \gamma_t^*$, where $\gamma_t(u) : [0, p] \to G$ is a t-path with

$$\gamma_t(u) = \begin{cases} \gamma(u) & \text{if } 0 \le u \le t, \\ \gamma(t) & \text{if } t \le u \le p. \end{cases}$$

Replacing γ by γ^*, we get also $(\gamma^* \vee \gamma) \sim_t \mathbf{0}$. □

Definition 3.4.4 ([BO2]). We call $\pi_1(G) = \langle \pi_1(G), \cdot, [\mathbf{0}] \rangle$ the *o-minimal fundamental group of G*.

Note: We could instead define $\pi_1(G, v) := \mathbb{L}(G, v)/ \sim_t$, for every $v \in G$, where $\mathbb{L}(G, v)$ is the set of all t-loops that start and end at v. As it turns out, this is not necessary, since G is t-connected and $\pi_1(G, v)$ is, up to definable isomorphism, independent of the choice of v (by identically applying the classical proof of the same fact, as in [Hat, Proposition 1.5], for example).

Definition 3.4.5 ([vdD], Chapter 8, (3.1)). Let $A \subseteq X \subseteq M^m$. We say that X *deformation retracts* to A if there is a homotopy $F(t,s) : X \times [0,r] \to X$ such that $F(X,0) = A$, $F_1 = \mathbf{1}_X$, and $\forall s \in [0,r]$, $F(\cdot, s) \restriction_A = \mathbf{1}_A$.

Lemma 3.4.6. *For every $r \in M$, the n-box $\mathcal{B}_0^n(r) = (-r,r)^n \subset M^n$ deformation retracts to $\{0\}$.*

Proof. Let $B_m := \mathcal{B}_0^m(r) = (-r,r)^m \subset M^m$, $m > 0$, and $B_0 = \{0\}$. By induction, it suffices to show that for $m > 0$, B_m deformation retracts to B_{m-1}. But this is witnessed by the following homotopy in M^m: $F(t,s) : B_m \times [0,r] \to B_m$, with

$$F\big((t_1,\dots,t_m),s\big) = \begin{cases} (t_1,\dots,t_m) & \text{if } |t_m| \leq s, \\ (t_1,\dots,t_{m-1},s) & \text{if } t_m > s, \\ (t_1,\dots,t_{m-1},-s) & \text{if } t_m < -s. \end{cases}$$

□

Corollary 3.4.7. *Let $\gamma : [0,p] \to M^n$ be a loop with $\gamma(0) = 0$. Then $\gamma \sim \mathbf{0} : [0,p] \to \{0\}$.*

Proof. Since $[0,p] \subset M$ is closed and bounded, $\mathrm{Im}(\gamma)$ is (closed and) bounded by [PeS, Corollary 2.4], and, thus, $\mathrm{Im}(\gamma) \subseteq \mathcal{B}_0(r) \subset M^n$, for some $r \in M$. By Lemma 3.4.6, there is a deformation retraction $F(t,s) : \mathcal{B}_0(r) \times [0,q] \to M^n$ of $\mathcal{B}_0(r)$ to $\{0\}$. It is then not hard to check that $G(t,s) := F\big(\gamma(t),s\big) : [0,p] \times [0,q] \to M^n$ is a homotopy between γ and $\mathbf{0} : [0,p] \to \{0\}$. □

We now proceed to show that $\pi_1(G) \cong L = \ker(\phi)$. Let us first prove a useful lemma about paths and t-paths.

Lemma 3.4.8. *(i) Let $\delta : [0,p] \to U$ be a path. Then there are some $h_1, \ldots, h_m \in H$ with definable slopes (Definition 2.3.2) and, $\forall i \in \{1, \ldots, m\}$, linear paths $h_i(t) \in H$ from 0 to h_i, such that*

$$\delta(t) = \big(\delta(0) + h_1(t)\big) \vee \big(\delta(0) + h_1 + h_2(t)\big) \vee \ldots \vee \big(\delta(0) + h_1 + \ldots + h_{m-1} + h_m(t)\big).$$

(ii) Let $\gamma : [0,p] \to G$ be a t-path starting at $c \in G$. Then there are some $h_1, \ldots, h_m \in H$ with definable slopes and, $\forall i \in \{1, \ldots, m\}$, linear paths $h_i(t) \in H$ from 0 to h_i, such that

$$\gamma(t) = \big(c \oplus h_1(t)\big) \vee \big(c \oplus h_1 \oplus h_2(t)\big) \vee \ldots \vee \big(c \oplus h_1 \oplus \ldots \oplus h_{m-1} \oplus h_m(t)\big).$$

Proof. (i) By Remark 2.3.3(ii), it suffices to show the statement for δ being linear. Let $\delta(p) \in H^k$ for some $k \in \mathbb{N}$. Then, easily, $\frac{\delta(p)}{k} \in H$, and δ is the concatenation of k linear paths $\delta \restriction [0, \frac{p}{k}]$.

(ii) Let $\gamma(t) : [0,p] \to G$ with $\gamma(0) = c \in G$ and $H_1 := \frac{1}{2}H$. By Lemma 3.2.28, finitely many $\oplus$-translates, $\{a_i \oplus H_1\}_{\{1 \leq i \leq m\}}$, $m \in \mathbb{N}$, of H_1 cover $\mathrm{Im}(\gamma)$. By o-minimality, and since H_1 is t-open, we may assume that there are $0 = t_0, t_1, \ldots, t_m \in [0,p]$, such that $\forall i \in \{1, \ldots, m-1\}$, $\gamma(t_i) \in (a_i \oplus H_1) \cap (a_{i+1} \oplus H_1)$, $\gamma(t_0) = c \in a_1 \oplus H_1$, $\gamma(p) \in a_m \oplus H_1$, and that for each $i \in \{0, \ldots, m-1\}$,

(a) $\gamma^{i+1} := \gamma \restriction_{[t_i, t_{i+1}]}$ lies in $a_i \oplus H_1$,

(b) $\gamma^{i+1} \restriction_{(t_i, t_{i+1})}$ is linear, and

(c) γ does not jump at any $t \in (t_i, t_{i+1})$.

By (b), for all $i \in \{0, \ldots, m-1\}$, there exists some linear path $h_{i+1} : [t_i, t_{i+1}] \to M^n$ such that $\forall t \in (t_i, t_{i+1})$, $\gamma^{i+1}(t) = \gamma(t_i) + h_{i+1}(t)$. We denote $h_{i+1} := h_{i+1}(t_{i+1}) \in M^n$.

We work by induction on i. Suppose that for some $i \in \{1, \dots, m-1\}$, $\gamma(t_i) = c \oplus h_1 \oplus \dots \oplus h_i$ and $h_1, \dots, h_i \in H$. We show that $\forall t \in (t_i, t_{i+1}]$, $\gamma(t) = c \oplus h_1 \oplus \dots \oplus h_i \oplus h_{i+1}(t)$ and $h_{i+1} \in H$. Let us assume that γ^{i+1} does not jump at t_i. The other case can be handled similarly. By (c), γ^{i+1} does not jump at any $t \in [t_i, t_{i+1})$.

By (a), $\forall t \in (t_i, t_{i+1})$, $\gamma(t_i) + h_{i+1}(t) \in a_i \oplus H_1$. Since also $\gamma(t_i) \in a_i \oplus H_1$, we have $\big(\gamma(t_i) + h_{i+1}(t)\big) \ominus \gamma(t_i) \in (a_i \oplus H_1) \ominus (a_i \oplus H_1) \subseteq H$. By Lemma 3.2.23(ii), we have $\forall t \in (t_i, t_{i+1})$, $\big(\gamma(t_i) + h_{i+1}(t)\big) \ominus \gamma(t_i) = \big(\gamma(t_i) \ominus \gamma(t_i)\big) + h_{i+1}(t) = h_{i+1}(t)$. This shows that

$$\forall t \in [t_i, t_{i+1}),\ \gamma(t) = \gamma(t_i) + h_{i+1}(t) = \gamma(t_i) \oplus h_{i+1}(t).$$

We thus have:

$$\gamma(t_{i+1}) = \lim_{t \to t_{i+1}^-} {}^t\gamma(t) = \lim_{t \to t_{i+1}^-} {}^t[\gamma(t_i) \oplus h_{i+1}(t)] = \gamma(t_i) \oplus h_{i+1}(t_{i+1}).$$

That $h_{i+1} \in H$ is then also clear, since

$$h_{i+1}(t_{i+1}) = \gamma(t_{i+1}) \ominus \gamma(t_i) \in (a_i \oplus H_1) \ominus (a_i \oplus H_1) \subseteq H.$$

□

Lemma 3.4.9. $\ker(\phi) = \{J_\gamma : \gamma$ *is a t-loop*$\}$.

Proof. $\subseteq$. This is just Lemma 3.2.33. For $x \in \ker(\phi)$ and γ as in that proof, we have $x = -J_\gamma = J_{\gamma^*}$.

$\supseteq$. Let $\gamma(t)$ be a t-loop starting and ending at $c \in G$, and $h_1, \dots, h_m \in H$ as in Lemma 3.4.8(ii). Since γ is a t-loop, we have $c \oplus h_1 \oplus \dots \oplus h_m = c$ and, thus,

$h_1 \oplus \ldots \oplus h_m = 0$. On the other hand, by Proposition 3.2.24, $c \oplus h_1 \oplus \ldots \oplus h_m = c + \sum_{i=0}^m h_i + J_\gamma$ and, thus, $J_\gamma = -\sum_{i=0}^m h_i$. Therefore, $\phi(J_\gamma) = \phi\Big(-\sum_{i=0}^m h_i\Big) = \ominus\phi\Big(\sum_{i=0}^m h_i\Big) = \ominus(h_1 \oplus \ldots \oplus h_m) = 0$. □

For a t-path $\gamma : [0,p] \to G$ starting at c, we fix some h_i and $[t_{i-1}, t_i] \ni t \mapsto h_i(t) \in H$, $i \in \{1, \ldots, m\}$, to be as in Lemma 3.4.8(ii).

Definition 3.4.10. Let $\gamma : [0,p] \to G$ be a t-path starting at $c \in G$. Let $d \in U$ such that $\phi(d) = c$. The *lifting of* γ *at* d is the following path $\hat{\gamma}_d : [0,p] \to U$,

$$\hat{\gamma}_d(t) = \big(d + h_1(t)\big) \vee \big(d + h_1 + h_2(t)\big) \vee \ldots \vee \big(d + h_1 + \ldots + h_{m-1} + h_m(t)\big).$$

Let γ as above be in addition a t-loop. By Proposition 3.2.24, $c = c \oplus h_1 \oplus \ldots \oplus h_m = c + h_1 + \ldots + h_m + J_\gamma$. It follows that

$$J_\gamma = 0 \Leftrightarrow h_1 + \ldots + h_m = 0 \Leftrightarrow \hat{\gamma}_d \text{ is a loop in } U.$$

Lemma 3.4.11. *(i) For any t-path $\gamma : [0,p] \to G$ starting at c, and $d \in U$ such that $\phi(d) = c$, we have $\phi \circ \hat{\gamma} = \gamma$.*

(ii) For any path $\delta : [0,p] \to U$, $J_{\phi\circ\delta} = \phi\big(\delta(p)\big) - \phi\big(\delta(0)\big) - \big(\delta(p) - \delta(0)\big)$. In particular, for any loop δ in U, $J_{\phi\circ\delta} = 0$.

Proof. (i) This is clear, since $\phi\big(d + h_1 + \ldots + h_{i-1} + h_i(t)\big) = c \oplus h_1 \oplus \ldots \oplus h_{i-1} \oplus h_i(t)$.

(ii) By Lemma 3.4.8(i), let $h_1, \ldots, h_m \in H$ have definable slopes and, $\forall i \in \{1, \ldots, m\}$, let $h_i(t) \in H$ be a linear path from 0 to h_i, such that $\delta(t) = \big(\delta(0) + h_1(t)\big) \vee \big(\delta(0) + h_1 + h_2(t)\big) \vee \ldots \vee \big(\delta(0) + h_1 + \ldots + h_{m-1} + h_m(t)\big)$. It is then $\delta = \hat{\gamma}_{\delta(0)}$, where $\gamma(t) = \big(c \oplus h_1(t)\big) \vee \big(c \oplus h_1 \oplus h_2(t)\big) \vee \ldots \vee \big(c \oplus h_1 \oplus \ldots \oplus h_{m-1} \oplus h_m(t)\big)$, with $c = \phi(\delta(0))$. By (i), $\phi \circ \delta = \gamma$, and then Proposition 3.2.24

gives $c \oplus h_1 \oplus \ldots \oplus h_m = c + \sum_{i=0}^{m} h_i + J_{\phi\circ\delta}$. Therefore, $J_{\phi\circ\delta} = (c \oplus h_1 \oplus \ldots \oplus h_m) - c - \sum_{i=0}^{m} h_i = \phi\big(\delta(p)\big) - \phi\big(\delta(0)\big) - \big(\delta(p) - \delta(0)\big)$. □

Lemma 3.4.12. *For every $\gamma \in \mathbb{L}(G)$, $\gamma \sim_t \mathbf{0} \Leftrightarrow J_\gamma = 0$.*

Proof. ($\Leftarrow$). Let $\gamma \in \mathbb{L}(G)$ with $J_\gamma = 0$. Then $\hat{\gamma}_0$ is a loop in U, homotopic to $\mathbf{0}$ by Corollary 3.4.7. Since ϕ is t-continuous, the image of the homotopy under ϕ is a t-homotopy between γ and $\mathbf{0}$.

($\Rightarrow$). Assume now $\gamma \sim_t \mathbf{0}$, witnessed by $F(t,s) : [0,p] \times [0,r] \to G$, say $\gamma(t) = F_r(t) = F(t,r)$. Since $F(0,s) = 0 = F(p,s)$ for all s, the paths $\widehat{F(0,s)}_0$, $\widehat{F(p,s)}_0$ should equal $\mathbf{0}$. This means that for all s, $\widehat{(F_s)}_0$ is a loop in U. By Lemma 3.4.11(i), $J_\gamma = J_{\phi\circ\widehat{\gamma}}$, and by Lemma 3.4.11(ii), $J_{\phi\circ\widehat{(F_r)}_0} = 0$. It follows, $J_\gamma = J_{\phi\circ\widehat{\gamma}} = J_{\phi\circ\widehat{(F_r)}_0} = 0$. □

Theorem 3.4.13. $\pi_1(G) \cong \ker(\phi) = L$.

Proof. By Lemma 3.4.9, we have to show that the map $j : \pi_1(G) \ni [\gamma] \mapsto J_\gamma \in \{J_\gamma : \gamma \text{ is a } t\text{-loop}\}$ is a group isomorphism. Note: $\forall \gamma, \delta \in \mathbb{L}(G)$, $J_{\gamma\vee\delta} = J_\gamma + J_\delta$ and $J_{\gamma^*} = -J_\gamma$. Now, j is well-defined and one-to-one, since for all $\gamma : [0,p] \to G$ and $\delta : [0,q] \to G$ in $\mathbb{L}(G)$,

$$[\gamma] = [\delta] \Leftrightarrow [\gamma] \cdot [\delta^*] = 0 \Leftrightarrow [\gamma \vee \delta^*] = 0 \Leftrightarrow J_{\gamma\vee\delta^*} = 0 \Leftrightarrow J_\gamma = J_\delta,$$

where the third equivalence is by Lemma 3.4.12. Trivially, j is onto, and it is a group homomorphism by the note above. □

Remark 3.4.14. The pair $\langle U, \phi \rangle$ can be considered as a universal covering space for G, in the sense that (i) there is a definable t-open covering $\{G_i\}$ of G such that every $\phi^{-1}(G_i)$ is a disjoint union of open sets in U, each of which is mapped by ϕ homeomorphically onto G_i, and (ii) U is 'definably' simply-connected. Indeed:

(i) Let $\{a_i \oplus H\}$ be a finite t-open covering of G by $\oplus$-translates of H. We show that for all i, $\phi^{-1}(a_i \oplus H) = \bigsqcup_{\phi(x)=a}(x + H)$ is a disjoint union of open sets in U. Let $x \neq y$ with $\phi(x) = \phi(y)$. We show $(x + H) \cap (y + H) = \emptyset$. If there were $h_1, h_2 \in H$ with $x + h_1 = y + h_2$, then $\phi(h_1 - h_2) = \phi(y - x) = 0$, and, thus, $\phi(h_1) = \phi(h_2)$. Since ϕ restricted to H is the identity, we have $h_1 = h_2$. Thus, $x = y$, a contradiction. It is also not hard to see that ϕ restricted to $x + H$ is a homeomorphism onto $\phi(x) \oplus H$.

(ii) U is easily definably path-connected. To see that it is also simply-connected, first notice that for every $k \in \mathbb{N}$, H^k deformation retracts to $\{0\}$, with a proof similar to the proof of Lemma 3.4.6. Then observe that for every loop γ in U, there is a homotopy in U between γ and $\mathbf{0}$, with the same proof as Corollary 3.4.7, after replacing $\mathcal{B}_0(r)$ by some H^k such that $\text{Im}(\gamma) \subseteq H^k$. It follows that U is simply-connected.

CHAPTER 4

THE GENERAL CASE

4.1 Introduction

In this chapter we generalize the work from Chapter 3 and prove Theorems 1, 2 and 4 from Chapter 1 in full. In Section 4.2, we extend our proofs to the case where G is not necessarily definably compact (Theorems 4.1.5, 4.2.25 and 4.2.28), and in Section 4.3 to the case where, moreover, $\mathcal{M}$ is any linear o-minimal expansion of an ordered group (Theorems 4.3.8, 4.3.6 and 4.3.11).

Theorem 1 is suggested as a structure theorem analogous to the following classical theorem (see, for example, [Bour]).

Fact 4.1.1. *Every connected abelian real Lie group is isomorphic to a direct sum of copies of the additive group $\langle \mathbb{R}, + \rangle$ of the reals and the circle topological group S^1.*

In addition to the remarks following Fact 3.1.1, the following observations explain why a model theoretic analogue of Fact 4.1.1 takes the form of Theorem 4.1.5 below.

Let $G = \langle G, \oplus, e_G \rangle$ be a definable group equipped with the t-topology.

Fact 4.1.2 ([PeS]). *If G is not definably compact, then it has an 1-dimensional torsion-free $\mathcal{M}$-definable subgroup.*

By induction on the dimension of G, we obtain:

Fact 4.1.3. *Assume G is abelian. Then there are $\mathcal{M}$-definable subgroups $\{e_G\} = G_0 < G_1 < \ldots < G_r \leqslant G$, such that G/G_r is definably compact and G_{i+1}/G_i is a 1-dimensional torsion-free group, for $i = 0, \ldots, r-1$.*

Until Section 4.3, we fix $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D}\rangle$ to be a big saturated ordered vector space over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1\rangle$.

Fact 4.1.4 ([EdEl1]). *Let G be a definable group. Then the torsion-free subgroup G_r of G from Fact 4.1.3 is definably isomorphic to $\langle M^r, +, 0\rangle$.*

Let $G = \langle G, \oplus, e_G\rangle$ be a definable group of dimension n. We identify $\langle G_r, \oplus, e_G\rangle$ with $M^r = \langle M^r, +, 0\rangle$. Moreover, we fix a definable complete set of representatives $K \subseteq G$ for G/M^r that contains e_G. We also identify K with G/M^r, and we let $K = \langle K, \oplus_K, e_G\rangle$ be the topological group with the structure induced by the canonical surjection $q : G \to G/M^r$. We call K the *compact part of* G. We have $G = \{a \oplus u : a \in M^r,\ u \in K\}$. The following is a short exact sequence:

$$0 \to M^r \to G \xrightarrow{q} K \to 0.$$

As we know by examples in [PeS] and [Str], we cannot always expect G to be definably isomorphic to the direct sum of M^r and K; that is, the above short exact sequence does not always definably split (see [PeSt] for more on definable splitting). We show:

Theorem 4.1.5. *Let G be an n-dimensional definable group which is t-connected. Assume that the compact part of G has dimension s. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant \langle M^n, +, 0\rangle$, and a lattice L of rank s.*

We obtain the following two corollaries.

Theorem 4.2.25. *Let G be as in Theorem 4.1.5. Then there is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s.*

Theorem 4.2.28. *Let G be as in Theorem 4.1.5. Then the o-minimal fundamental group of G is isomorphic to L.*

The structure of this chapter is as follows. Section 4.2 handles the case where G is not necessarily definably compact. The proofs are generalizations of the corresponding ones from Chapter 3. Several arguments run in complete analogy and we are then brief. We outline the main differences with Chapter 3 at the beginning of Section 4.2.1.

Section 4.3 handles the case where $\mathcal{M}$ is any linear o-minimal expansion of an ordered group.

4.2 G not necessarily definably compact

In Section 4.2, we fix a $\emptyset$-definable group $G = \langle G, \oplus, e_G \rangle$ which is t-connected and has dimension n. We fix a definable complete set of representatives $K \subseteq G$ for G/M^r as above. Let $s := \dim(K) = n - r$.

By Corollary 3.2.5, G is abelian.

4.2.1 The Structure Theorem

Outline. The proof of Theorem 4.1.5 is a generalization of the proof of Theorem 3.1.2, presented in Section 3.2. We keep the same structure; namely, the proof runs in three steps, as follows. Step I contains the main tools arising from a local analysis of G and a comparison of the two group operations $\oplus$ and $+$. In Step II, we define the $\bigvee$-definable subgroup $U \leqslant M^n$, and a group homomorphism ϕ from U onto G. In Step III, we show that $L := \ker(\phi)$ is a lattice of rank s and finish the proof.

The main idea of the generalization is described next. In Section 3.2, in Step II, the existence of a generic open n-parallelogram $H \subseteq W^G$ in G was proved, and the subgroup $U \leqslant M^n$ was generated by H. Here, in Step II, the existence of a suitable generic open s-parallelogram H in K is proved, and the subgroup $U \leqslant M^n$ is generated by $H^G := M^r \times H$. Many of the definitions and arguments following in Steps II and III are then identical with the corresponding ones from Section 3.2, after replacing H by H^G.

The main differences from Section 3.2 are described next. Since G is not necessarily definably compact, it may not be bounded, and therefore we cannot always assume that all λ_i's in Lemma 4.2.3 below are equal to $\mathbb{I}_n$ (which was the case in Lemma 3.2.11. The consequences are: (i) the main tools from Step I need to be restated, (ii) the lattice $L = \ker(\phi)$ no longer consists of elements of the form J_γ, for γ t-loop, but rather, of the form W_γ, for γ t-loop, and W_γ defined in Step III.

Before we start with the main body of our proof, we include some introductory lemmas that will be used throughout without mentioning. Recall that

$$G = \{a \oplus u : a \in M^r,\, u \in K\},$$

where $M^r = \langle M^r, +, 0\rangle$ is the torsion-free subgroup of G, such that $K = \langle K, \oplus_K, 0\rangle = G/M^r$ is a definably compact group.

Lemma 4.2.1. *For all $x_1, x_2 \in K$, there is $a \in M^r$, such that*

$$x_1 \oplus_K x_2 = x_1 \oplus x_2 \oplus a.$$

Proof. Since $q(x_1 \oplus x_2) = x_1 \oplus_K x_2 = q(x_1 \oplus_K x_2)$, we have $(x_1 \oplus_K x_2) \ominus (x_1 \oplus x_2) \in \ker(q) = M^r$. □

Lemma 4.2.2. $M^r \cap K = \{0\}$.

Proof. This is because K is a complete set of representatives for G/M^r. Indeed, let $x \in M^r \cap K$. Then $\ominus x \in M^r$ and $x \ominus x = 0 \in K$. Thus, x is in the same class with 0 and must be equal to 0. □

It follows that G is definably bijective to $M^r \times K$. By Lemma 2.3.7, K may be assumed to be a subset of M^s. Therefore, we may assume that

$$G = M^r \times K \subseteq M^n.$$

Moreover, we identify an element $a \in M^r$ with the tuple $(a, 0) \in G$, and an element $u \in K$ with the tuple $(0, u) \in G$.

STEP I. Comparing $\oplus$ with $+$. Let us start with several facts that were shown in Section 3.2 without (using) the assumption that G is definably compact. First of all, $V := V^G$ is large in G. By cell decomposition, V is the disjoint union of finitely many open (t-)connected components $V_0, \ldots, V_N$, that is, $V = \bigsqcup_{i \in I} V_i$, for a fixed index set $I := \{0, \ldots, N\}$. Also, it may be assumed that $e_G = 0 \in V_0$, using the translation

$$f : G \ni x \mapsto (x \oplus c) - c \in f(G) \subseteq M^n, \tag{4.1}$$

as in Lemma 3.2.9. Since V is large in G, we may also assume that $G \subseteq \overline{V}$ (Lemma 3.2.15). We have $G = \bigcup_{i \in I} G_i$, where $G_i := G \cap \overline{V_i}$. Finally, it was shown in Lemma 3.2.8:

Lemma 4.2.3. *There are invertible $\lambda_0, \dots, \lambda_N \in \mathbb{M}(n, D)$ such that for any $i, j \in I = \{0, \dots, N\}$, $u \in V_i$ and $v \in V_j$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$, we have $u + \lambda_i \varepsilon, v + \lambda_j \varepsilon \in G$, and*

$$(u + \lambda_i \varepsilon) \ominus u = (v + \lambda_j \varepsilon) \ominus v.$$

In particular, $\lambda_0 = \mathbb{I}_n$.

Assuming G is definably compact, we can suitably 'scale' the G_i's so that all λ_i's in Lemma 4.2.3 may be assumed to be equal to $\mathbb{I}_n$ (Proposition 3.2.11). Since we do not assume definable compactness here, we need to carry all λ_i's along our proof. For example, the next corollary follows from Lemma 4.2.3 in an analogous way to the one that Corollary 3.2.13 follows from Lemma 3.2.8.

Corollary 4.2.4. *For all $u \in V_i$, $v \in G$, such that $u \oplus v \in V_k$, $i, k \in I$, there is $r > 0$ in M, such that for all $\varepsilon \in (-r, r)^n$,*

$$(u + \lambda_i \varepsilon) \oplus v = (u \oplus v) + \lambda_k \varepsilon. \tag{4.2}$$

Moreover, it is a straightforward exercise to check that analogous versions of the statements in Section 3.2 from Lemma 3.2.18 up to Proposition 3.2.24 go through here. We thus obtain our two final statements of this first step, regarding the validity of the equation (4.2) for u, v and $u \oplus v$ in G, and ε arbitrary in M^n. First, Lemma 3.2.23 generalizes to the following:

Lemma 4.2.5. *Let $u, v \in G$, such that $u \in G_i$ and $u \oplus v \in G_k$, $i, k \in I$. Let also $\gamma(t) = u + \lambda_i \varepsilon(t) : [0, p] \to G_i$, $\varepsilon(0) = 0$, be a t-path with no jumps, such that*

(i) $(u \oplus v) + \lambda_k \varepsilon(t)$ is a t-path in G_k,

or

(ii) $\big(u + \lambda_i \varepsilon(t)\big) \oplus v$ is a t-path in G_k with no jumps.

Then:

$$\big(u + \lambda_i \varepsilon(p)\big) \oplus v = (u \oplus v) + \lambda_k \varepsilon(p).$$

Given a t-path $\gamma(t) = u + \lambda_i \varepsilon(t) : [0, p] \to G_i$, $\varepsilon(0) = 0$, with no jumps, and $v \in G$, we let $0 = t_0 < t_1 < \ldots < t_l = p$ be such that if $\gamma_j := \gamma \restriction_{(t_{j-1}, t_j)}$, $1 \leq j \leq l$, then every $\gamma_j \oplus v$ lies entirely in some $G_{k(j)}$ and has no jumps. Denote $\varepsilon_j := \varepsilon(t_j) - \varepsilon(t_{j-1})$, $1 \leq j \leq l$. Under this notation, Proposition 3.2.24 generalizes to the following:

Proposition 4.2.6. *Let $u, v \in G$, such that $u \in G_i$ and $u \oplus v \in G_k$, $i, k \in I$. Let also $\gamma(t) = u + \lambda_i \varepsilon(t) : [0, p] \to G_i$, $\varepsilon(0) = 0$, be a t-path with no jumps. Then:*

$$\big(u + \lambda_i \varepsilon(p)\big) \oplus v = (u \oplus v) + \lambda_{k(1)} \varepsilon_1 + \ldots + \lambda_{k(l)} \varepsilon_l + J_{\gamma \oplus v}.$$

STEP II. A generic open s-parallelogram of K. In what follows, if X is a definable group and $Y \subseteq X$, by the *X-topology on Y* we mean the subspace topology on A induced by the t-topology on X. A subset of Y is then called *X-open* if it is open in the X-topology of Y.

In particular, we distinguish between the following two subspace topologies on $K \subseteq M^s$: the G-topology, induced by the t-topology of G, and the $\mathcal{M}$-topology, induced by the product topology of M^s. Moreover, K has its own t_K-topology as a definable group.

Observe that the t_K-topology is the same as the quotient topology induced by the canonical surjection $q : G \to G/M^r$, by Fact 2.1.1. In particular, it is clear that a subset $A \subseteq K$ is t_K-open if and only if $M^r \times A \subseteq G$ is t-open.

Lemma 4.2.7. *On a large subset W of K the $\mathcal{M}$-, the G- and the t_K- topologies coincide.*

Proof. This is because all these three topologies on K are definable manifold topologies. Clearly the $\mathcal{M}$- and the t_K- topologies are definable, and it is easy to see that the G-topology is definable as well. We provide the details.

We show that for every dim-generic element a of K, there is a G-open neighborhood of a on which all three topologies coincide. Assuming the notation of the t-topology on G from Section 2.1, let $\Sigma_i := S_i \cap K$ and $U_i := \phi_i(\Sigma_i)$. By cell decomposition, and since $\dim(K) = s$, we may assume that all U_i's are open subsets of M^s. That is, $\{\langle \Sigma_i, \phi_i \restriction_{\Sigma_i} \rangle : i \in J\}$ is a definable atlas on K for the G-topology. By o-minimality, if $a \in \Sigma_i$ is a dim-generic element of K, then ϕ_i must be a homeomorphism with respect to all of the $\mathcal{M}$-, the G- and the t_K-topologies on some G-open subset A of K that contains a, and, thus, on A all these three topologies coincide. $\square$

We fix $W \subseteq M^s$ as in Lemma 4.2.7. Since on W the $\mathcal{M}$- and G- topologies coincide, every path is a t-path and vice versa. Since on W the $\mathcal{M}$- and t_K-topologies coincide, every open subset of W is t_K-open and vice versa.

Since W is large in K, it is also generic, by Fact 2.3.9(i). By the Linear Cell Decomposition Theorem, W is a finite union of linear cells, and by Lemma 2.3.10, one of them, call it Y, must be generic. By Fact 2.3.9(ii), Y has dimension s, and by Lemma 2.3.7, it is bounded. Therefore, by Lemma 2.3.6, $\overline{Y}$ is a finite union of closed s-parallelograms, say $W_1, \ldots, W_l$. For $i \in \{1, \ldots, l\}$, let $Y_i := Y \cap W_i$.

Then $Y = Y_1 \cup \ldots \cup Y_l$. By Lemma 2.3.10 again, one of the Y_i's must be generic, say Y_1. Let $H := \text{Int}(Y_1)$. Since on W the $\mathcal{M}$- and t_K- topologies coincide, $H = \text{Int}(Y_1)^{t_K}$. By Fact 2.3.9(iii), H is generic. Since W_1 is a closed s-parallelogram and $\text{Int}(W_1) = \text{Int}(W_1 \cap \overline{Y}) = \text{Int}(W_1 \cap Y) = \text{Int}(Y_1) = H$, we have that H is an open s-parallelogram.

Our next goal is to show that H may be assumed to have center $0 = e_G$. Let c be the center of H. Consider the following two definable bijections:

$$f_G : G \ni x \mapsto (x \oplus c) - c \in f_G(G) \subseteq M^n,$$

$$f_K : K \ni x \mapsto (x \oplus_K c) - c \in f_K(K) \subseteq M^n.$$

Without loss of generality, we may assume that the translation f used in (4.1) from Step I was $f = f_G$ (and, thus, the set V remains unaltered).

Now, let $G' := f_G(G)$, $R' := f_G(M^r)$, $K' := f_K(K)$, $H' := f_K(H \ominus_K c) = H - c$. Let $G' = \langle G', +_{G'}, 0 \rangle$ and $K' = \langle K', +_{K'}, 0 \rangle$ be the induced topological group structures induced by f_G and f_K, respectively. By Remark 2.1.2, f_G and f_K are definable isomorphisms, for all $x, y \in G'$,

$$x +_{G'} y = [(x + c) \ominus c \oplus (y + c)] - c,$$

for all $x, y \in K'$,

$$x +_{K'} y = [(x + c) \ominus_K c \oplus_K (y + c)] - c,$$

and it suffices to show the following.

Lemma 4.2.8. *(i)a) R' is a definable torsion-free subgroup of G' definably isomorphic to M^r, such that G'/R' is a definably compact group, b) $K' \subseteq G'$ is a definable complete set of representatives for G'/R', and c) $+_{K'}$ coincides with the operation induced by the canonical surjection $q : G' \to G'/R'$.*

(ii) $H' \subseteq K'$ is a G'-open, $t_{K'}$-open, open s-parallelogram, with center $0 = e_{G'}$, generic in K', and on which the $\mathcal{M}$-, G'- and $t_{K'}$- topologies coincide.

Proof. (i)a) holds because $f_G : G \to G'$ is a definable isomorphism.

For (i)b), we first show that $K' \subseteq G'$. Let $g \in K$. Let $g_1 = (g \oplus_K c) \ominus c$. Then $(g \oplus_K c) - c = (g_1 \oplus c) - c \in G'$.

Next we show that K' is a definable set of representatives for G'/R'. Let $g' = f_G(g) = (g \oplus c) - c \in G'$, for some $g \in G$. Since $G = M^r \oplus K$, there are $a \in M^r$ and $k \in K$ such that $g \oplus c = a \oplus (k \oplus_K c) \in G$. Then $f_G(a) +_{G'} f_K(k) = [(a \oplus c) - c] +_{G'} (k - c) = [a \oplus c \ominus c \oplus (k \oplus_K c)] - c = [a \oplus (k \oplus_K c)] - c = g \oplus c) - c = g'$.

Finally, K' is complete: assume $f_K(k_1) = f_G(a) +_{G'} f_K(k_2)$, for some $k_1, k_2 \in K$ and $a \in M^r$. We show $k_1 = k_2$ and, thus, $f_K(k_1) = f_K(k_2)$. We have, $f_K(k_1) = k_1 - c$ and $f_G(r) +_{G'} f_K(k_2) = [(a \oplus c) - c] +_{G'} [(k_2 \oplus_K c) - c] = [a \oplus (k_2 \oplus_K c)] - c$. Thus, $k_1 \oplus_K c = a \oplus (k_2 \oplus_K c)$. Since K is a complete set of representatives for G/M^r, $k_1 \oplus_K c = k_2 \oplus_K c$ and, thus, $k_1 = k_2$.

For (i)c), we show that for every $x, y \in K'$, there is $r \in R'$ such that $x +_{K'} y = x +_{G'} y +_{G'} r$. We have $x +_{K'} y = [(x + c) \ominus_K c \oplus_K (y + c)] - c = [(x + c) \ominus c \oplus (y + c) \oplus m] - c$, for some $m \in M^r$. Let $r := (m \oplus c) - c = f_G(m) \in R'$. Then we have $x +_{G'} y +_{G'} +r = \big([(x + c) \ominus c \oplus (y + c)] - c\big) +_{G'} [(m \oplus c) - c] = [(x + c) \ominus c \oplus (y + c) \ominus c \oplus m \oplus c] - c = x +_{K'} y$.

(ii) It is clear that $H' = H - c$ is an open s-parallelogram with center 0. Since H is generic in K, $H \ominus_K c$ is generic in K, and, thus, $H' = f_K(H \ominus_K c)$ is generic in K'. For the rest, consider the definable automorphism

$$\overline{f_K} : M^n \ni x \mapsto x - c \in M^n.$$

To see that H' is G'-open, and that on H' the $\mathcal{M}$- and G'- topologies coincide, notice that $\overline{f_K} \restriction_G$ is in fact a homeomorphism from G to G', since for all $x \in G$, $\overline{f_K}(x) = f_G(x \ominus c)$.

To see that H' is $t_{K'}$-open, and that on H' the $\mathcal{M}$- and $t_{K'}$- topologies coincide, notice that $\overline{f_K} \restriction_K$ is also a homeomorphism from K to K', since for all $x \in K$, $\overline{f_K}(x) = f_K(x \ominus_K c)$. □

Summarizing, we may assume that:

- H is a generic in K, G-open, t_K-open, open s-parallelogram, with center 0, on which the $\mathcal{M}$-, G- and t_K- topologies coincide.

The next three lemmas are the same as Lemmas 3.2.25-3.2.27, respectively, and we omit their proofs. For Lemmas 4.2.9 and 4.2.11, in particular, the convexity of H is essential.

Lemma 4.2.9. *Let $a, b \in H$, such that $a + b \in H$. There is a path $\varepsilon(t)$ in H from 0 to a, such that the path $\varepsilon(t) + b$ lies entirely in H, as well.*

Since on H on which the $\mathcal{M}$- and G- topologies coincide, the paths $\varepsilon(t)$ and $b + \varepsilon(t)$ from Lemma 4.2.9 are also t-paths.

Lemma 4.2.10. *Let $x_1, \ldots, x_l \in H$ be such that for any subset σ of $\{1, \ldots, l\}$, $\sum_{j \in \sigma} x_j \in H$. Then $x_1 + \ldots + x_l = x_1 \oplus \ldots \oplus x_l$.*

Proof. The proof is the same as for Lemma 3.2.26, using Lemma 4.2.5(i) in place of Lemma 3.2.23(i), for $\lambda_i = \lambda_k = \lambda_0 = \mathbb{I}_n$. □

Lemma 4.2.11. *For every $x_1, \ldots, x_l, y_1, \ldots, y_m \in H$, if $x_1 + \ldots + x_l = y_1 + \ldots + y_m$, then $x_1 \oplus \ldots \oplus x_l = y_1 \oplus \ldots \oplus y_m$.*

We let

$$H^G := \{a \oplus u : a \in M^r,\ u \in H\} = M^r \times H.$$

Since H is generic in K, it is easy to see that H^G is generic in G.

Lemma 4.2.12. *Let $(H^G)_1 := \frac{1}{2}H^G = \{\frac{1}{2}x : x \in H^G\}$. Then $(H^G)_1$ is generic in G.*

Proof. Let $H_1 := \frac{1}{2}H$. By Lemma 3.2.28, H_1 is generic in K. Since $(H^G)_1 = M^r \times H_1$, it is easy to see that $(H^G)_1$ is generic in G. □

Lemma 4.2.13. *There is $\Xi \in \mathbb{N}$, such that $G = \underbrace{H^G \oplus \ldots \oplus H^G}_{\Xi-times}$.*

Proof. Since H_1 is t_K-open, $(H^G)_1$ is t-open. The proof is then the same with the proof of Lemma 3.2.29, after replacing H_1 by $(H^G)_1$. □

Definition 4.2.14. Let U_H be the subgroup of M^s generated by H as in Section 2.4; that is, $U_H =< H >= \bigcup_{k<\omega} H^k$. Let U denote the subgroup U_{H^G} of M^n generated by H^G; that is,

$$U =< H^G >= \bigcup_{k<\omega} (H^G)^k.$$

Equivalently, $U := M^r \times U_H$. By Lemma 4.2.11, the following function $\phi : U \to G$ is well-defined. For all $x_1 = (a_1, u_1), \ldots, x_k = (a_k, u_k) \in H^G = M^r \times H$, if $x = x_1 + \ldots + x_n$, then

$$\phi(x) = x_1 \oplus \ldots \oplus x_k = (a_1 + \ldots + a_k) \oplus u_1 \oplus \ldots \oplus u_k,$$

where the second equation is obtained using the identifications we assumed in the discussion before Step I.

Since M^r and $U_H = \langle U_H, +_{\restriction U_H}, 0\rangle$ are subgroups of M^n, so is their direct product $U = M^r \times U_H$. Easily, U is a $\bigvee$-definable group, and convexity of H implies convexity of U. Moreover, the proof of Proposition 3.2.31, after replacing H by H^G, shows the following.

Proposition 4.2.15. *ϕ is a t-continuous group homomorphism from U onto G.*

Thus, if we let $L := \ker(\phi)$, we know that $U/L \cong G$ as abstract groups.

STEP III. L is a lattice of rank s. Let $\{\lambda_i\}_{i \in I}$ be as in Step I. For $i, j \in I$, let $\lambda_{ij} := \lambda_j \lambda_i^{-1}$.

Lemma 4.2.16. *The set*

$$\{(b' - b) - \lambda_{ij}(a' - a) : i, j \in I,\ a, a' \in \overline{G_i},\ b, b' \in \overline{G_j},\ a \sim_G b, a' \sim_G b'\}$$

is finite.

Proof. Suppose not. Since $\sim_G$ is definable, by o-minimality it is not very hard to see that the following must be true:

(A) there are $i, j \in I$ and two non-constant paths γ, δ with $\mathrm{Im}(\gamma) \subseteq \mathrm{bd}(V_i)$ and $\mathrm{Im}(\delta) \subseteq \mathrm{bd}(V_j)$, such that:

- every element in $\mathrm{Im}(\gamma)$ is $\sim_G$-equivalent with an element in $\mathrm{Im}(\delta)$, and vice versa, and

- for all $a, a' \in \mathrm{Im}(\gamma)$, $b, b' \in \mathrm{Im}(\delta)$ such that $a \sim_G b$ and $a' \sim_G b'$,

$$b' - b \neq \lambda_{ij}(a' - a).$$

By o-minimality again, we may assume that $\gamma(t) = a + \varepsilon(t) : [0, p_\gamma] \to M^n$ and $\delta(t) = b + \zeta(t) : [0, p_\delta] \to M^n$, for some paths $\varepsilon(t)$ and $\zeta(t)$ in H with $\varepsilon(0) = 0$, $\varepsilon := \varepsilon(p_\gamma) \neq 0$, $\zeta(0) = 0$, $\zeta := \zeta(p_\delta) \neq 0$, and $a \sim_G b$. Moreover, we may assume that there is a path $\rho(s) : [0, q] \to M^n$, with $\rho(0) = 0$, such that $\forall s > 0$, $a + \rho(s)$ and $a + \varepsilon + \rho(s)$ are in G, and $a + \rho(s) + \varepsilon(t) : [0, p_\gamma] \to G$ is a t-path, with no jumps, from $a + \rho(s)$ to $a + \rho(s) + \varepsilon$. Similarly, we may assume that there is a path $\sigma(s) : [0, p] \to M^n$, with $\sigma(s) = 0$, such that $\forall s > 0$, $b + \sigma(s)$ and $b + \zeta + \sigma(s)$ are in G, and $b + \sigma(s) + \zeta(t) : [0, p_\delta] \to G$ is a t-path, with no jumps, from $b + \sigma(s)$ to $b + \sigma(s) + \zeta$. We show that if $a + \varepsilon \sim_G b + \zeta$, then $\lambda_{ij}\varepsilon = \zeta$, which contradicts (A). By Lemma 4.2.5(i), we have that for any $s \in (0, p_\gamma] \cap (0, p_\delta]$,

$$\big(a + \rho(s) + \varepsilon\big) \ominus \big(a + \rho(s)\big) = \lambda_i^{-1}\varepsilon \text{ and } \big(b + \sigma(s) + \zeta\big) \ominus \big(b + \sigma(s)\big) = \lambda_j^{-1}\zeta.$$

On the other hand, since $a \sim_G b$ and $a + \varepsilon \sim_G b + \zeta$,

$$\lim_{s \to 0} \big[\big(a + \rho(s)\big) \ominus \big(b + \sigma(s)\big)\big] = 0 \text{ and } \lim_{s \to 0} \big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big)\big] = 0.$$

Since in a t-neighborhood of 0 the $\mathcal{M}$- and t- topologies coincide,

$$\lim_{s \to 0}{}^{t} \big[\big(a + \rho(s)\big) \ominus \big(b + \sigma(s)\big)\big] = 0 \text{ and } \lim_{s \to 0}{}^{t} \big[\big(a + \varepsilon + \rho(s)\big) \ominus \big(b + \zeta + \sigma(s)\big)\big] = 0,$$

and, thus,

$$
\begin{aligned}
(\lambda_i^{-1}\varepsilon) \ominus (\lambda_j^{-1}\zeta) &= \lim_{s\to 0}{}^{t}[(\lambda_i^{-1}\varepsilon) \ominus (\lambda_j^{-1}\zeta)] \\
&= \lim_{s\to 0}{}^{t}\big[\big(a+\varepsilon+\rho(s)\big) \ominus \big(a+\rho(s)\big) \ominus \big(b+\zeta+\sigma(s)\big) \oplus \big(b+\sigma(s)\big)\big] \\
&= \lim_{s\to 0}{}^{t}\big[\big(a+\varepsilon+\rho(s)\big) \ominus \big(b+\zeta+\sigma(s)\big)\big] \ominus \lim_{s\to 0}{}^{t}\big[\big(a+\rho(s)\big) \ominus \big(b+\sigma(s)\big)\big] \\
&= 0.
\end{aligned}
$$

It follows $\lambda_i^{-1}\varepsilon = \lambda_j^{-1}\zeta$, hence $\lambda_{ij}\varepsilon = \zeta$. □

Let $\gamma : [0,p] \to G$ be a t-path starting at $c \in G$. Recall, $H^G = M^r \times H$. Then there are $x_1, \dots, x_l \in H^G$ of definable slopes and, $\forall j \in \{1, \dots, l\}$, linear paths $x_j(t) \in H^G$ from 0 to x_j, such that

$$
\gamma(t) = \big(c \oplus x_1(t)\big) \vee \big(c \oplus x_1 \oplus x_2(t)\big) \vee \ldots \vee \big(c \oplus x_1 \oplus \ldots \oplus x_{l-1} \oplus x_l(t)\big). \tag{4.3}
$$

This is just Lemma 3.4.8(ii), with the identical proof, after replacing H by H^G, and Lemma 3.2.23(ii) by Lemma 4.2.5(ii). Moreover, by that proof, we may assume that there are $0 = t_0 < \ldots < t_l = p$ such that $\forall j \in \{1, \dots, l\}$, if γ^j denotes the t-path

$$
\gamma^j : [t_{j-1}, t_j] \ni t \mapsto c \oplus x_1 \oplus \ldots \oplus x_{j-1} \oplus x_j(t),
$$

then $\gamma \restriction_{(t_{j-1},t_j)}$ lies entirely in some $G_{k(j)}$ and has no jumps (where $x_0 := c$). We refer to the form of equation (4.3) as a *decomposition of* γ. By Proposition 4.2.6,

$$
x_1 \oplus \ldots \oplus x_l = x_1 + \ldots + x_l + \sum_{j=1}^{l} \big(\lambda_{k(j)} - \mathbb{I}_n\big)\, x_j + J_\gamma. \tag{4.4}
$$

We let

$$W_\gamma := \sum_{j=1}^{l} \left(\lambda_{k(j)} - \mathbb{I}_n\right) x_j + J_\gamma.$$

It is not hard to verify the following.

Remark 4.2.17. (i) W_γ does not depend on the choice of x_j and $x_j(t)$.

(ii) For any two t-paths γ_1 and γ_2, $W_{\gamma_1 \vee \gamma_2} = W_{\gamma_1} + W_{\gamma_2}$.

(iii) Let $\gamma : [0,p] \to G$ and $\gamma' : [0,p'] \to G$ be two t-paths that pass through the same components of G, in the same order, and have the same jumps. More precisely, assume that $0 = t_0 < \ldots < t_l = p$ and $0 = t'_0 < \ldots < t'_l = p'$ are as above for γ and γ', respectively, and that the following hold for all $j \in \{1, \ldots, l\}$:

- γ^j and $(\gamma')^j$ lie in the same component $G_{k(j)}$, and
- $\gamma(t_j) = \gamma'(t'_j)$, $\lim_{t \to t_j^-} \gamma(t) = \lim_{t \to t_j'^-} \gamma'(t)$, and
  $\lim_{t \to t_j^+} \gamma(t) = \lim_{t \to t_j'^+} \gamma'(t)$.

Then $W_\gamma = W_{\gamma'}$.

(iv) If $\gamma^*(t) = \gamma(p-t)$ then $W_{\gamma^*} = -W_\gamma$.

For all $i, j \in \{1, \ldots, N\}$, fix some $a_i \in \overline{G_i}$ and $b_j \in \overline{G_j}$ such that $a_i \sim_G b_j$. Then Lemma 4.2.16 says that the set

$$U_{ij} := \{b' - b_j - \lambda_{ij}(a' - a_i) : \overline{G_i} \ni a' \sim_G b' \in \overline{G_j}\},$$

is finite.

Denote $u_{ij} := b_j - a_i$. Fix one element c_i in each G_i, and for every $i, j \in I$, a t-path δ_{ij} starting at c_i and ending at c_j which has a unique jump equal to u_{ij}.

Denote by W the subgroup of M^n generated by the finite set

$$\{W_{\delta_{ij}} + \lambda_j^{-1} w : w \in U_{ij}, i, j \in I\}.$$

Lemma 4.2.18. *Let $\gamma : [0,p] \to G$ be a t-path starting at $c \in G$. Then $W_\gamma \in W$.*

Proof. Assume that γ passes through $G_{k(1)}, \ldots, G_{k(l)}$. Consider

$$\delta := \delta_{k(1)k(2)} \vee \ldots \vee \delta_{k(l-1)k(l)}.$$

By Remark 4.2.17(iii), we may assume that γ passes through all $c_{k(j)}$, $1 \leq j \leq l$. Let $s_j \in [0,p]$ be such that $\gamma(s_j) = c_{k(j)}$. According to Remark 4.2.17(ii), it suffices to show that $\forall j \in \{2, \ldots, l\}$, $W_{\gamma\restriction_{[s_{j-1}, s_j]}} = W_{\delta_{k(j-1)k(j)}} + \lambda_{k(j)}^{-1} w$, for some $w \in U_{k(j-1)k(j)}$. We may thus assume that $l = 2$. Then $\delta = \delta_{12}$. To simplify notation, let us also assume, without loss of generality, that γ starts at $c \in G_1$, it has only one jump, which is equal to u, and that γ ends at $\gamma(p) \in G_2$. Let

$$\gamma(t) = \big(c \oplus x_1(t)\big) \vee \big(c \oplus x_1 \oplus x_2(t)\big)$$

be a decomposition of γ. On the other hand, consider a decomposition of $\delta : [0,q] \to G$,

$$\delta(t) = \big(c \oplus y_1(t)\big) \vee \big(c \oplus y_1 \oplus y_2(t)\big),$$

$y_i, y_i(t) \in H^G$, with a unique jump equal to u_{12}. We show:

$$W_\gamma = W_{\delta_{12}} + \lambda_2^{-1} w,$$

for some $w \in U_{12}$. We have

$$W_\gamma = (\lambda_1 - \mathbb{I}_n)\, x_1 + (\lambda_2 - \mathbb{I}_n)\, x_2 + u,$$

$$W_{\delta_{12}} = (\lambda_1 - \mathbb{I}_n)\, y_1 + (\lambda_2 - \mathbb{I}_n)\, y_2 + u_{12}.$$

Assume that the unique jump u_{12} of δ occurs at t_δ. Then

$$a_1 = \lim_{t \to t_\delta^-} \delta(t) = \lim_{t \to t_\delta^-} \big(c \oplus y_1(t)\big) = \lim_{t \to t_\delta^-} \big(c + y_1(t)\big) = c + \lambda_1 y_1,$$

where the third equality is by Lemma 4.2.5. Also,

$$b_2 = \lim_{t \to t_\delta^+} \delta(t) = c + \lambda_1 y_1 + u_{12}.$$

Similarly, if we assume that the unique jump u of γ occurs at t_γ, then

$$a' := \lim_{t \to t_\gamma^-} \gamma(t) = c + \lambda_1 x_1$$

and

$$b' := \lim_{t \to t_\gamma^+} \gamma(t) = c + \lambda_1 x_1 + u.$$

Let $w \in U_{12}$ be as follows:

$$w = b' - b_2 - \lambda_{12}(a' - a_1) = \lambda_1 x_1 + u - (\lambda_1 y_1 + u_{12}) - \lambda_{12}(\lambda_1 x_1 - \lambda_1 y_1).$$

We have:

$$\lambda_1 x_1 + u - (\lambda_1 y_1 + u_{12}) = \lambda_{12}(\lambda_1 x_1 - \lambda_1 y_1) + w = \lambda_2(x_1 - y_1) + w, \tag{4.5}$$

that is,

$$u - u_{12} = (\lambda_2 - \lambda_1)(x_1 - y_1) + w. \tag{4.6}$$

On the other hand, since $\gamma(p) = \delta(q)$, we know, using Proposition 4.2.6, that

$$c + \lambda_1 x_1 + \lambda_2 x_2 + u = c \oplus x_1 \oplus x_2 = c \oplus y_1 \oplus y_2 = c + \lambda_1 y_1 + \lambda_2 y_2 + u_{12},$$

and, thus,

$$\lambda_1 x_1 + u - (\lambda_1 y_1 + u_{12}) = \lambda_2 (y_2 - x_2). \tag{4.7}$$

By (4.5) and (4.7),

$$x_2 - y_2 = y_1 - x_1 - \lambda_2^{-1} w.$$

Using also (4.6),

$$W_\gamma - W_{\delta_{12}} = (\lambda_1 - \mathbb{I}_n)(x_1 - y_1) + (\lambda_2 - \mathbb{I}_n)(x_2 - y_2) + u - u_{12} =$$

$$(\lambda_1 - \mathbb{I}_n)(x_1 - y_1) + (\lambda_2 - \mathbb{I}_n)(y_1 - x_1 - \lambda_2^{-1} w) + (\lambda_1 - \lambda_2)(y_1 - x_1) + w = \lambda_2^{-1} w.$$

□

Lemma 4.2.19. $\ker(\phi) \subseteq W$.

Proof. For an element $x \in U$, fix $x_1, \ldots, x_l \in H$ with definable slopes, so that $x = x_1 + \ldots + x_l \in H^G$. Furthermore, for all $j \in \{1, \ldots, l\}$, fix a linear path $x_i(t)$ in H from 0 to x_i, and let $\gamma_x : [0, p] \to G$ be the following t-path:

$$\big(x_1(t)\big) \vee \big(x_1 \oplus x_2(t)\big) \vee \ldots \vee \big(x_1 \oplus \ldots \oplus x_{l-1} \oplus x_l(t)\big).$$

We may assume that x_j and $x_j(t)$ are so that the above is a decomposition of γ, essentially by Remark 4.2.17(ii). By Proposition 4.4,

$$\phi(x) = x_1 \oplus \ldots \oplus x_l = x_1 + \ldots + x_l + W_{\gamma_x}.$$

Thus, if $x \in \ker(\phi)$, then

$$x = x_1 + \ldots + x_l = x_1 + \ldots + x_l = -W_{\gamma_x}.$$

By Lemma 4.2.18, $x \in W$. $\square$

It follows that $\ker(\phi)$ is a free abelian subgroup of U generated by some k elements. In Claims 4.2.22 and 4.2.23 below we show that $k = s$.

The following lemma is Lemma 3.2.35 after replacing H by H^G.

Lemma 4.2.20. *(i)* $\ker(\phi) \cap H^G = \{0\}$.

(ii) Let Ξ *be as in Lemma 4.2.13. Then* $\forall x \in U, \exists y \in (H^G)^\Xi$, $y - x \in ker(\phi)$.

Recall now that, since H is generic in K it must have dimension s, and therefore we can obtain a standard part map $st_H : U_H \to \mathbb{R}^s$ as in Section 2.4. We define another group homomorphism $st^G : U \to \mathbb{R}^s$ as follows. For every $x = (a, u) = M^r \times U_H$, let

$$st^G(x) = st_H(u).$$

Moreover, we let

$$||x|| := |st^G(x)|_{\mathbb{R}},$$

where $|\cdot|_{\mathbb{R}}$ is the Euclidean norm in $\mathbb{R}^n$. As in Lemma 3.2.34, we can show the following.

Lemma 4.2.21. *For all $x = (a, u) \in M^r \times U_H$ and $m \in \mathbb{N}$,*

$$x \in (H^G)^m \Leftrightarrow ||u|| < m\sqrt{s}.$$

Fix a set $\{v_1, \dots, v_k\}$ of generators for L. Then the reader can check that the following two claims have identical proofs with the ones of Claims 3.2.36, 3.2.37, respectively, after replacing H be H^G, st by st^G, and n by s.

Claim 4.2.22. $k \geq s$.

Claim 4.2.23. $k \leq s$.

Proof of Theorem 4.1.5. In Definition 4.2.14, we defined a convex $\bigvee$-definable subgroup $U \leqslant M^n$, and an onto group homomorphism $\phi : U \to G$ (Proposition 4.2.15). In Claims 4.2.22 and 4.2.23, we showed that $L := \ker(\phi) \leqslant U$ is a lattice of rank s.

Let $\Sigma := (H^G)^\Xi$, where Ξ is as in Lemma 4.2.13. Then Σ and $\phi_{\restriction\Sigma}$ are definable. Moreover, E_L^Σ is definable, since, for all $x, y \in \Sigma$, we have $xE_L^\Sigma y \Leftrightarrow x - y \in L \Leftrightarrow \phi_{\restriction\Sigma}(x) = \phi_{\restriction\Sigma}(y)$. By Lemma 4.2.20(ii), Σ contains a complete set S of representatives for E_L^U, and, by definable choice, there is a definable such set S. By Claim 2.2.4, $U/L = \langle S, +_S\rangle$ is a definable quotient group. The restriction of ϕ on S is a definable group isomorphism between $\langle S, +_S\rangle$ and G. By Remark 2.1.2(ii), we are done. □

The following is immediate.

Corollary 4.2.24. *For every $k \in \mathbb{N}$, the k-torsion subgroup of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$.*

4.2.2 On Pillay's Conjecture

In [BOPP], the existence of G^{00} was established for a group G definable in any o-minimal structure. Here, we compute the dimension of the compact Lie group G/G^{00}, for our fixed G and $\mathcal{M}$. The special case where G is definably compact constitutes Pillay's Conjecture for $\mathcal{M}$, proved in Theorem 3.3.1.

Theorem 4.2.25. *There is a smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s.*

Proof. For $i < \omega$, we define H_i inductively as follows: $H_0 = H$, and $H_{i+1} = \frac{1}{2}H_i$. Let also for every $i < \omega$, $\left(H^G\right)_i := M^r \times H_i$. Denote

$$B = \bigcap_{i<\omega} \left(H^G\right)_i = \bigcap_{i<\omega} (M^r \times H_i) = M^r \times \left(\bigcap_{i<\omega} H_i \right).$$

By Lemma 4.2.10, it is easy to see that B is a subgroup of G. As in Theorem 3.3.1, one can show that for all $i < \omega$, the set $(H^G)_i$ is generic in G, and, thus, B has bounded index in G. Moreover, it is not hard to see that B is torsion-free, and, thus, by [BOPP], it must be the smallest type-definable subgroup G^{00} of G of bounded index, and G/G^{00} with the logic topology is a connected compact abelian Lie group. Since G^{00} is torsion free and divisible ([BOPP]), it follows that for all k, the k-torsion subgroup of G/G^{00} is isomorphic to the k-torsion subgroup of G, which is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^s$, by Corollary 4.2.24. Thus, G/G^{00} is isomorphic to the real s-torus and has dimension s. □

4.2.3 O-minimal fundamental group

The entire Section 3.4 goes through word-by-word here, after replacing H by H^G, and J_γ by W_γ whenever γ is a t-path. In particular, with notation from that section, the following are true.

Lemma 4.2.26. $\ker(\phi) = \{W_\gamma : \gamma \textit{ is a } t\textit{-loop}\}$.

Lemma 4.2.27. *For every* $\gamma \in \mathbb{L}(G)$, $\gamma \sim_t \mathbf{0} \Leftrightarrow W_\gamma = 0$.

Theorem 4.2.28. $\pi_1(G) \cong \ker(\phi) = L$.

Proof. By Lemma 4.2.27, the map $j : \pi_1(G) \ni [\gamma] \mapsto W_\gamma \in \{W_\gamma : \gamma \text{ is a } t\text{-loop}\}$ is a group isomorphism. □

Remark 4.2.29. The pair $\langle U, \phi\rangle$ can be considered as a universal covering space for G.

4.3 $\mathcal{M}$ any linear o-minimal expansion of an ordered group

Here we show that Theorems 4.1.5, 4.2.25 and 4.2.28 hold for a group G definable in a saturated linear o-minimal expansion of an ordered group (see Theorems 4.3.8, 4.3.6 and 4.3.11, respectively). We recall the following definition from [LP].

Definition 4.3.1 ([LP]). An o-minimal expansion $\mathcal{M} = \langle M, +, <, \ldots\rangle$ of an ordered group is called *linear* if for every $\mathcal{M}$-definable function $f : A \subseteq M^n \to M$, there is a partition of A into finitely many definable A_i, such that for each i, if $x, y, x+t, y+t \in A_i$, then

$$f(x+t) - f(x) = f(y+t) - f(y).$$

Fact 4.3.2 ([LP]). *Let $\mathcal{M} = \langle M, +, <, 0, \ldots \rangle$ be a linear o-minimal expansion of an ordered group. Then $\mathcal{M}$ can be elementarily embedded into a reduct of an ordered vector space $\mathcal{N} = \langle N, +, <, 0, \{\lambda\}_{\lambda \in D} \rangle$ over an ordered division ring D.*

Let $\mathcal{M}$ and $\mathcal{N}$ be as above, saturated, and G a t-connected, $\mathcal{M}$-definable group of dimension n. We may assume that $\mathcal{M}$ is a reduct of $\mathcal{N}$, and, thus, G is also $\mathcal{N}$-definable. Then Theorems 4.1.5 and 4.2.25 are true but with all definability stated with respect to $\mathcal{N}$. Namely, since H is $\mathcal{N}$-definable, $U =< M^r \times H >$ is $\bigvee$-definable over $\mathcal{N}$, and $G^{00} = \bigcap_{i<\omega} (M^r \times H_i)$ is type-definable over $\mathcal{N}$. We show however in Theorem 4.3.6 below that G^{00} is 'absolute'.

For a group G definable in a saturated o-minimal structure $\mathcal{M}$, we denote by $G^{00}_{\mathcal{M}}$ the smallest type-definable over $\mathcal{M}$ subgroup of G of bounded index (which exists by [BOPP, Theorem 1.1]). The following fact was pointed out by Pillay. (See [HPP] for any terminology.)

Fact 4.3.3 ([HPP]). *Let T be an o-minimal theory, $\mathcal{M}$ a saturated model of T, and G a group definable in $\mathcal{M}$. Assume:*

(1) For all definable $X \subseteq G$, either X or $G \setminus X$ is generic.

(2) There is a left-invariant Keisler measure on G.

Then (G^{00} exists and) G^{00} is torsion-free.

The following fact has already been used in the proofs of Theorems 3.3.1 and 4.2.25.

Fact 4.3.4 ([BOPP], Corollary 1.2). *Let G be a group definable in some saturated o-minimal structure $\mathcal{M}$. Assume that X is a torsion-free, type-definable over $\mathcal{M}$, subgroup of G of bounded index. Then $X = G^{00}_{\mathcal{M}}$.*

Corollary 4.3.5. *Let K be an abelian, definably compact group, definable in a saturated o-minimal expansion $\mathcal{M}$ of an ordered group. Let $\mathcal{N}$ be a saturated o-minimal expansion of $\mathcal{M}$. Then $K^{00}_{\mathcal{M}} = K^{00}_{\mathcal{N}}$.*

Proof. We first verify that the assumptions of Fact 4.3.3 hold for K: (1) holds by Lemma 2.3.10, and (2) holds because K is abelian. It follows that $K^{00}_{\mathcal{M}}$ is torsion-free. By Fact 4.3.4, $K^{00}_{\mathcal{M}} = K^{00}_{\mathcal{N}}$. □

For the rest of Section 4.3, let $\mathcal{M} = \langle M, +, <, 0, \ldots \rangle$ be a big saturated linear o-minimal expansion of an ordered group, G a t-connected, $\mathcal{M}$-definable group of dimension n, and $\mathcal{N}$ a big saturated ordered vector space over an ordered division ring expanding $\mathcal{M}$ as in Fact 4.3.2.

By Fact 4.1.3, there is a torsion-free $\mathcal{M}$-definable subgroup G_r of G such that $K = G/G_r$ is a definably compact $\mathcal{M}$-definable group of dimension s.

Theorem 4.3.6. $G^{00}_{\mathcal{M}} = G^{00}_{\mathcal{N}}$. *Therefore $G/G^{00}_{\mathcal{M}}$ equipped with the logic topology is a compact Lie group of dimension s.*

Proof. Since G and K are also $\mathcal{N}$-definable, by Fact 4.1.4, $G = M^r \times K$. Moreover, we can find $H \subseteq K$ as in Step II of Section 2, which is $\mathcal{N}$-definable. By Theorem 3.3.1, $K^{00}_{\mathcal{N}} = \bigcap_{i<\omega} H_i$, and by the proof of Theorem 4.2.25, $G^{00}_{\mathcal{N}} = M^r \times K^{00}_{\mathcal{N}}$. Since K is abelian, $K^{00}_{\mathcal{M}}$ is torsion-free, as in the proof of Corollary 4.3.5. Therefore, $M^r \times K^{00}_{\mathcal{M}}$ is torsion-free. Since $K^{00}_{\mathcal{M}}$ has bounded index in K, easily $M^r \times K^{00}_{\mathcal{M}}$ has bounded index in G. By Fact 4.3.4, $G^{00}_{\mathcal{M}} = M^r \times K^{00}_{\mathcal{M}}$. But, by Corollary 4.3.5, $K^{00}_{\mathcal{M}} = K^{00}_{\mathcal{N}}$. It follows that $G^{00}_{\mathcal{M}} = M^r \times K^{00}_{\mathcal{M}} = M^r \times K^{00}_{\mathcal{N}} = G^{00}_{\mathcal{N}}$.

The rest follows by Theorem 4.2.25. □

In case G is definably compact, we obtain Pillay's Conjecture.

Corollary 4.3.7 (Pillay's Conjecture). *Assume G is a t-connected, definably compact, $\mathcal{M}$-definable group of dimension s. Then there is a smallest type-definable over $\mathcal{M}$ subgroup G^{00} of G of bounded index, and G/G^{00} equipped with the logic topology is a compact Lie group of dimension s.*

Theorem 4.3.8. *$U =< M^r \times H >$ is $\bigvee$-definable over $\mathcal{M}$. Therefore, G is definably isomorphic to a definable quotient group U/L, where U is a $\bigvee$-definable over $\mathcal{M}$ subgroup of M^n and L is a lattice of rank s.*

Proof. Since K^{00} is type-definable over $\mathcal{M}$ and it is contained in the $\mathcal{N}$-definable H, by compactness, there exists some $\mathcal{M}$-definable subset X of H that contains K^{00}. On the other hand, since $K^{00} = \bigcap_{i<\omega} H_k$ is contained in X, by compactness again, there exists some H_k contained in X. We have $H_k \subseteq X \subseteq H$, and therefore $U_H =< X >$ is a $\bigvee$-definable over $\mathcal{M}$ subgroup of M^s. We have that $U =< M^r \times X >$ is a $\bigvee$-definable over $\mathcal{M}$ subgroup of M^n.

The rest follows from Theorem 4.1.5. □

We finally turn to Theorem 4.2.28 in the linear setting. Lemmas 4.2.26 and 4.2.27 are still true with definability taken in $\mathcal{N}$. By $\sim_t^{\mathcal{M}}$ we denote a t-homotopy between two $\mathcal{M}$-definable paths, where the definability of the homotopy is taken in $\mathcal{M}$. If $\mathbb{L}^{\mathcal{M}}(G)$ denotes the set of all $\mathcal{M}$-definable t-loops that start and end at 0, then let $\pi_1^{\mathcal{M}}(G) := \mathbb{L}^{\mathcal{M}}(G)/\sim_t^{\mathcal{M}}$ and $[\gamma]^{\mathcal{M}}$:= the class of $\gamma \in \mathbb{L}^{\mathcal{M}}(G)$.

Lemma 4.3.9. $\ker(\phi) = \{W_\gamma : \gamma$ *is an $\mathcal{M}$-definable t-loop*$\}$.

Proof. $\subseteq$. Let X be as in the proof of Theorem 4.3.8. We may assume that X is t-connected. It is then not hard to see that the path γ_x in the proof of Lemma 4.2.19 can be chosen to be $\mathcal{M}$-definable.

$\supseteq$. By Lemma 4.2.26. □

Lemma 4.3.10. *For every $\gamma \in \mathbb{L}^{\mathcal{M}}(G)$, $\gamma \sim_t^{\mathcal{M}} \mathbf{0} \Leftrightarrow W_\gamma = 0$.*

Proof. ($\Leftarrow$). For G definably compact, the proof boils down to the observation that the homotopy in Corollary 3.4.7 is $\mathcal{M}$-definable. The non-compact case follows after replacing H by H^G, and J_γ by W_γ whenever γ is a t-path.

($\Rightarrow$). By Lemma 4.2.27. □

Theorem 4.3.11. $\pi_1^{\mathcal{M}}(G) \cong \ker(\phi) = L$.

Proof. By Lemma 4.3.10, the map

$$j : \pi_1(G) \ni [\gamma]^{\mathcal{M}} \mapsto W_\gamma \in \{W_\gamma : \gamma \text{ is an } \mathcal{M}\text{-definable } t\text{-loop}\}$$

is a group isomorphism. □

CHAPTER 5

COMPACT DOMINATION

5.1 Introduction

The notion of compact domination arose in [HPP] in connection with the solution of Pillay's Conjecture for groups definable in o-minimal expansions of real closed fields ([Pi2]). The intuition behind the Compact Domination Conjecture is that the canonical homomorphism $\pi : G \to G/G^{00}$ from Pillay's Conjecture is a kind of intrinsic 'standard part map'. Recall, by definition, a set $A \subseteq G/G^{00}$ is closed in the logic topology if and only if $\pi^{-1}(A) \subseteq G$ is type-definable ([LaPi]).

When working over a saturated o-minimal expansion $\mathcal{M}$ of an ordered field, standard part maps have already appeared, among others, in the following two occasions. In [BO3, Definition 4.1], a standard part map is defined from the 'finite part' $Fin(M^n)$ of M^n onto $\mathbb{R}^n$, for $n \in \mathbb{N}_+$. In [PePi, Section 4], if $G(\mathbb{R})$ is a compact group of dimension n definable in an o-minimal expansion $\mathcal{M}_0$ of $\mathbb{R}$, and G is the realization of $G(\mathbb{R})$ in a saturated elementary extension $\mathcal{M}$ of $\mathcal{M}_0$, then a standard part map is defined from G onto $G(\mathbb{R})$. In both cases, the standard part map st has the desired properties so that a notion of measure can be defined for the definable subsets of $Fin(M^n)$ and G, respectively. Namely, if λ denotes the Lebesque measure on $\mathbb{R}^n$, or the Haar measure on $G(\mathbb{R})$, respectively, then a definable set $X \subseteq Fin(M^n)$, or $X \subseteq G$, respectively, is given measure $\lambda\big(st(X)\big)$.

The main idea of this chapter is to observe the correspondence between our standard part map $st : U \to \mathbb{R}^n$ defined in Chapter 3 and the standard part map $st : Fin(M^n) \to \mathbb{R}^n$ from [BO3] above.

We next recall some terminology from [HPP] and give the definition of compact domination. The context is more general than the one of definable groups that we consider here. So, for the purposes of Definition 5.1.1 and Fact 5.1.2 below, let $\mathcal{N}$ be any saturated structure, and let 'definable' mean 'definable in $\mathcal{N}$ (with parameters)'. By a *small* set or a set of *bounded cardinality* we mean a set of cardinality less than $|N|$. By a type-definable set we mean an intersection of a small collection of definable sets. For a type-definable set X, by $Def(X)$ we denote the set of all subsets of X which are definable in $\mathcal{N}$.

Let X be a type-definable over A set, and C a compact Hausdorff space of bounded cardinality. A map $f : X \to C$ is called *A-definable* if for every closed set $C_1 \subseteq C$, $f^{-1}(C_1) \subseteq X$ is type-definable over A.

A *Keisler measure on X* is a finitely additive probability measure on $Def(X)$, that is, a map $\mu : Def(X) \to [0,1]$ such that $\mu(\emptyset) = 0$, $\mu(X) = 1$, and for $Y, Z \in Def(X)$, $\mu(Y \cup Z) = \mu(Y) + \mu(Z) - \mu(Y \cap Z)$.

Definition 5.1.1 ([HPP]). Suppose X is a type-definable over $\emptyset$ set, C is a compact Hausdorff space of bounded cardinality equipped with a probability measure μ, and $\sigma : X \to C$ is a $\emptyset$-definable surjective map. We say that X is *compactly dominated by* (C, μ, σ) if for all $Y \in Def(X)$,

$$\mu\big(\{c \in C : \sigma^{-1}(c) \cap Y \neq \emptyset \text{ and } \sigma^{-1}(c) \cap (X \setminus Y) \neq \emptyset\}\big) = 0.$$

Let G be a type-definable over $\emptyset$ group. We say that G is *compactly dominated (as a group)* if G is compactly dominated by $(H, \mathbf{m}, \sigma)$, where H is a compact

Hausdorff group, $\mathbf{m}$ is the unique normalized $\big(\mathbf{m}(H) = 1\big)$ Haar measure on H, and σ is a group homomorphism.

When we work with a type-definable group, we always refer to compact domination in the group sense.

Fact 5.1.2 ([HPP], Proposition 9.3, Theorem 9.5). *Let G be a type-definable over $\emptyset$ group which is compactly dominated by $(H, \mathbf{m}, \sigma)$. Then*

(i) G^{00} exists and equals $\ker(\sigma)$.

(ii) G has a unique left (and right) invariant Keisler measure μ', given by: for all $X \in Def(G)$, $\mu'(X) = \mathbf{m}\big(\sigma(X)\big)$.

From now on $\mathcal{M} = \langle M, <, +, 0, \ldots \rangle$ again denotes a saturated o-minimal expansion of an ordered group, and 'definable' means 'definable in $\mathcal{M}$ with parameters'.

For the rest of Chapter 5, if G is a definable group, we assume that the language contains constants for the parameters that are used to define G, that m denotes the unique normalized Haar measure on G/G^{00}, and that π denotes the $\emptyset$-definable group homomorphism from G onto G/G^{00}. (The fact that G^{00} always exists and that G/G^{00} is a compact Lie group is by [BOPP].)

Compact Domination Conjecture ([HPP]). *Assume that G is a definably compact definable group that satisfies Pillay's Conjecture. Then G is compactly dominated by $(G/G^{00}, \mathbf{m}, \pi)$.*

Fact 5.1.3 ([HPP], Lemma 10.5). *Suppose G is a definably compact definable group of dimension n, such that, for all $X \in Def(G)$,*

$$\dim(X) < n \Rightarrow \mathbf{m}\big(\pi(X)\big) = 0. \tag{5.1}$$

Then G is compactly dominated by $(G/G^{00}, \mathbf{m}, \pi)$.

It is not hard to see that the converse of Fact 5.1.3 is also true, and, thus, the Compact Domination Conjecture stated above is equivalent to its restatement in Chapter 1.

Let us note that (5.1) was a crucial property that implicitly held for st in place of π in both accounts [BO3] and [PePi] mentioned above, for A a '$\mathbb{Q}$-bounded' definable subset of $Fin(M^n)$ in [BO3], and $A \in Def(G)$ in [PePi], respectively. Additionally, st resembled π in that a bounded set $A \subseteq \mathbb{R}^n$, or $A \subseteq G$, respectively, is closed if and only if $st^{-1}(A)$ is type-definable. In fact, the combination of these two properties for the group example in [PePi] imply that G is compactly dominated. This same idea was generalized in [HPP] to show Fact 5.1.4(ii)(a) below.

By Fact 5.1.2(ii), Facts 5.1.4(i) and (ii)(a) below can be seen as a generalization of the existence of a measure from [BO3] and [PePi], respectively.

Fact 5.1.4. *(i) [HPP, Theorem 10.4] The unit n-cube $I^n \subseteq M^n$ is compactly dominated by $\big(I^n(\mathbb{R}), \lambda, st\big)$, where $I^n(\mathbb{R}) := st(I^n)$. $\big(I^n$ and st are defined after a copy of $\mathbb{R}$ in M is fixed; see [HPP, Section 10]$\big)$.*

(ii) [HPP, Theorem 10.7] Let G be a definably compact definable group. Then G is compactly dominated in either of the cases:

(a) G has a 'very good reduction'.

(b) $\dim(G) = 1$.

In this chapter, we give a positive answer to the Compact Domination Conjecture in case G is defined in a saturated linear o-minimal expansion $\mathcal{M}$ of an ordered group. Since $\mathcal{M}$ is essentially a reduct of a saturated ordered vector space $\mathcal{N}$ over an ordered division ring, and $G^{00}_{\mathcal{M}} = G^{00}_{\mathcal{N}}$ (see Section 4.3), we may assume that $\mathcal{M}$ *is* an ordered vector space over an ordered division ring.

In fact, our proof shows the following stronger version of compact domination.

Theorem 5.1.5. *Let $\mathcal{M}$ be a saturated ordered vector space over an ordered division ring, and G a definably compact group definable in $\mathcal{M}$. Then for all definable sets $X \subseteq G$ defined in* any *o-minimal expansion of $\mathcal{M}$, property (5.1) holds; that is,*

$$\dim(X) < n \Rightarrow \mathbf{m}\big(\pi(X)\big) = 0.$$

The strategy of our proof is to use the standard part map $st : U \to \mathbb{R}^n$ defined in Chapter 3 in order to define a standard part map $st_G : G \to (S^1)^n$, $n = \dim(G)$, from G onto the real n-torus $(S^1)^n$ that has the following two properties. First, st_G 'resembles' $\pi : G \to G/G^{00}$ in that: $\ker(st_G) = G^{00}$ and, for all $A \subseteq (S^1)^n$, A is closed if and only if $st_G^{-1}(A)$ is type-definable. Second, st_G satisfies property (5.1). We can then conclude that π satisfies (5.1), and, thus, G is compactly dominated by $(G/G^{00}, \mathbf{m}, \pi)$.

For the rest of Chapter 5, we fix $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda \in D}\rangle$ to be a big saturated ordered vector space over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1\rangle$, and $G = \langle G, \oplus, e_G\rangle$ a t-connected, definably compact, $\emptyset$-definable group of dimension n.

It is easy to see that the assumption of definable connectedness is at no loss of generality, by [Pi1].

We also fix our notation regarding the following objects from the proof of Theorem 3.1.2 in Chapter 3:

- $H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$
- $U =< H >= \bigcup_{k<\omega} H^k$
- $\phi : U \to G$
- $L = \ker(\phi)$
- $\Sigma = H^{\Xi}$
- $st : U \to \mathbb{R}^n$

Moreover, it follows from the proof of Theorem 3.1.2 that we may assume up to definable isomorphism that

- $e_G = 0 \in H \subseteq G \subseteq \Sigma \subseteq U$

This assumption does not affect our proof of compact domination for G, since, easily, property (5.1), that we are aiming to show, remains invariant under definable isomorphisms.

Finally, G^{00} was defined in the proof of Theorem 3.3.1. We observe the following.

Lemma 5.1.6. $\ker(st) = G^{00}$.

Proof. For all $x \in G$,

$$\begin{aligned} x \in G^{00} &\Leftrightarrow x = \lambda_1\chi^1 + \ldots + \lambda_n\chi^n, \text{ for some } \chi^i \text{ with } \forall k \in \mathbb{N},\ -\frac{1}{2^k}e_i < \chi^i < \frac{1}{2^k}e_i, \\ &\Leftrightarrow x = \lambda_1\chi^1 + \ldots + \lambda_n\chi^n, \text{ for some } \chi^i \text{ with } \forall q \in \mathbb{Q},\ -qe_i < \chi^i < qe_i, \\ &\Leftrightarrow st(x) = 0. \end{aligned}$$

□

5.2 G is compactly dominated

We start with defining a standard part map for G. Recall $L = \ker(\phi)$ has rank n. By Lemma 2.4.5, $st(L) \subseteq \mathbb{R}^n$ is a lattice of rank n. Therefore, $\mathbb{R}^n/st(L)$ is isomorphic to the real n-torus $(S^1)^n$.

Let q denote the canonical homomorphism from $\mathbb{R}^n$ onto $\mathbb{R}^n/st(L)$.

We define a standard part map $st_G : G \to \mathbb{R}^n/st(L)$ as follows. For all $x \in G$, let

$$st_G(x) = q\big(st(x)\big) = [st(x)]^{\mathbb{R}^n}_{st(L)}.$$

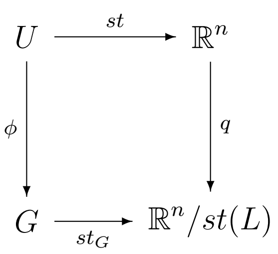
Since st is a group homomorphism, so is st_G. Indeed, for all $x, y \in G$, we have $x \oplus y = \phi(x + y) \in x + y + L$, and

$$\begin{aligned} (x \oplus y) - (x + y) \in L &\Rightarrow st(x \oplus y) - \big(st(x) + st(y)\big) \in st(L) \\ &\Leftrightarrow \big[st(x \oplus y)\big]^{\mathbb{R}^n}_{st(L)} = \big[st(x) + st(y)\big]^{\mathbb{R}^n}_{st(L)} = \big[st(x)\big]^{\mathbb{R}^n}_{st(L)} +_{\mathbb{R}^n/st(L)} \big[st(y)\big]^{\mathbb{R}^n}_{st(L)} \\ &\Leftrightarrow st_G(x \oplus y) = st_G(x) + st_G(y). \end{aligned}$$

Also, $\ker(st_G) = G^{00}$. Indeed, for all $x \in G$, $st_G(x) = [0]^{\mathbb{R}^n}_{st(L)} \Leftrightarrow st(x) \in st(L) \Leftrightarrow$

$x \in (G^{00} + L) \cap G = G^{00}$, since $G^{00} \subseteq G$.

Lemma 5.2.1. *The following diagram commutes,*



namely, $q \circ st = st_G \circ \phi$. *Thus, in particular, for all* $A \subseteq \mathbb{R}^n/st(L)$,

$$st^{-1}\big(q^{-1}(A)\big) = \phi^{-1}\big(st_G^{-1}(A)\big).$$

Proof. First, notice that for all $x, y \in U$, if $[x]_L^U = [y]_L^U$, then $[st(x)]_{st(L)}^{\mathbb{R}^n} = [st(y)]_{st(L)}^{\mathbb{R}^n}$. This is because st is a group homomorphism:

$$x - y \in L \Rightarrow st(x - y) \in st(L) \Leftrightarrow st(x) - st(y) \in st(L).$$

Now, let $x \in U$. On the one hand, we have $q\big(st(x)\big) = [st(x)]_{st(L)}^{\mathbb{R}^n}$. On the other, $\phi(x) \in G$ with $[x]_L^U = [\phi(x)]_L^U$, and, thus, $[st(x)]_{st(L)}^{\mathbb{R}^n} = \big[st\big(\phi(x)\big)\big]_{st(L)}^{\mathbb{R}^n} = st_G\big(\phi(x)\big)$. Hence, $q\big(st(x)\big) = st_G\big(\phi(x)\big)$. □

Lemma 5.2.2. $\overline{\Sigma}^{\mathbb{R}} = st(\Sigma) \subseteq \mathbb{R}^n$ *contains a set of representatives for* $E_{st(L)}^{\mathbb{R}^n}$. *Thus, for all* $A \subseteq \mathbb{R}^n/st(L)$, A *is closed (in the quotient topology) if and only if* $\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A) \subseteq \mathbb{R}^n$ *is closed.*

Proof. Let $y \in \mathbb{R}^n$. Pick $x \in U$ such that $st(x) = y$. Let $g \in G$ such that $g - x \in L$. Then $st(g) - y = st(g - x) \in st(L)$. But $st(g) \in st(G) \subseteq \overline{\Sigma}^{\mathbb{R}}$.

For the second claim, if A is closed, then $q^{-1}(A) \subseteq \mathbb{R}^n$ is closed, and, thus, $\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A)$ is closed. Conversely, if $\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A)$ is closed, then $\big(\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A)\big) +$

$st(L)$ is closed. But since $\overline{\Sigma}^{\mathbb{R}}$ contains a set S of representatives for $E^{\mathbb{R}^n}_{st(L)}$, we have

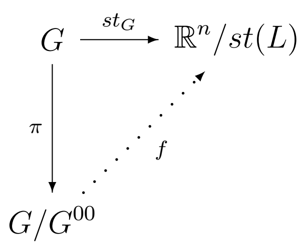
$$q^{-1}(A) = \big(S \cap q^{-1}(A)\big) + st(L) \subseteq \big(\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A)\big) + st(L) \subseteq q^{-1}(A),$$

that is, $q^{-1}(A) = \big(\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A)\big) + st(L)$ is closed, and, thus, A is closed. $\square$

By Pillay's Conjecture, G/G^{00} (equipped with the logic topology) is a connected, compact, abelian Lie group of dimension n and, therefore, it is isomorphic to $\mathbb{R}^n/st(L)$. The following lemma implies that the alleged isomorphism is indeed witnessed by the map

$$f : G/G^{00} \ni x \oplus G^{00} \mapsto st_G(x) \in \mathbb{R}^n/st(L).$$

(As a side remark, f is not an isomorphism if seen as the induced quotient map where G/G^{00} has the quotient topology; that would be the case if f were open. In any case, such an f would not be what we need here, since the logic topology on G/G^{00} is different from the quotient one, [Pi2, Remark 3.3].) We denote by $\pi : G \to G/G^{00}$ the canonical surjective homomorphism; then f is by definition the unique map that makes the following diagram commute:



Lemma 5.2.3. *For all $A \subseteq \mathbb{R}^n/st(L)$, A is closed if and only if $st_G^{-1}(A)$ is type-definable.*

Proof. By Lemma 2.4.9(v), a bounded set $A \subseteq \mathbb{R}^n$ is closed if and only if $st^{-1}(A)$ is type-definable. Now, let $A \subseteq \mathbb{R}^n/st(L)$. Then, A is closed

if and only if (Lemma 5.2.2) $\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A) \subseteq \mathbb{R}^n$ is closed

if and only if $st^{-1}\big(\overline{\Sigma}^{\mathbb{R}} \cap q^{-1}(A)\big)$ is type-definable

if and only if $st^{-1}\big(\overline{\Sigma}^{\mathbb{R}}\big) \cap st^{-1}\big(q^{-1}(A)\big)$ is type-definable

if and only if $(\Sigma + G^{00}) \cap st^{-1}\big(q^{-1}(A)\big)$ is type-definable

if and only if $\phi\Big((\Sigma + G^{00}) \cap st^{-1}\big(q^{-1}(A)\big)\Big)$ is type-definable,

where the last equivalence is because $\phi_{\restriction \Sigma + G^{00}}$ is type-definable. Indeed, the 'only if' part is clear, whereas for the 'if' part, let $B := (\Sigma + G^{00}) \cap st^{-1}\big(q^{-1}(A)\big)$. We show that if $\phi(B)$ is type-definable, then B is as well. To this aim, we show that

$$B = \{y \in \Sigma + G^{00} : \phi_{\restriction \Sigma + G^{00}}(y) \in \phi_{\restriction \Sigma + G^{00}}(B)\}.$$

To see this, let $y \in \Sigma + G^{00}$ such that $\phi(y) = \phi(b)$, for some $b \in B$. Then

$$q\big(st(y)\big) = st_G\big(\phi(y)\big) = st_G\big(\phi(b)\big) = q\big(st(b)\big) \in A,$$

by Lemma 5.2.1, showing that $y \in st^{-1}\big(q^{-1}(A)\big)$, and, thus, $y \in B$. This completes the proof of the last 'if and only if'.

Therefore, we will be done if we show that

$$\phi\Big(\big(\Sigma + G^{00}\big) \cap st^{-1}\big(q^{-1}(A)\big)\Big) = st_G^{-1}(A). \tag{5.2}$$

First, we observe that

$$\phi\Big(\big(\Sigma + G^{00}\big) \cap st^{-1}\big(q^{-1}(A)\big)\Big) = \phi\Big(st^{-1}\big(q^{-1}(A)\big)\Big).$$

Indeed, for the non-trivial inclusion ($\supseteq$), let $\phi(x) \in \phi\Big(st^{-1}\big(q^{-1}(A)\big)\Big)$, for some $x \in U$ with $q\big(st(x)\big) \in A$. Then we find $y \in \Sigma + G^{00}$ with $\phi(x) = \phi(y)$ and $q\big(st(y)\big) \in A$, as follows. Let $y \in \Sigma$ such that $x - y \in L$. Then, on the one hand, $\phi(x) = \phi(y)$, and on the other, $st(x) - st(y) \in st(L)$ and, thus, $q\big(st(y)\big) = q\big(st(x)\big) \in A$.

Now, by Lemma 5.2.1, $st^{-1}\big(q^{-1}(A)\big) = \phi^{-1}\big(st_G^{-1}(A)\big)$. Since ϕ is onto,

$$\phi\Big(st^{-1}\big(q^{-1}(A)\big)\Big) = \phi\Big(\phi^{-1}\big(st_G^{-1}(A)\big)\Big) = st_G^{-1}(A).$$

This proves (5.2). $\square$

Corollary 5.2.4. *The map* $f : G/G^{00} \ni x \oplus G^{00} \mapsto st_G(x) = [st(x)]^{\mathbb{R}^n}_{st(L)} \in \mathbb{R}^n/st(L)$ *is an isomorphism between topological groups.*

Proof. f is well-defined and it is injective, since for $x, y \in G$,

$$x \oplus G^{00} = y \oplus G^{00} \Leftrightarrow x \ominus y \in G^{00} \Leftrightarrow st_G(x \ominus y) = 0 \Leftrightarrow st_G(x) = st_G(y).$$

Easily, f is a group homomorphism, since st_G is. That it is surjective, is essentially Lemma 5.2.2: given $st(z) \in \mathbb{R}^n$, we can find $g \in G$, such that $st(g) - st(z) \in st(L)$.

It remains to show that f is a homeomorphism. We note that this can also be obtained by [HPP, Remark 2.3(i)] and Lemma 5.2.3; we provide a direct proof here (still using Lemma 5.2.3). Let $A \subseteq G/G^{00}$. We show that A is closed (in the logic topology) if and only if $f(A) = st_G\big(\pi^{-1}(A)\big)$ is closed. By definition of the logic topology, we have that A is closed if and only if $\pi^{-1}(A)$ is type-definable. By Lemma 5.2.3, it remains to show that $st_G^{-1}\Big(st_G\big(\pi^{-1}(A)\big)\Big) = \pi^{-1}(A)$. For the non-trivial inclusion ($\subseteq$), let

$$z \in st_G^{-1}\Big(st_G\big(\pi^{-1}(A)\big)\Big) =$$

$$\{z \in G : \exists y \in G,\ \pi(y) \in A \,\&\, st_G(z) = st_G(y)\}.$$

But then $z - y \in \ker(st_G) = G^{00}$, thus, $\pi(z) = \pi(y) \in A$, and $z \in \pi^{-1}(A)$. $\square$

The compact Lie group G/G^{00} has a unique normalized Haar measure $\mathbf{m}$. Thus, if $\mathbf{m}'$ is a Haar measure on $\mathbb{R}^n/st(L)$, then there is a positive $r \in \mathbb{R}$, such that for all $A \subseteq G/G^{00}$, A is $\mathbf{m}$-measurable if and only if $f(A)$ is $\mathbf{m}'$-measurable, and, if they are, then

$$\mathbf{m}(A) = r\mathbf{m}'\big(f(A)\big). \tag{5.3}$$

Since for all $X \subseteq G$, $f\big(\pi(X)\big) = st_G(X)$, in order to show property (5.1) for π it is thus equivalent to show it for st_G, that is, to show, for all definable $X \subseteq G$,

$$\dim(X) < n \Rightarrow \mathbf{m}'\big(st_G(X)\big) = 0. \tag{5.4}$$

On the other hand, a Haar measure $\mathbf{m}'$ on $\mathbb{R}^n/st(L)$ can be defined using the Lebesque measure λ on $\mathbb{R}^n$, as follows. Let $S \subseteq \mathbb{R}^n$ be the fundamental domain for $E^{\mathbb{R}^n}_{st(L)}$. Then, for $X \subseteq \mathbb{R}^n/st(L)$, let

$$\mathbf{m}'(X) := \lambda\big(S \cap q^{-1}(X)\big),$$

assuming that $S \cap q^{-1}(X)$ is a Lebesque measurable subset of $\mathbb{R}^n$. It is an easy classical fact that, if $A \subseteq \mathbb{R}^n$ is Lebesque measurable, then for all $B \subseteq S$ with

$q(B) = q(A)$, B is Lebesque measurable and $\lambda(B) \leq \lambda(A)$. Therefore, for every $X \subseteq G$, such that $st(X)$ is Lebesque measurable, $S \cap q^{-1}\big(st_G(X)\big)$ is Lebesque measurable and

$$\mathbf{m}'\big(st_G(X)\big) \leq \lambda\big(st(X)\big). \tag{5.5}$$

Indeed, it is not hard to see by Lemma 5.2.1 that

$$q\Big(S \cap q^{-1}\big(st_G(X)\big)\Big) = q\big(st(X)\big).$$

Namely, for $\subseteq$, if $y \in S \cap q^{-1}\big(st_G(X)\big)$, then $q(y) = st_G(x) = q\big(st(x)\big)$, for some $x \in X$, and, thus, $q(y) \in q\big(st(X)\big)$. For $\supseteq$, if $z = q\big(st(x)\big)$, $x \in X$, then let $y \in S$ such that $q(y) = q\big(st(x)\big) = st_G(x)$. Thus, $y \in q^{-1}\big(st_G(x)\big)$, and $z = q(y) \in q\Big(S \cap q^{-1}\big(st_G(X)\big)\Big)$.

It follows that in order to show (5.4), it suffices to show that for all definable $X \subseteq U$, if $\dim(X) < n$, then $\big(st(X)$ is Lebesque measurable and$\big)$ $\lambda\big(st(X)\big) = 0$. We prove the following stronger statement.

Lemma 5.2.5. *Let $\mathcal{N}$ be any saturated o-minimal expansion of $\mathcal{M}$. Then for all $\mathcal{N}$-definable $X \subseteq U$, if $\dim(X) < n$, then $\lambda\big(st(X)\big) = 0$.*

Proof. Here we imitate the proof of Fact 5.1.4(i). The compact domination of the unit n-cube I^n was already known by [BO3] if the ambient o-minimal structure expanded an ordered field. In [HPP, Theorem 10.4] a different proof was given, including the case that the ambient structure expanded an ordered divisible abelian group.

To succeed in better analogy with the account from [HPP], we make the following convention. Let $I = \{1, \ldots, n\}$. We assume:

$$U = U_1 \times \ldots \times U_n \subseteq N^n,$$

where for each $d \in I$,

$$U_d = \{x \in N : \exists q \in \mathbb{Z},\ -qe_d < x < qe_d\},$$

and $st : U \to \mathbb{R}^n$ is defined as follows: for all $u = (u_1, \ldots, u_n) \in U$,

$$st(u) = \big(st_1(u_1), \ldots, st_n(u_n)\big).$$

This convention is at no loss of generality, since by the construction of st in Section 2.4, the following function is a definable bijection

$$g : U \ni \lambda_1 u_1 + \ldots + \lambda_n u_n \mapsto (u_1, \ldots, u_n) \in g(U) \subseteq N^n.$$

Recall that a weakly o-minimal structure is a totally ordered structure such that every definable subset of the universe is a finite union of convex sets. We are going to make use of the following fact.

Fact 5.2.6 ([BP]). *If the saturated o-minimal structure $\mathcal{N}$ is expanded by any number of convex subsets of N then the resulting structure is weakly o-minimal.*

Let $\overline{\mathcal{N}} = \langle \mathcal{N}, \{U_d\}_{d \in I}, \{\ker(st_d)\}_{d \in I} \rangle$ be the structure $\mathcal{N}$ equipped with unary predicates for U_d and $\ker(st_d)$, for all $d \in I$. By Fact 5.2.6, $\overline{\mathcal{N}}$ is weakly o-minimal.

Each quotient $U_d/\ker(st_d)$ is interpretable in $\overline{\mathcal{N}}$, and each st_d induces a canonical bijection $i_d : U_d/\ker(st_d) \to \mathbb{R}$.

By $\mathbb{R}_{ind}$ we then mean the structure whose universe is $\mathbb{R}$ and whose relations are exactly the images under i_d of subsets of $U_d/\ker(st_d)$ which are definable in $\overline{\mathcal{N}}$, for all $d \in I$. As in [HPP, Lemma 10.2], one can see the following.

Claim 1. $\mathbb{R}_{ind}$ *is an o-minimal expansion of* $\langle \mathbb{R}, <, + \rangle$.

Proof of Claim 1. Clearly, $<$ and the graph of $+$ are among the basic relations of $\mathbb{R}_{ind}$. Now let $X \subseteq \mathbb{R}$ be definable in $\mathbb{R}_{ind}$. Then $X = X_1 \cup \ldots \cup X_n$, for some $X_d \subseteq \mathbb{R}$, $d \in I$, such that each $st^{-1}(X_d) \subseteq U_d$ is definable in $\overline{\mathcal{N}}$ and, therefore, is a finite union of convex sets. It follows that X has finitely many connected components. Thus, $\mathbb{R}_{ind}$ is o-minimal. □

Easily, if X is an $\mathcal{N}$-definable subset of U, then $st(X)$ is definable in $\mathbb{R}_{ind}$. As in [HPP, Lemma 10.3], we can see the following:

Claim 2. *Let* $X \subseteq U$ *be* $\mathcal{N}$*-definable with* $\dim(X) < n$*. Then* $\dim\big(st(X)\big) < n$*.*

Proof of Claim 2. By induction on n. If $n = 1$, then X and $st(X)$ are finite. Let $n > 1$. We may assume that $\dim(X) = n - 1$. By cell decomposition, Lemma 2.4.9(i), and additivity of λ, we may assume that X is the graph of some continuous $\mathcal{N}$-definable function $f : C \to U_n$, where C is a definable open subset of $U_1 \times \ldots \times U_{n-1}$, after perhaps rearranging coordinates. Assume, towards a contradiction, that $\dim\big(st(X)\big) = n$. By Claim 1, $st(X)$ must contain the closure of a subset $B \times (q_1, q_2)$, for an open rectangular box $B \subseteq \mathbb{R}^{n-1}$ of rational coordinates, and $q_1, q_2 \in \mathbb{Q}$. We may assume that $B(\mathcal{N})$ is contained in C.

Consider now an arbitrary $x \in B(\mathcal{N})$ and a rational number r in (q_1, q_2). By assumptions, there exist $y, z \in B(\mathcal{N})$ such that for every $d = 1, \ldots, n-1$, $st_d(y_d) =$

$st_d(z_d) = st(x_d)$, and $st_n\big(f(y)\big) = q_1 = st_n(q_1 e_n)$ and $st_n\big(f(z)\big) = q_2 = st_n(q_2 e_n)$. By continuity, there exists $x' \in B(\mathcal{N})$, such that for every $d = 1, \ldots, n-1$, $st_d(x'_d) = st(x_d)$ and $f(x') = re_n$. It follows that $st\big(\{x \in B(\mathcal{N}) : f(x) = re_n\}\big) = B$, which by induction implies that the set $\{x \in B(\mathcal{N}) : f(x) = re_n\}$ has a non-empty interior in N^{n-1}. This can be done for any rational r in (q_1, q_2), deriving a contradiction. $\square$

We are done, since every definable in $\mathbb{R}_{ind}$ subset of $\mathbb{R}^n$ of dimension less than n has Lebesque measure 0. $\square$

Proof of Theorem 5.1.5. For every $\mathcal{N}$-definable $X \subseteq G \subseteq U$, by (5.3) and (5.5) we have:

$$\mathbf{m}\big(\pi(X)\big) = r\mathbf{m}'\Big(f\big(\pi(X)\big)\Big) = r\mathbf{m}'\big(st_G(X)\big) \leq \lambda\big(st(X)\big).$$

Therefore, by Lemma 5.2.5, we obtain (5.1). $\square$

Corollary 5.2.7. *(i) G has a unique left (and right) invariant Keisler measure μ', given by: for all $X \in Def(G)$, $\mu'(X) = \mathbf{m}\big(\pi(X)\big)$.*

(ii) For all $X \in Def(G)$, $\mu'(X) > 0$ if and only if X is generic.

(iii) Every definable generic subset of G contains a torsion point.

Proof. (i) is by Fact 5.1.2(ii). For (ii) see Claim 3 in the proof of [HPP, Proposition 9.3], and for (iii) see [HPP, Proposition 10.6]. $\square$

CHAPTER 6

RESTRICTIONS ON L

6.1 Introduction

In Chapter 6, we fix $\mathcal{M} = \langle M, +, <, 0, \{\lambda\}_{\lambda\in D}\rangle$ to be an ordered vector space over an ordered division ring $D = \langle D, +, \cdot, <, 0, 1\rangle$.

The Structure Theorem can be seen as a procedure for recovering a lattice L, given the definable group G. In this chapter we investigate a partial 'converse' to this procedure, namely we address the following question.

Question. *Given a lattice $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n \leqslant M^n$ of rank n, is there a convex $\bigvee$-definable subgroup U of M^n such that U/L is a t-connected definably compact definable quotient group of dimension n?*

As it was pointed out in [PeS, Example 5.2], for $M = D = \mathbb{R}$ the answer is positive. In general, the answer is negative, and we provide here a counterexample (Example 6.2.5). Moreover, we give necessary and sufficient conditions that a lattice L must satisfy so that the answer is positive. The conditions are stated in terms of the archimedean pre-order defined next.

Definition 6.1.1. (i) Let $a, b \in M$. We define

$$a \preccurlyeq b \Leftrightarrow \exists n \in \mathbb{N},\ |a| < n|b|.$$

We let $a \sim b \Leftrightarrow a \preccurlyeq b \,\&\, a \preccurlyeq b$. Then $\sim$ is an equivalence relation on $\mathcal{M}$. We let $a \prec b$ if $a \preccurlyeq b$ but not $b \preccurlyeq a$. It follows that

$$a \prec b \Leftrightarrow \forall n \in \mathbb{N},\ n|a| < |b|.$$

In this chapter we fix an indexed set $I := \{1, \dots, n\}$. If $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n \leqslant M^n$ is a lattice of rank n, then for all $j, i \in I$, a_j^i denotes the i-th coordinate of the j-th generator a_j of L.

By a *linear transformation of* M^n we mean a definable map

$$M^n \ni x \mapsto Bx \in M^n,$$

where B is an invertible matrix with entries from D.

We show the following (see also Theorem 6.2.4 below):

Theorem 6.1.2. *Let $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n \leqslant M^n$ be a lattice of rank n. Then the following are equivalent:*

(a) There is a convex $\bigvee$-definable subgroup $U \leqslant M^n$ containing L such that U/L is a t-connected definably compact definable quotient group of dimension n.

(b) There are positive $e_1, \dots, e_n \in M$, such that, up to a linear transformation of M^n (applied to the generators of L), the following hold:

(i) for all $x = (x^1, \dots, x^n) \in L \setminus \{0\}$, there is $i \in I$, such that $e_i \leq |x^i|$, and

(ii) for all $j, i \in I$, $a_j^i \preccurlyeq e_i$.

In the rest of this section we fix our notation coming from the proof of Theorem 3.1.2 in Chapter 3. Let $G = \langle G, \oplus, e_G \rangle$ be an n-dimensional definable group which is t-connected and definably compact. By Step II of Section 3.2, there is a generic, open n-parallelogram $H \subseteq M^n$ with center 0, and some $x^G \in G$ such that $x^G + H$

is generic in G. Fix such an H. Using the definable bijection

$$f_G : G \ni x \mapsto (x \oplus x^G) - x^G \in f(G) \subseteq M^n,$$

we may assume that

$$e_G = 0 \text{ is the center of } H \subseteq G. \tag{6.1}$$

For such an H, let $U^G := U_H$ be as in Section 2.4, and $L^G := \ker(\phi)$, where $\phi : U^G \to G$ is as in Definition 3.2.30. By Lemma 3.2.35(i),

$$L^G \cap H = \{0\}. \tag{6.2}$$

6.2 The criteria

We begin with a useful lemma:

Lemma 6.2.1. *Let $L \leqslant U \leqslant M^n$, with L a lattice of rank n. Assume that U/L is a t-connected definably compact definable quotient group of dimension n, with a definable complete set of representatives S for E^U_L. Let $G = \langle S, +_S\rangle = U/L$. Then $U = U^G$ and $L = L^G$.*

Proof. We may assume that (6.1) holds for G. Indeed, denote $G' := f_G(G)$, and let $G' = \langle G', +_{G'}, 0\rangle$ be the topological group structure on G' induced by f_G. Observe that G' is a complete set of representatives for E^U_L, since every $x \in S$ is equivalent with $f_G(x) = (x +_S x^G) - x^G = (x + x^G + z) - x^G = x + z$, for some $z \in L$; it is also straightforward to see that G' is complete. It suffices then to show that $+_{G'}$ coincides with the group operation induced by the canonical surjection $q : U \to G'$. We show that for all $x, y \in G'$, there is $z \in L$ such that

$x +_{G'} y = x + y + z$. We have

$$\begin{aligned} x +_{G'} y = & \quad f_G\big(f_G^{-1}(x) +_S f_G^{-1}(y)\big) \\ = & \quad \big([(x + x^G) -_S x^G] +_S [(y + x^G) -_S x^G] +_S x^G\big) - x^G \\ = & \quad \big([(x + x^G) -_S x^G] +_S [y + x^G]\big) - x^G \\ = & \quad (x + x^G - x^G + y + x^G + z) - x^G \\ = & \quad x + y + z, \end{aligned}$$

for some $z \in L$.

It follows that $U^G \subseteq U$. Indeed, on the one hand, $U^G =< H >$, and on the other, by the proof of Claim 2.2.4(i), $U =< S + L_1 >$. Hence, since by (6.1) $H \subseteq G = S$, we are done.

Thus, if q denotes the canonical surjection from U onto S, then q is also defined on U^G.

Claim 1. $q_{\restriction U^G} = \phi_{\restriction U^G}$.

Proof of Claim 1. Clearly, $q_{\restriction H^G} = \phi_{\restriction H^G} = id_{H^G}$. Now, let $x \in U^G$. By convexity of H^G, there is $m \in \mathbb{N}$ such that $\frac{x}{m} \in H^G$. We have,

$$q(x) = \underbrace{q\left(\frac{x}{m}\right) \oplus \ldots \oplus q\left(\frac{x}{m}\right)}_{m-\text{times}} = \underbrace{\phi\left(\frac{x}{m}\right) \oplus \ldots \oplus \phi\left(\frac{x}{m}\right)}_{m-\text{times}} = \phi(x).$$

□

In particular, the kernel of ϕ agrees with the kernel of q on U^G, that is,

$$L^G = \ker(\phi) = L \cap U^G. \tag{6.3}$$

It follows that $L = L'$. Indeed, let $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n$ and $L^G = \mathbb{Z}a_1^G + \ldots + \mathbb{Z}a_n^G$ (since, by Theorem 3.1.2, L^G has rank n). By (6.3), there are $l_i^j \in \mathbb{Z}$, $i, j = 1, \ldots, n$, such that

$$a_1^G = l_1^1 a_1 + \ldots + l_1^n a_n$$
$$\vdots$$
$$a_n^G = l_n^1 a_1 + \ldots + l_n^n a_n$$

Since $a_1^G, \ldots, a_n^G$ are $\mathbb{Z}$-independent, we can solve for $a_1, \ldots, a_n$ in terms of $a_1^G, \ldots, a_n^G$ over $\mathbb{Q}$. Since U^G is convex, this implies that $a_1, \ldots, a_n \in U^G$. That is, $L \subseteq U^G$, and by (6.3), $L = L^G$.

Now let $\Sigma = H^{\Xi}$ be as in the proof of Theorem 3.1.2.

Claim 2. *$\forall x \in U$, $\exists y \in \Sigma$, $x - y \in L$.*

Proof of Claim 2. Consider $q(x) \in S = G$. By Lemma 3.2.29, there are $x_1, \ldots, x_\Xi \in H$ such that $q(x) = x_1 \oplus \ldots \oplus x_\Xi$. Let $y = x_1 + \ldots + x_\Xi \in \Sigma$. Then

$$q(x) = \phi(y) = q(y),$$

where the second equation is by Claim 1. □

It follows that $U \subseteq U^G$. Indeed, by the proof of Claim 2.2.4(i), $U^G = \Sigma + L^G = \Sigma + L$. Then apply Claim 2. □

Corollary 6.2.2. *The set U^G is independent of the choice of the generic subset $x^G + H$ of G in Section 6.1.*

The key lemma of this chapter is the following.

Lemma 6.2.3. *Let $L = \mathbb{Z}a_1 + \ldots + \mathbb{Z}a_n \leqslant M^n$ be a lattice of rank n. Then the following are equivalent:*

(a) There is a convex $\bigvee$-definable subgroup $U \leqslant M^n$ containing L such that U/L is a t-connected definably compact definable quotient group of dimension n.

(b) There is an open n-parallelogram $H \subset M^n$ of dimension n, such that

(i) $L \cap H = \{0\}$, and

(ii) $a_1, \ldots, a_n \in H^k$ for some $k \in \mathbb{N}$.

Proof. (a) $\Rightarrow$ (b). Let $G = U/L = \langle S, +_S \rangle$. By Lemma 6.2.1, $L = L^G$ and $U = U^G$. So if H is as in (6.1), then by (6.2), we are done.

(b) $\Rightarrow$ (a). Let $H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$ have dimension n, and let U denote the convex $\bigvee$-definable subgroup $U_H =< H >\leqslant M^n$. We show that (A) $L \subseteq U$ and (B) U/L is a definable quotient. Note that since $\dim(H) = n$, all notation and facts from Section 2.4 apply.

(A) By (b)(ii) and equation (2.2) from Section 2.4, it follows that, for all $j, i \in I$, there are $\alpha_j^i \in (-ke_i, ke_i)$, such that

$$a_j = \lambda_1 \alpha_j^1 + \ldots + \lambda_n \alpha_j^n.$$

Thus, $\forall l_1, \ldots, l_n \in \mathbb{Z}$, we have $l_1 a_1 + \ldots + l_n a_n \in H^{(l_1+\ldots+l_n)k} \subset U$.

(B) **Claim 1.** $\forall k \in \mathbb{N}$, $H^k \cap L$ is finite.

Proof of Claim 1. Consider $st_H : U \ni x \mapsto st_H(x) \in \mathbb{R}^n$ as in Section 2.4. Property (b)(i) guarantees that $\forall x, y \in L$, $x \neq y \Rightarrow st_H(x) \neq st_H(y)$. It follows that the set $H^k \cap L$ is bijective with $st_H(H^k \cap L) \subset \mathbb{R}^n$. By property (b)(i) again, $st_H(H^k \cap L)$ is discrete, and as a subset of the compact set $[-k, k]^n$, it is finite. Thus, $H^k \cap L$ is finite. □

Claim 2. $\forall k \in \mathbb{N}$, $E_L^{H^k}$ is definable.

Proof of Claim 2. Let $k \in \mathbb{N}$. By Claim 1, $H^{2k} \cap L$ is finite. Let $x, y \in H^k$. Then $x - y \in H^{2k}$, and $x E_L^{H^k} y \Leftrightarrow x - y \in H^{2k} \cap L$. $\square$

Now fix $\Xi \in \mathbb{N}$, such that $a_1, \ldots, a_n \in H^\Xi$. To find a definable complete set of representatives for E_L^U, by definable choice, it suffices to show:

Claim 3. $\forall x \in U, \exists y \in H^{n\Xi}, x - y \in L$.

Proof of Claim 3. Recall that for all $j, i \in I$,

$$a_j = \lambda_1 \alpha_j^1 + \ldots + \lambda_n \alpha_j^n,$$

where $-\Xi e_i < \alpha_j^i < \Xi e_i$, and that $st_H(a_j) = \big(st_1(\alpha_j^1), \ldots, st_n(\alpha_j^n)\big)$. Assume also

$$x = \lambda_1 \chi^1 + \ldots + \lambda_n \chi^n \in U_H,$$

for some $\chi^i \in U_i$. Then $st_H(x) = \big(st_1(\chi^1), \ldots, st_n(\chi^n)\big)$.

Since, by Lemma 2.4.5, $st_H(L)$ is a lattice of rank n, we can find $l_1, \ldots, l_n \in \mathbb{Z}$ and real numbers $r_1, \ldots, r_n \in [0, 1)$ such that $st_H(x) = (l_1 + r_1) st_H(a_1) + \ldots + (l_n + r_n) st_H(a_n)$. It follows that for all $i \in I$,

$$\big|st_i(\chi^i) - \big(l_1 st_i(\alpha_1^i) + \ldots + l_n st_i(\alpha_n^i)\big)\big| < |st_i(\alpha_1^i)| + \ldots + |st_i(\alpha_n^i)|.$$

Since st_i is a group homomorphism, as well as since for all $t, s \in U_i$, $|st_i(t)| = st_i(|t|)$, and $st_i(t) < st_i(s) \Rightarrow t < s$, it follows easily that:

$$|\chi^i - (l_1 \alpha_1^i + \ldots + l_n \alpha_n^i)| < |\alpha_1^i| + \ldots + |\alpha_n^i| < n\Xi e_i,$$

that is, $x-(l_1a_1+\ldots+l_na_n)\in H^{n\Xi}$. $\square$

We have shown (A) and (B). Now, by Claim 2.2.4(ii), U/L is a definable quotient group. Let S be a definable complete set of representatives for E_L^U contained in $H^{n\Xi}$, by Claim 3.

We may assume that $\frac{1}{2}H\subseteq S$, and therefore U/L has dimension n.

Note that, by Claim 2.2.4(iii), the quotient topology on S coincides with the t-topology on S. That is, the canonical surjection $q:U\to S$ is a t-continuous map.

Now, to see that U/L is t-connected, consider two points in S, and take a definable path γ between them. Then $q(\gamma)$ is a t-path between the two points.

To see that U/L is definably compact, first observe that the closure $\overline{H^{n\Xi}}$ of $H^{n\Xi}$ is a closed and bounded subset of M^n containing S. Thus, if $\gamma:(0,p)\to S$ is a definable t-continuous map, then it has a limit $x:=\lim_{t\to p^-}\gamma(t)$ inside $H^{n\Xi}$. It follows that $\lim^t_{t\to p^-}\gamma(t)=q(x)\in S$. $\square$

For $\mu=(\mu^1,\ldots,\mu^n)\in D^n$, and $x=(x^1,\ldots,x^n)\in M^n$, we denote $\mu x:=\mu^1x^1+\ldots+\mu^nx^n$.

Theorem 6.2.4. *Let $L=\mathbb{Z}a_1+\ldots+\mathbb{Z}a_n\leqslant M^n$ be a lattice of rank n. Then the following are equivalent:*

(a) There is a convex $\bigvee$-definable subgroup $U\leqslant M^n$ containing L such that U/L is a t-connected definably compact definable quotient group of dimension n.

(b) There is an $n\times n$ invertible matrix $B=\begin{pmatrix}\mu_1\\ \vdots\\ \mu_n\end{pmatrix}$ with entries from D, and positive $e_1,\ldots,e_n\in M$, such that:

(i) for all $x\in L\setminus\{0\}$, there is $i\in I$, such that $e_i\leq|\mu_ix|$, and

(ii) for all $j,i\in I$, $\mu_ia_j\preccurlyeq e_i$.

Proof. (a) $\Rightarrow$ (b). Let $H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : -e_i < t_i < e_i\}$ be the open n-parallelogram of dimension n as in Lemma 6.2.3(b). By Corollary 2.4.3, the matrix $A = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix}$ is invertible. Let

$$B = A^{-1} = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_n \end{pmatrix} = \begin{pmatrix} \mu_1^1 & \cdots & \mu_1^n \\ \vdots & \cdots & \vdots \\ \mu_n^1 & \cdots & \mu_n^n \end{pmatrix}.$$

Then for every $x = \lambda_1 \chi^1 + \ldots + \lambda_n \chi^n \in M^n$, we have, for $i \in I$, $\chi^i = \mu_i x$. By equation (2.2), (i) corresponds to Lemma 6.2.3(b)(i), and (ii) corresponds to Lemma 6.2.3(b)(ii).

(b) $\Rightarrow$ (a). We assume (b) and show Lemma 6.2.3(b). Let

$$A = B^{-1} = \begin{pmatrix} \lambda_1 & \cdots & \lambda_n \end{pmatrix} = \begin{pmatrix} \lambda_1^1 & \cdots & \lambda_n^1 \\ \vdots & \cdots & \vdots \\ \lambda_1^n & \cdots & \lambda_n^n \end{pmatrix}.$$

Let $H = \{\lambda_1 t_1 + \ldots + \lambda_n t_n : t_i \in -e_i < t_i < e_i\}$. Note that every $x \in M^n$ can be written as $x = \lambda_1 \chi^1 + \ldots + \lambda_n \chi^n \in M^n$, where for every $i \in I$, $\chi^i = \mu_i x$. By equation (2.2), (i) corresponds to Lemma 6.2.3(b)(i), and (ii) corresponds to Lemma 6.2.3(b)(ii). □

We conclude with an example of a lattice L for which the above criterion is not satisfied.

Example 6.2.5. Let $\mathcal{M}$ be a big saturated ordered vector space over $\mathbb{Q}$. Let $a_1 = (a_1^1, a_1^2)$, $a_2 = (a_2^1, 0) \in M^2$, such that $0 < a_2^1 \prec a_1^2 \prec a_1^1$. We show that condition (b) of Theorem 6.2.4 is not satisfied. Assume, towards a contradiction, that it is.

Following the notation from that theorem, let for $i = 1, 2$, $\mu_i = (\mu_i^1, \mu_i^2) \in \mathbb{Q}^2$. We have, for $i = 1, 2$,

$$\mu_i a_2 = \mu_i^1 a_2^1 \prec a_1^2 \preccurlyeq \mu_i^1 a_1^1 + \mu_i^2 a_1^2 = \mu_i a_1,$$

that is, $\mu_i a_2 \prec \mu_i a_1$. Therefore, if (b)(ii) is satisfied for $j = 1$, then (b)(i) cannot hold for $x = a_2$, a contradiction.

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