PREGEOMETRIES AND IMAGINARIES

PANTELIS E. ELEFTHERIOU

Abstract. We give two alternative proofs of the fact that a modular, surgical pregeometric theory admits geometric elimination of imaginaries.

1. Introduction

A surgical pregeometric theory T, introduced in [G], generalizes both o-minimal and strongly minimal theories. In [Pi1] it is shown that if an o-minimal theory is modular, then weak elimination of imaginaries (w.e.i.) follows. In the strongly minimal case, it is known that, again under modularity, geometric elimination of imaginaries (g.e.i.) holds (see e.g. [Pi3, Ch.2, Lemma 5.2]). In this paper we prove:

Proposition 1.1. *G.e.i. holds for any modular, surgical pregeometric theory.*

We show the proposition in two ways, adapting each time the corresponding argument from the above two cases. Notably, the first way makes no reference to T^{eq}, whereas the second (and much shorter) one employs a proposition from [G] that establishes the anti-reflexivity property for $\dim^{eq}$. The result can be seen as a partial converse to [G, Cor. 3.6], that every pregeometric theory with g.e.i. must be surgical.

1.1. **Structure of the paper.** We split Section 2 into four parts. In the first two, we recall definitions and facts about pregeometric theories and elimination of imaginaries, that are used in the rest of the paper. In part three, we give equivalent definitions for elimination of imaginaries that do not refer to $\mathcal{M}^{eq}$, whereas in part four, we present a dimension in $\mathcal{M}^{eq}$ following [G].

We split Section 3 in two parts. In the first, we show Proposition 1.1 using the results from part three of Section 2, resembling the proof of w.e.i. for modular o-minimal theories in [Pi1]. In the second, we show Proposition 1.1 based on part four of Section 2, resembling the proof of g.e.i. for modular strongly minimal theories.

1.2. **Notation.** $\mathcal{M}$ denotes a structure. We only allow finite tuples $\bar{b} \subset M$. By convention, for every $n \in \mathbb{N}$, there is an *empty tuple* $\bar{\emptyset} \in M^n$ that satisfies: $a \in acl(\emptyset) \Leftrightarrow a \in acl(\bar{\emptyset})$. If $\bar{b}, A \subseteq M$, we write $\bar{b}A$ for $\{\bar{b}\} \cup A$.

$\mathrm{Aut}(M)$ denotes the set of automorphisms of M, and for $A \subseteq M$, $\mathrm{Aut}_A(M) := \{f \in \mathrm{Aut}(M) : f \restriction_A = i_A\}$.

We assume familiarity with the construction and basic properties of $\mathcal{M}^{eq}$, as presented e.g. in [Pi2, Chapter 1]. We omit the bar from tuples in $\mathcal{M}^{eq}$. We often

Date: May 2005.

This is a paper for the 'Topics in Logic' class, instructed in Spring 2005 by David Lippel, at Notre Dame. The subject was suggested to me by Jerry Gagelman, whom I thank. Great thanks to David for being so helpful and patient throughout.

abuse notation and use the logical symbols for abbreviations, such as $\exists$ for 'there is'.

2. Pregeometries and imaginaries

2.1. Pregeometric theories.

Definition 2.1. A (finitary) *pregeometry* is a pair (S, cl), where S is a set and $cl : P(S) \to P(S)$ is a *closure operator* satisfying, for all $A, B \subseteq S$ and $a, b \in S$:
(i) $A \subseteq cl(A)$
(ii) $A \subseteq B \Rightarrow cl(A) \subseteq cl(B)$
(iii) $cl\big(cl(A)\big) = cl(A)$
(iv) $cl(A) = \{cl(B) : B \subseteq A \text{ finite}\}$
(v) (Exchange) $a \in cl(bA) \setminus cl(A) \Rightarrow b \in cl(aA)$.

Definition 2.2. If $\mathcal{M}$ is a structure, the *algebraic closure* operator $acl_{\mathcal{M}} : P(M) \to P(M)$ is defined as:

$$acl_{\mathcal{M}}(A) = \{a \in M : \text{there are } \bar{b} \subseteq A \text{ and } \phi(x, \bar{y}), |\phi(\mathcal{M}, \bar{b})| < \omega \,\&\, \mathcal{M} \vDash \phi(a, \bar{b})\},$$

whereas the *definable closure* $dcl_{\mathcal{M}} : P(M) \to P(M)$ is defined in the same way by replacing ω by 2. For simplicity we omit the index $\mathcal{M}$.

A complete theory T is called *pregeometric* if for every model $\mathcal{M} \vDash T$, $(\mathcal{M}, acl)$ is a pregeometry.

For a structure $\mathcal{M}$, a subset $A \subseteq M$ with $acl(A) = A$ is called *algebraically closed*.

Lemma 2.3. *(i) For any structure $\mathcal{M}$, (M, acl) satisfies 2.1(i)-(iv).*
(ii) If T is o-minimal or strongly minimal, then T is a pregeometric theory.

Proof. (i) Easy.
(ii) See [Mac, p.102] or [PS] for original proof, and [Hart, p.134], respectively. □

Lemma 2.4. *The intersection of algebraically closed sets is algebraically closed.*

Proof. Let $\{A_i\}_{i\in I}$ be a collection of algebraically closed sets. We show that $acl(\cap A_i) = \cap A_i$. By 2.1(i), we get $\supseteq$. On the other hand, $\cap A_i \subseteq A_i$, and thus by 2.1(ii), $acl(\cap A_i) \subseteq acl(A_i) = A_i$, for all $i \in I$. □

For the rest of this paper, T **denotes a pregeometric complete theory and** $\mathcal{M}$ **is a sufficiently saturated model of** T**.** Definability always means definability in $\mathcal{M}$ with parameters.

Definition 2.5. Let $A, B, C \subseteq M$. We say that B is *A-independent* if for all $b \in B$, $b \notin acl\big(A \cup (B \setminus \{b\})\big)$. We say that B *is independent from C over A*, denoted by $B \mathop{\smile\!\!\!\!\!\!|}_A C$, if every finite A-independent subset of B is $A \cup C$-independent.

A maximal A-independent subset of B is called *a basis for B over A*.

We refer to the above notion of independence as *algebraic independence*, and denote it by $\mathop{\smile\!\!\!\!\!\!|}$. It can be verified that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies all properties of an 'independence relation'. We are going to make use of the 'extension' property:

$$\forall A, B, \bar{c} \subset M, \exists \bar{c}' \models tp(\bar{c}/A),\ \bar{c}' \mathop{\smile\!\!\!\!\!\!|}_A B.$$

By the Exchange property in a pregeometric theory, (i) the independence relation is symmetric, and (ii) any two bases for B over A have the same cardinality. By (ii) we can define the *algebraic dimension*:

$$\dim(B/A) = \text{ the cardinality of any basis of } B \text{ over } A.$$

In particular, the dimension of tuples in M satisfies several nice properties, among which we distinguish the following.

Lemma 2.6. *For all $\bar{b}, \bar{c}, A, B \subseteq M$:*
(i) Additivity: $\dim(\bar{b}\bar{c}/A) = \dim(\bar{b}/\bar{c}A) + \dim(\bar{c}/A)$.
(ii) Transitivity: $A \subseteq B \Rightarrow \dim(\bar{c}/A) \geq \dim(\bar{c}/B)$.
(iii) Anti-reflexivity: $\dim(\bar{b}/A) = 0 \Rightarrow \bar{b} \in acl(A)$.

Note, (iii)($\Leftarrow$) is clearly also true.

Independence and dimension are also related via:

$$\bar{b} \underset{A}{\perp} C \Leftrightarrow \dim(\bar{b}/C) = \dim(\bar{b}/AC).$$

Definition 2.7. Let p be a partial type over $A \subset M$. Then,

$$\dim(p) := \max\{\dim(\bar{c}/A) : \bar{c} \subset M\}.$$

The dimension of a definable set is then the dimension of its defining formula.

Definition 2.8. T is called *modular* if for all algebraically closed $A, B \subseteq M$, $A \perp_{A\cap B} B$.

2.2. $\mathcal{M}^{eq}$ **and elimination of imaginaries.** Recall the construction of $\mathcal{M}^{eq}$. Its elements are called *imaginaries*. The corresponding algebraic (definable) closure operator is denoted by acl^{eq} (dcl^{eq}). We will need the following facts.

Fact 2.9. *(i) Every definable set X has a* code $e \in \mathcal{M}^{eq}$*:*

$$\textit{for all } f \in \mathrm{Aut}(M),\ f(e) = e \Leftrightarrow f(X) = X,$$

which is unique up to interdefinability.
(ii) Every $e \in M^{eq}$ is the code for some definable set.
(iii) For every $e \in M^{eq}$, there is $\bar{b} \subset M$, such that $e \in dcl^{eq}(\bar{b})$.
(iv) Every $f \in \mathrm{Aut}(M)$ extends uniquely to an $\bar{f} \in \mathrm{Aut}(M^{eq})$.

Proof. The proofs can be found in [Pi2]. More precisely in that reference, (i) is on p.7, (ii) and (iii) are by construction of $\mathcal{M}^{eq}$, and (iv) is Lemma 1.7. □

By (iv), we obtain that acl^{eq} coincides with acl on M, that is, for every $X \subseteq M$ and $A \subseteq M^{eq}$,

$$acl^{eq}(A) \cap X = acl(A) \cap X.$$

We can thus always write acl for acl^{eq} without any confusion. Similarly, we write dcl for dcl^{eq}.

Definition 2.10 (Forms of elimination of imaginaries)**.** We say that $\mathcal{M}$, or T, has:
(i) *elimination of imaginaries (e.i.)* if $\forall e \in M^{eq}, \exists \bar{c} \subset M,\ e \in dcl(\bar{c})\ \&\ \bar{c} \in dcl(e)$.
(ii) *weak elimination of imaginaries (w.e.i.)* if $\forall e \in M^{eq}, \exists \bar{c} \subset M,\ e \in dcl(\bar{c})\ \&\ \bar{c} \in acl(e)$.
(iii) *geometric elimination of imaginaries (g.e.i.)* if $\forall e \in M^{eq}, \exists \bar{c} \subset M,\ e \in acl(\bar{c})\ \&\ \bar{c} \in acl(e)$.

The intuition behind this definition is that the different forms of elimination of imaginaries express different powers of 'coding' imaginaries in $\mathcal{M}$. Since definable sets can be coded in $\mathcal{M}^{eq}$, the intuition becomes 'how well definable sets can be coded in $\mathcal{M}$'. Since definable sets in $\mathcal{M}$ do not seem to require anything from $\mathcal{M}^{eq}$ in their definition, this intuition should be expressible in a '$\mathcal{M}^{eq}$-free' way. Lemma 2.18 below verifies that.

Example 2.11. (i) The theory of an ACF has e.i.

(ii) The theory of the pure set has w.e.i. but not e.i.

(iii) The theory of an equivalence relation with two infinite equivalence classes has g.e.i. but not w.e.i. (Recall our convention about the existence of an empty tuple $\bar{\emptyset} \in M^n$, for each $n \in \mathbb{N}$.)

(iv) The theory of an equivalence relation with infinitely many infinite equivalence classes does not have g.e.i.

It is plausible to ask if all examples where g.e.i. fails have something in common with (iv) above. This common property is captured in the following definition from [G], in the sense of Proposition 2.13 below and our Proposition 1.1.

Definition 2.12 (Definition 2.5 in [G])**.** A pregeometric theory T is called *surgical* if for any definable set X and any definable equivalence relation E on X, at most finitely many E-classes have the same dimension as X.

For example, any o-minimal theory is surgical, [Pi1, Proposition 2.1]. A strongly minimal theory is surgical, as well. Our Proposition 1.1 can be seen as a partial converse to:

Proposition 2.13 (Corollary 3.6 in [G])**.** *Any pregeometric theory with g.e.i. must be surgical.*

Recall that a pregeometric theory is called *geometric*, if for every definable family of sets, there is a uniform bound on the size of finite fibers. Although being surgical may seem to be related with being geometric, it is shown via four examples in [G, p.316] that the two notions are totally independent. However, a connection of a different kind can be drawn in the following way:

Proposition 2.14 (Proposition 2.6 in [G])**.** *If a surgical theory T is geometric, then T^{eq} is geometric.*

2.3. **Equivalent forms of elimination of imaginaries.**

Definition 2.15. Let $X \subseteq M^n$ be a definable set. We say that $A \subseteq M$ is a *defining set* for X if X is A-definable, or, equivalently, if for all $f \in \mathrm{Aut}_A(M)$, $f(X) = X$. We say that X is *almost over* $A \subseteq M$ if there is an A-definable equivalence relation E on M^n with only finitely many classes such that X is a union of some of the classes. Equivalently ([Pi2, Lemma 1.5]), X is almost over A if $\{f(X) : f \in \mathrm{Aut}_A(M)\}$ is finite. In this case, we say that A is an *almost-defining set* for X.

Lemma 2.16. *For all $f \in \mathrm{Aut}(M)$, X (almost) B-definable $\Rightarrow f(X)$ (almost) $f(B)$-definable.*

Proof. Straightforward. □

Lemma 2.17. *Let $C = acl(C_0) \subseteq M$, C_0 finite, and $X \subseteq M$ definable such that $\forall f \in \mathrm{Aut}(M)$, $f(X) = X \Rightarrow f(C) = C$. Let $e \in M^{eq}$ be a code for X. Then, $C \subseteq acl(e)$.*

Proof. Since C_0 is finite, $|C| \leq |T|$ is bounded ($< |M|$). Let $c \in C$, $p = tp_{\mathcal{M}}(c/e)$ and $p(\mathcal{M}) = \{b_\alpha : \alpha < \lambda\}$. By compactness, $\lambda < \omega$ or $\lambda > |T|$: indeed, if not, the type $p \cup \{x \neq b_\alpha\}_{\alpha<\lambda}$ should be realized in the $|T|^+$-saturated $\mathcal{M}$, a contradiction. Now, since $C \supseteq \{f(c) : f \in \mathrm{Aut}_{\{e\}}(M)\} = p(\mathcal{M})$, we see that $\{f(c) : f \in \mathrm{Aut}_{\{e\}}(M)\}$ is finite, that is, $c \in acl(e)$. □

The proofs of (ii) and (iii) of the following lemma proceed in absolute analogy, but we include them both in the interests of completeness. As far as Proposition 1.1 is concerned, only (iii) will be used in Section 3.1.

Lemma 2.18. *$\mathcal{M}$ has:*

(i) e.i. $\iff$ for every definable X there is $C = dcl(C_0) \subseteq M$, C_0 finite, such that $\forall f \in \mathrm{Aut}(M)$, $f \in \mathrm{Aut}_C(M) \Leftrightarrow f(X) = X$.

(ii) w.e.i. $\Leftrightarrow$ every definable set has a smallest algebraically closed defining set $C = acl(C_0) \subseteq M$ with C_0 finite.

(iii) g.e.i. $\Leftrightarrow$ every definable set has a smallest algebraically closed almost-defining set $C = acl(C_0) \subseteq M$ with C_0 finite.

Proof. (i) ($\Rightarrow$). Assume M has e.i., and let X be a definable set with some code e. Let $\bar{c} = (c_1, \ldots, c_n)$ as in Definition 2.10(i), and define $C_0 := \{c_1, \ldots, c_n\}$.

($\Leftarrow$). Let $e \in M^{eq}$, and X some definable set with code e. Define $\bar{c} := (c_1, \ldots, c_n)$, where $\{c_1, \ldots, c_n\} = C_0$.

(ii) ($\Rightarrow$). Assume $\mathcal{M}$ has w.e.i., and let X be a definable set with some code $e \in M^{eq}$ and $\bar{c} = (c_1, \ldots, c_n) \in M^n$ such that $e \in dcl(\bar{c}) \,\&\, \bar{c} \in acl(e)$. Let $C_0 := \{c_1, \ldots, c_n\}$ and $C := acl(C_0)$. We show that C is the smallest algebraically closed defining set for X. It is a defining set for X, since $e \in dcl(\bar{c})$. For some index set I, let $\{B_i\}_{i\in I}$ be the set of all algebraically closed defining sets for X, and $B := \bigcap_{i\in I} B_i$. It suffices to show that $C \subseteq B$. By Lemma 2.4, B is algebraically closed, and since $C = acl(C_0)$, it suffices to show that $C_0 \subseteq B$, that is, that $C_0 \subseteq B_i$ for all $i \in I$. Since X is B_i-definable, we have $\forall f \in \mathrm{Aut}_{B_i}(M), f(X) = X$, that is, $e \in dcl(B_i)$. Since $\bar{c} \in acl(e)$, we have $\bar{c} \in acl(B_i)$, and hence $C_0 \subseteq acl(B_i) = B_i$.

($\Leftarrow$). Let $e \in M^{eq}$, and $X \subseteq M^n$ some definable set with code e. By hypothesis there is a smallest algebraically closed defining set for X, $C = acl(C_0)$, with C_0 finite. Then, if we let $\{B_i\}_{i\in I}$ be all the algebraically closed defining sets for X, it must be $C = \bigcap_{i\in I} B_i$.

Claim. For all $f \in \mathrm{Aut}(M)$, $f(X) = X \Rightarrow f(C) = C$.

Proof. Let $f \in \mathrm{Aut}(M)$ with $f(X) = X$. By Lemma 2.16, for all i, X is $f(B_i)$-definable, that is, f permutes $\{B_i\}_{i\in I}$. On the other hand, since f is one-to-one, $f\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f(B_i)$. It follows, $f(C) = C$.

Lemma 2.17 applies, to give $C \subseteq acl(e)$. Now, pick some $\bar{c} \subseteq C$ such that X is definable by a formula with parameters $\bar{c}$. Clearly, $e \in dcl(\bar{c})$, and by the above, $\bar{c} \in acl(e)$.

(iii) ($\Rightarrow$). Assume $\mathcal{M}$ has g.e.i., and let X be a definable set with some code $e \in M^{eq}$ and $\bar{c} = (c_1, \ldots, c_n) \in M^n$ such that $e \in acl(\bar{c}) \,\&\, \bar{c} \in acl(e)$. Let $C_0 := \{c_1, \ldots, c_n\}$ and $C := acl(C_0)$. We show that C is the smallest algebraically closed almost-defining set for X. It is an almost-defining set for X, since $e \in acl(\bar{c})$. Let $\{B_i\}_{i\in I}$ be the set of all algebraically closed almost-defining sets for X, and

$B := \bigcap_{i\in I} B_i$. It suffices to show that $C \subseteq B$. By Lemma 2.4, B is algebraically closed, and since $C = acl(C_0)$, it suffices to show that $C_0 \subseteq B$, that is, that $C_0 \subseteq B_i$ for all $i \in I$. Since X is almost over B_i, we have that $\{f(X) : f \in \mathrm{Aut}_{B_i}(M)\}$ is finite, and since e is a code for X, $e \in acl(B_i)$. Since $\bar{c} \in acl(e)$, it follows $\bar{c} \in acl(B_i)$ and hence $C_0 \subseteq acl(B_i) = B_i$.

($\Leftarrow$). Let $e \in M^{eq}$, and $X \subseteq M^n$ some definable set with code e. By hypothesis there is a smallest algebraically closed almost-defining set for X, $C = acl(C_0)$, with C_0 finite. Then, if we let $\{B_i\}_{i\in I}$ be all the algebraically closed almost-defining sets for X, it must be $C = \bigcap_{i\in I} B_i$.

Claim. For all $f \in \mathrm{Aut}(M)$, $f(X) = X \Rightarrow f(C) = C$.

Proof. Let $f \in \mathrm{Aut}(M)$ with $f(X) = X$. By Lemma 2.16, for all i, X is almost over $f(B_i)$, that is, f permutes $\{B_i\}_{i\in I}$. On the other hand, since f is one-to-one, $f\left(\bigcap_{i\in I} B_i\right) = \bigcap_{i\in I} f(B_i)$. It follows, $f(C) = C$.

Lemma 2.17 applies, to give $C \subseteq acl(e)$. Now, pick some $\bar{c} \subseteq C$ such that X is almost over $\bar{c}$. Clearly, $e \in acl(\bar{c})$, and by the above, $\bar{c} \in acl(e)$. $\square$

2.4. **Dimension in $\mathcal{M}^{eq}$.** It is easy to see that $(\mathcal{M}^{eq}, acl^{eq})$ is not always a pregeometry. Consider, e.g., Example 2.11(iv), and notice that if $a \in M$ belongs to some class with code $e \in M^{eq}$, then $e \in acl(a) \setminus acl(\emptyset)$ but $a \notin acl(e)$.

Therefore, acl^{eq} cannot give rise to a meaningful notion of dimension in the same way that *acl* did for tuples in M, but it *is* possible to define a good notion of dimension in M^{eq}, in the way we describe below. The definition goes back to [HP].

Lemma 2.19 (Lemma 3.1 in [G])**.** *acl satisfies the Exchange property for elements in M over imaginary parameters, that is: for all $\bar{a}, \bar{b} \subset M$ and $A \subset M^{eq}$, $\bar{a} \in acl(\bar{b}A) \setminus acl(A) \Rightarrow \bar{b} \in acl(A\bar{a})$.*

It follows that for any two maximal A-independent subtuples of $\bar{b} \subset M$ have the same cardinality and thus we can again define, for $\bar{b}, C \subseteq M$ and $A \subseteq M^{eq}$,

$\dim(\bar{b}/A)$:= the cardinality of a maximal A-independent subtuple of $\bar{b}$, and

$\bar{b} \mathop{\smile\!\!\!\!\!|}_A C \;:\Leftrightarrow\; \dim(\bar{b}/C) = \dim(\bar{b}/AC)$.

Remark 2.20. Lemma 2.6 is still true for $A, B \subseteq M^{eq}$.

Recall that (Fact 2.9(iii)) every imaginary tuple $e \in M^{eq}$ is definable over M.

Definition 2.21. Let $e \in M^{eq}$ and $A \subset M^{eq}$. We let:

$$\dim^{eq}(e/A) = |\bar{b}| - \dim(\bar{b}/Ae),$$

where $\bar{b}$ is some/any tuple in M which is A-independent and $e \in acl(A\bar{b})$.

A justification why $|\bar{b}| - \dim(\bar{b}/Ae)$ does not depend on the choice of the tuple $\bar{b}$ is given in [G, Lemma 3.3]. In [G] it is also shown that in a pregeometric theory $\dim^{eq}$ satisfies additivity and transitivity, whereas more interestingly:

Proposition 2.22 (Proposition 3.5 in [G])**.** *A pregeometric theory is surgical if and only if for all $e \in M^{eq}$ and $A \subseteq M^{eq}$, $\dim^{eq}(e/A) = 0 \Rightarrow e \in acl(A)$.*

Easily, $\dim^{eq}(e/A) = 0 \Leftarrow e \in acl(A)$ is also true.

Remark 2.23. Notice that the property in Proposition 2.22 is 'anti-reflexivity' but *not* in the same sense as in Lemma 2.6, since $\dim^{eq}(e/A)$ is not algebraic dimension.

Accordingly, we can define $e \mathop{\smile\!\!\!\!\!\!|}^{eq}_A C :\Leftrightarrow \dim^{eq}(e/C) = \dim^{eq}(e/AC)$, but again, $\mathop{\smile\!\!\!\!\!\!|}^{eq}$ is *not* algebraic independence.

One can check that for $\bar{a}, \bar{b} \subset M$ and $A \subseteq M^{eq}$, $\dim^{eq}(\bar{a}/A) = \dim(\bar{a}/A)$, and $a \mathop{\smile\!\!\!\!\!\!|}^{eq}_A C \Leftrightarrow a \mathop{\smile\!\!\!\!\!\!|}_A C$. We thus omit the index '*eq*', keeping however in mind the last remark.

3. Proofs of Proposition 1.1

Let $T = Th(\mathcal{M})$ be a modular, surgical pregeometric theory.

3.1. **Imitating** [Pi1] **- and avoiding $\mathcal{M}^{eq}$.**

The argument is at times 'almost' word-by-word the argument in [Pi1]. Namely, the following lemma corresponds to [Pi1, Proposition 2.2] and the rest of the proof to [Pi1, Proposition 3.2].

Lemma 3.1. *Let $A, B, C \subseteq M$, with $B \mathop{\smile\!\!\!\!\!\!|}_A C$. If $X \subseteq M^n$ is almost over B and almost over C, then it is almost over A.*

Proof. Let $\phi(\bar{x}, \bar{y}, \bar{b})$ be a formula over B and $\psi(\bar{x}, \bar{y}, \bar{c})$ a formula over C, defining the equivalence relations E_ϕ and E_ψ, respectively, such that each of them has finitely many classes, and X is the union of some of the E_ϕ-classes, as well as the union of some of the E_ψ-classes.

We assume that $c = (c_1, \ldots, c_k)$, for some $\{c_1, \ldots, c_k\}$ which is A-independent (or replace $\{c_1, \ldots, c_k\}$ by a maximal such subset of it). It is then easy to see that E_ψ is defined by some formula $\chi(\bar{x}, \bar{y}, \bar{c}, \bar{a})$, $\bar{a} \subseteq A$. For $\bar{w} = (w_1, \ldots, w_k)$ and $\bar{z} = (z_1, \ldots, z_k)$, let

$$E(\bar{w}, \bar{z}) : \forall \bar{x} \forall \bar{y} \big(\chi(\bar{x}, \bar{y}, \bar{w}, \bar{a}) \leftrightarrow \chi(\bar{x}, \bar{y}, \bar{z}, \bar{a}) \big).$$

Then E defines an equivalence relation on M^k over A.

Furthermore, consider the equivalence relation $E_\phi \cap E_\psi$, and pick elements $d_i \in X$, one in each of the finitely many $(E_\phi \cap E_\psi)$-classes whose union is X. Let $\bar{d}$ be a tuple consisting of the d_i's. Then, each of

$$\sigma_\phi(\bar{x}, \bar{d}, \bar{b}) := \bigvee_{d_i \in \bar{d}} \phi(\bar{x}, d_i, \bar{b})$$

and

$$\sigma_\chi(\bar{x}, \bar{d}, \bar{c}, \bar{a}) := \bigvee_{d_i \in \bar{d}} \chi(\bar{x}, d_i, \bar{c}, \bar{a})$$

defines X. For $\bar{w} = (w_1, \ldots, w_k)$ and $\bar{z} = (z_1, \ldots, z_k)$, let

$$E'(\bar{w}, \bar{z}) : \forall x \big(\sigma_\chi(\bar{x}, \bar{d}, \bar{w}, \bar{a}) \leftrightarrow \sigma_\chi(\bar{x}, \bar{d}, \bar{z}, \bar{a}) \big).$$

Then E' defines an equivalence relation on M^k over $A\bar{d}$. Note that E is a refinement of E'. Let Z be the E'-class of $\bar{c}$. Then Z is the union of some E-classes.

Claim. Z is almost over A.

Proof of Claim. First notice that Z is defined by the formula

$$\forall \bar{x} \big(\sigma_\phi(\bar{x}, \bar{d}, \bar{b}) \leftrightarrow \sigma_\chi(\bar{x}, \bar{d}, \bar{z}, \bar{a}) \big),$$

that is, Z is $(AB\bar{d})$-definable. Now, $(c_1, \ldots, c_k) \in Z$, and on the other hand, since $\{c_1, \ldots, c_k\}$ is A-independent and $B \mathop{\smile\!\!\!\!\!\mid}_A C$, $\{c_1, \ldots, c_k\}$ is AB-independent. Moreover, by the extension property of $\mathop{\smile\!\!\!\!\!\mid}$, we can replace $\bar{c}$ by some $\bar{c}' \models tp(\bar{c}/AB)$ such that $\bar{c}' \mathop{\smile\!\!\!\!\!\mid}_{AB} \bar{d}$. Then, $\bar{c}' \in Z$ $\big($since $E(\bar{c}, \bar{c}')\big)$ and $\bar{c}'$ is $AB\bar{d}$-independent. It follows that $\dim(Z) = k$.

Since E refines E', Z is the union of some E-classes. Since T is surgical, only finitely many classes of E on M^k can have dimension equal to $\dim(M^k) = k$. Since any $f \in \mathrm{Aut}_A(M)$ permutes the classes of E, and preserves their dimension, $\{f(Z) : f \in \mathrm{Aut}_A(M)\}$ is finite, that is, Z is almost over A. $\square$

Let $F(\bar{w}, \bar{z})$ denote an A-definable equivalence relation on M^k with finitely many classes, such that Z is the union of some of them. Then the following equivalence relation witnesses that X is almost over A: for all $\bar{x}, \bar{y} \in M^n$,

$$\bar{x} \sim \bar{y} : \ \forall \bar{w}, \bar{z}\big(F(\bar{w}, \bar{z}) \to [\chi(\bar{x}, \bar{y}, \bar{z}) \leftrightarrow \chi(\bar{x}, \bar{y}, \bar{w})]\big).$$

$\square$

Proof of Proposition 1.1. Let X be a definable set and $B_0 = \{b_0, \ldots, b_k\}$ a $\emptyset$-independent set, almost-defining X and with k least possible. We show that $B = acl(B_0)$ is the least algebraically closed almost-defining set for X. Suppose not. Then for some $C = acl(C_0)$, C_0 finite, with $B \not\subseteq C$, X is almost over C. Let $A = B \cap C$. Then A is algebraically closed and $A \subsetneq B$. By Exchange, $A = acl(\{a_0, \ldots, a_{k'}\})$ for some $k' < k$. Since T is modular, $B \mathop{\smile\!\!\!\!\!\mid}_A C$. By Lemma 3.1, X is almost over A, contradicting the choice of B. $\square$

3.2. **By analysis in** [G]**.**

Lemma 3.2. *Let $a, b, c \in M^{eq}$, $D \subseteq M^{eq}$ with $b \mathop{\smile\!\!\!\!\!\mid}_D c$. If $a \in acl(b)$ and $a \in acl(c)$, then $a \in acl(D)$.*

Proof. We first show that $b \mathop{\smile\!\!\!\!\!\mid}_D c$ implies $ba \mathop{\smile\!\!\!\!\!\mid}_D ca$. Assume $\dim(b/D) = \dim(b/cD)$. We have:

$$\dim(ba/D) =^{a \in acl(b)} \dim(b/D) = \dim(b/cD) =^{a \in acl(c)} \dim(b/caD) = \dim(ba/caD).$$

We now show that $ba \mathop{\smile\!\!\!\!\!\mid}_D ca$ implies $\dim(a/D) = 0$. Assume $\dim(ba/D) = \dim(ba/caD)$. We have:

$$\dim(ba/D) = \dim(ba/caD) \leq^{transitivity} \dim(ba/aD) = \dim(b/aD).$$

On the other hand, by additivity:

$$\dim(ba/D) = \dim(b/aD) + \dim(a/D).$$

It follows that $\dim(a/D) = 0$. By Proposition 2.22, $a \in acl(D)$. $\square$

Proof of Proposition 1.1. Let $e \in M^{eq}$, and $\bar{b} \subset M$ such that $e \in acl(\bar{b})$. By saturation, there is $\bar{b}' \subset M$ with $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_e \bar{b}'$ and $tp(\bar{b}/e) = tp(\bar{b}'/e)$. By modularity, if $X := acl(\bar{b}) \cap M$ and $Y := acl(\bar{b}') \cap M$, we have $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_{X \cap Y} \bar{b}'$. By Lemma 3.2, $e \in acl(X \cap Y)$.

Let $\bar{d} \in X \cap Y$ such that $e \in acl(\bar{d})$. Since $\bar{d} \in acl(\bar{b})$, $\bar{d} \in acl(\bar{b}')$ and $\bar{b} \mathop{\smile\!\!\!\!\!\mid}_e \bar{b}'$, Lemma 2.6 and Remark 2.20 (or simply Lemma 3.2) give that $\bar{d} \in acl(e)$. $\square$

References

[G] J. Gagelman, *Stability in geometric theories*, Annals of Pure and Applied Logic 132 (2005), 313-326.

[Hart] B. Hart, *Stability theory and its variants*, pp. 131-149 in Model theory, algebra, and geometry, edited by D. Haskell, A. Pillay and C. Steinhorn, Math. Sci. Res. Inst. Publ. 39, Cambridge Univ. Press, New York, 2000.

[HP] E. Hrushovski, A. Pillay, *Groups definable in local fields and pseudo-finite fields*, Israel Journal of Mathematics 85 (1994), 203-262.

[Mac] D. Macpherson, *Notes on o-minimality and variations*, pp. 97-130 in Model theory, algebra, and geometry, edited by D. Haskell, A. Pillay and C. Steinhorn, Math. Sci. Res. Inst. Publ. 39, Cambridge Univ. Press, New York, 2000.

[Pi1] A. Pillay, *Some remarks on definable equivalence relations in o-minimal structures*, The Journal of Symbolic Logic, Volume 51, Number 3 (1986), 709-714.

[Pi2] A. Pillay, *Lecture notes - Stability theory*, Spring 2003, available online: http://www.math.uiuc.edu/People/pillay/lecturenotes.stability.pdf

[Pi3] A. Pillay, Geometric stability theory, Oxford University Press, 1996.

[PS] A. Pillay and C. Steinhorn, *Definable sets in ordered structures I*, Trans. AMS, 295 (1986), 565-592.

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556, USA
E-mail address: `pelefthe@nd.edu`