GROUPS DEFINABLE IN O-MINIMAL STRUCTURES

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Abstract. In this series of lectures, we will a) introduce the basics of o-minimality, b) describe the manifold topology of groups definable in o-minimal structures, and c) present a structure theorem for the special case of semi-linear groups, exemplifying their relation with real Lie groups.

The structure of these lectures is as follows:

(1) Basics of o-minimality: definition, Cell Decomposition Theorem, dimension of definable sets. The standard reference is [vdD]. A recent survey is [Pet].
(2) Definable groups: definable manifold topology, uniqueness, questions about (a) affine embedding and (b) resemblance with real Lie groups (Pillay's Conjecture). The standard reference is [Pi1], and a recent survey is [Ot].
(3) Semi-linear groups: Structure Theorem and answers to the above questions for semi-linear groups. Reference: [El].

1. Basics of o-minimality

Let $\mathcal{L}$ be a first-order language, and $\mathcal{M}$ an $\mathcal{L}$-structure. Then $X \subseteq M^n$ is called *definable* (in $\mathcal{M}$ over $\bar{a}$) if for some formula $\phi \in \mathcal{L}$,

$$X = \{\bar{b} \in M^n : \mathcal{M} \models \phi(\bar{b}, \bar{a})\}.$$

For example, the unit circle on the real plane $\{(x, y) \in \mathbb{R}^2 : x^2+y^2 = 1\}$ is definable in $\langle \mathbb{R}, +, \cdot \rangle$ but not in $\langle \mathbb{R}, + \rangle$.

A function $f : A \subseteq M^m \to M^n$ is called *definable* if its graph $\Gamma(f) \subseteq M^m \times M^n$ is definable. A group $G = \langle G, \oplus, e_G \rangle$ is called *definable* if $G \subseteq M^n$ and $\oplus : M^{2n} \to M^n$ are definable. For example, the following group is definable in $\langle \mathbb{R}, <, + \rangle$: let $S^1 = \langle [0, 1), \oplus, 0 \rangle$, with

$$x \oplus y = \begin{cases} x + y & \text{if } x + y < 1, \\ x + y - 1 & \text{if } x + y \geq 1. \end{cases}$$

The study of definability has been a powerful tool in stability theory. Indeed, Shelah's classification of models of a given theory up to isomorphism turned out to be intimately related to the classification of definable sets in the models of the theory. On the other hand, the definition of an o-minimal structure was given in terms of the definable sets in the structure. The creation of o-minimality can be viewed as an attempt of developing model theory for structures that do not fall under the scope of Shelah's classification theory.

Definition 1.1 (Knight-Pillay-Steinhorn, [KPS, PiS], 1986)**.** A structure $\mathcal{M} = \langle M, <, \dots \rangle$ is called *o-minimal* if

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(i) $\langle M, <\rangle$ is a dense linear order without endpoints, and
(ii) every definable subset of the universe M is a finite union of open intervals (a, b), $a, b \in \mathcal{M} \cup \{\pm\infty\}$ and points.

Remark.

- (ii) is equivalent to: every definable subset of the universe M can be defined in $\langle M, <\rangle$.
- There is an analogy with strongly minimal structures. A structure $\mathcal{M} = \langle M, \dots\rangle$ is called *strongly minimal* if every definable subset of the universe (of any $\mathcal{N} \equiv \mathcal{M}$) is finite or co-finite; equivalently, it can be defined just using $=$.
- An o-minimal structure is not stable.

Primary examples.

- O-minimal structure: $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$. The definable sets are the *semi-algebraic sets.*
- Strongly minimal: an algebraically closed field of characteristic p. The definable sets are the *constructible sets.*

Examples - Trichotomy.

- $\langle M, <\rangle$, "trivial": dense linear order without endpoints.
- $\langle M, <, +, 0, \{d\}_{d\in D}\rangle$, "linear": ordered vector space over an ordered division ring D.
- $\langle M, <, +, \cdot, 0, 1, \dots\rangle$, o-minimal expansion of a real closed field.

Trichotomy Theorem [Peterzil-Starchenko, [PeS], 1998](roughly): The above three cases are what an o-minimal structure can *locally* look like (where "locally" refers to the product topology of M^n described below).

Groups definable in o-minimal expansions of real closed fields have been studied by a number of people including Berarducci, Edmundo, Hrushovski, Otero, Peterzil, Pillay. Groups definable in a linear o-minimal structure are called *semi-linear groups*, they were studied in [El], and they are the topic of the last part of these lectures.

For the rest of this section, let $\mathcal{M} = \langle <, \dots\rangle$ be an o-minimal structure.

O-minimal structures are "nice" topological structures. Topological because:

- M is equipped with the order $<$-topology: the topology whose basic open sets are the open intervals.
- M^n is equipped with the product topology.

They are "nice" because of the following two theorems:

Monotonicity Theorem. *For every definable function $f : (a, b) \to M$, $a, b \in \mathcal{M} \cup \{\pm\infty\}$, there are points $a = a_0 < a_1 < \cdots < a_k < a_{k+1} = b$ such that on each subinterval (a_j, a_{j+1}), $j = 0, \dots, k$, the function f is either constant, or strictly monotone and continuous.*

Definition 1.2. The collection of *cells* is the least collection of sets that contains:
(A) *Cells in M*: points, open intervals,
and is closed under the following two constructions.
(B) If X is cell in M^n, then the following are *cells in M^{n+1}*:
(1) $\Gamma(f)$ and

(2) $(f,g) = \{(\bar{x},y) \in X \times M : f(\bar{x}) < y < g(\bar{x})\}$,

where f, g are definable continuous functions defined on X.

Cell Decomposition Theorem (CDT). *For every definable function $f : X \subseteq M^n \to M$, there is a partition of $X = X_1 \cup \cdots \cup X_k$ into finitely many cells X_i, such that each $f \restriction_{X_i}$ is continuous.*

In fact, a more refined version of CDT can be proved, which guarantees that each projection of the partition of X onto M^m, $m \le n$, is a partition of the projection of X into cells.

The proof of the CDT is given by induction on n, and it is carried out simultaneously with the following statement:

Uniform Finiteness. *Let $S \subseteq M^{n+1}$ be a definable set, and for each $\bar{b} \in M^n$, denote $S_{\bar{b}} := \{y \in M : (\bar{b}, y) \in M^{n+1}\}$. Then there is $N \in \mathbb{N}$, such that for every $\bar{b} \in M^n$,*

$$|S_{\bar{b}}| \text{ is finite } \Rightarrow |S_{\bar{b}}| < N.$$

Corollary 1.3. *If $\mathcal{M}$ is o-minimal and $\mathcal{N} \equiv \mathcal{M}$, then $\mathcal{N}$ is o-minimal.*

Proof. Let $\phi(x, \bar{y})$ be an $\mathcal{L}$-formula. We show that for every $\bar{b} \in M^n$, $\phi(\mathcal{M}, \bar{b})$ is a finite union of points and open intervals. Let

$$\psi(z, \bar{y}) : \ x \text{ is in the boundary of } \phi(x, \bar{y}).$$

Then there is $N \in \mathbb{N}$, such that $\mathcal{M} \models \forall \bar{y}\, \big(|\psi(\mathcal{M}, \bar{y})| < N\big)$. Therefore,

$$\mathcal{N} \models \forall \bar{y}\, \big(|\psi(\mathcal{N}, \bar{y})| < N\big).$$

□

Definition 1.4 (Dimension)**.** If $X \subseteq M^n$ is a cell, we define the dimension of X as follows:

$$\dim X = \text{ the number of times we apply } (f,g) \text{ in its construction.}$$

(The dimension of an open interval is 1).

If $X \subseteq M^n$ is a definable set, we define the dimension of X as follows:

$$\dim X = \max\{\dim X_i : X = X_1 \cup \cdots \cup X_k, X_i \text{ cell from CDT}\}.$$

Remark.

(i) if $X \subseteq M^n$ is a cell, then

$$\dim X = \max\{m \in \mathbb{N} : \text{ the projection of } X \text{ onto some } m \text{ coordinates is open}\}.$$

(ii) The definition of $\dim(X)$ does not depend on the cell decomposition of X.

Observe, by (i):

- If $X \subseteq M^n$ is a cell, then $\dim(X) = n \Leftrightarrow X$ is open.
- If $X \subseteq M^n$ is a definable set, then $\dim(X) = n \Leftrightarrow X$ has non-empty interior.

Lemma 1.5 (Properties of dimension)**.** *(i) For any definable sets X, Y and definable bijection $f : X \to Y$,*

$$\dim X = \dim f(X).$$

(ii) For any definable sets $X_1, \ldots, X_k$,

$$\dim(X_1 \cup \cdots \cup X_k) = \max\{\dim(X_i) : i = 1, \ldots, k\}.$$

Definition 1.6. Let $X \subseteq M^n$ be an A-definable set, and let $\bar{a} \in X$. We say that $\bar{a}$ is a *generic element of* X *over* A if it does not belong to any A-definable set of dimension $< \dim(X)$.

In a sufficiently saturated o-minimal structure, *generic elements always exist.* Indeed, by Compactness and Lemma 1.5(ii), given an A-definable set X, the type consisting of all formulas that express that x belongs to X but not to any A-definable set of dimension $< \dim(X)$ is consistent.

A definable subset V of a definable set X is called *large in* X if $\dim(X \setminus V) < \dim(X)$. In a sufficiently saturated o-minimal structure, V is large in X if and only if for every A over which V and X are defined, V contains every generic element a in X over A.

We extend the definition of a large set to possibly not definable subsets of X: a subset $V \subseteq X$ is called *large in* X if for every A over which X is defined, V contains every generic element a in X over A.

Note: The notions of dimension and genericity can alternatively be defined through the 'algebraic closure operator' *acl*, which in o-minimal structures as well as, strongly minimal structures, defines a pregeometry (see [El2]). On the one hand, this alternative way yields the same notion of dimension, and on the other, it indicates that there is an implicit 'geometry' going on in o-minimality.

2. Definable groups

In this section, $\mathcal{M}$ denotes an o-minimal structure, and 'definable' means 'definable in $\mathcal{M}$ over some parameters'.

Definition 2.1. Let $G \subseteq M^m$ be a definable group of dimension n. A *definable manifold topology on* G is a topology on G satisfying the following: there is a finite set $\mathcal{A} = \{\langle G_i, \phi_i\rangle : i \in I\}$ such that

(i) $\forall i \in I$, G_i is a definable open subset of G (in the manifold topology),

(iii) $G = \cup_{i\in I} G_i$,

(ii) $\forall i \in I$, $\phi_i : G_i \to \phi_i(G_i) \subseteq M^n$ is a definable homeomorphism,

(iv) for all $i, j \in I$, if $G_i \cap G_j \neq \emptyset$, then $G_{ij} := \phi_i(G_i \cap G_j)$ is a definable open set and $\phi_j \circ \phi_i^{-1} \restriction_{G_{ij}} : \phi_i(G_i \cap G_j) \to \phi_j(G_i \cap G_j)$ is a definable homeomorphism.

We fix our notation as above and refer to each ϕ_i as *a chart map*, to each $\langle G_i, \phi_i\rangle$ as *a chart on* G, and to $\mathcal{A}$ as *a definable atlas on* G.

A *topological group* is a group equipped with some topology in a way that makes its multiplication and inverse operations continuous.

Theorem 2.2 (Pillay, [Pi1], 1988)**.** *Every definable group* $G = \langle G, \oplus, e_G\rangle$ *can be equipped with a unique definable manifold topology that makes it into a topological group. Call it* G-topology.

This theorem as well as its proof are in analogy with the theorem by L. van den Dries and E. Hrushovski that every definable group in an algebraically closed field is algebraic. We only provide the proof of the uniqueness of the G-topology below.

Let us call a subset of G *G-open* if it is open with respect to the G-topology. By the definition of the G-topology, it follows that:

(v) $X \subseteq G$ is G-open if and only if $\forall i \in I$, $\phi_i(X \cap G_i)$ is open.

Questions:

(1) **Affine embedding.** There are two topologies on G: the G-topology and the subspace topology induced by M^m. Below we will show that the G-topology coincides with the subspace topology on a large subset of G. When does it coincide on the whole of G (up to definable isomorphism)?
(2) **Pillay's Conjecture (roughly).** Every definable group is a topological group. What is the relation between definable groups and real Lie groups (such as the definable groups in $\langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$)? For example, given a definable group G, is there a canonical group homomorphism from G onto a real Lie group?
(3) Assume that $\mathcal{M} = \langle M, <, +, \dots \rangle$ is an o-minimal expansion of an ordered group. How do $\oplus$ and $+$ relate?

Example 2.3. Let $\mathcal{M} = \langle \mathbb{R}, <, +\rangle$, and $S^1 = \langle [0, 1), \oplus, 0\rangle$ as in Section 1, that is:

$$x \oplus y = z \Leftrightarrow x + y - z \in \mathbb{Z}.$$

- Two definable charts: $G_1 = (0, 1)$, $G_2 = \left(\frac{1}{2}, 1\right) \cup \left(0, \frac{1}{2}\right)$, with
  $\phi_1 : (0, 1) \to (0, 1)$, $\phi_1(x) = x$, and
  $\phi_2 : \left(\frac{1}{2}, 1\right) \cup \left[0, \frac{1}{2}\right) \to (0, 1)$,
  $$\phi_2(x) = \begin{cases} x - \frac{1}{2} & \text{if } \frac{1}{2} < x < 1, \\ x + \frac{1}{2} & \text{if } 0 \le x < \frac{1}{2}. \end{cases}$$
- On $(0, 1)$, the G-topology coincides with the subspace topology.
- S^1 admits an affine embedding by 'bending' $[0, 1)$ into a square, gluing the two endpoints.

For the rest of this section we assume that
(1) $\mathcal{M}$ is sufficiently saturated.
(2) G is $\emptyset$-definable, and by a 'generic element of G' we mean a generic element of G over $\emptyset$.
(3) $G \subseteq M^n$, where $n = \dim(G)$.

The last assumption is not legitimate, but we make it in order to simplify the arguments. Without that assumption all statements still go through, but one needs to work out their proofs by 'projecting' G onto some n coordinates.

Claim 2.4. *For every generic element $g \in G$, there is an open subset $U \subseteq G$ containing g such that $\forall X \subseteq U$,*

$$X \text{ is open} \Leftrightarrow X \text{ is } G\text{-open}.$$

Proof. For every $i \in I$, such that $g \in G_i$, we show that there is an open neighborhood U_i of g in G_i, such that $\phi_{i\restriction U_i} : U_i \to \phi_i(U)$ is a homeomorphism with respect to the subspace topology in the domain. Indeed, by the CDT, $G_i = X_1 \cup \cdots \cup X_k$, where each X_j is a cell, on which ϕ_i is continuous. Since g is generic, $g \in X_j$, for some cell X_j that has dimension n and hence is open. Thus $\phi_{i\restriction X_j} : X_j \to \phi_i(X_j)$ is a continuous bijection. By a similar argument for ϕ_i^{-1} (since $\phi_i(g)$ is a generic element of $\phi_i(X_j)$), we can reduce the domain of ϕ_i further into a definable open set U_i on which ϕ_i is a homeomorphism.

Now, let $U = \bigcap_{g\in G_i} U_i$. Let $X \subseteq U$. We have:

$$X \text{ is open } \Leftrightarrow \forall i \in I, X \cap U_i \text{ is open} \Leftrightarrow \forall i \in I, \phi_i(X \cap G_i) \text{ is open } \Leftrightarrow X \text{ is } G\text{-open},$$

where the third equivalence is by the definition of G-topology. □

Corollary 2.5. *The G-topology is unique.*

Proof. In a topological group $\langle K, +\rangle$, the topology is determined by the topology in an open neighborhood U of any point $a \in K$. Indeed, let $T = \{X \subseteq U : X \ni a, X \text{ is open in } K\}$. Let $V \subseteq K$. Then V is open in K if and only if $\forall g \in V$, there is $X \in T$, $g - a + X \subseteq V$.

So any two group topologies on G are determined by the corresponding topologies in an open neighborhood of some generic element. But the latter ones coincide; namely, they are both equal to the subspace topology by Claim 2.4. □

Corollary 2.6. *The G-topology and the subspace topology coincide on a large subset V of G.*

Proof. We show that they coincide on the set V of all generics of G. Let $Y \subseteq V$, and $g \in Y$. We want to show that there is an open neighborhood $X \subseteq Y$ containing g if and only if there is a G-open neighborhood $X' \subseteq Y$ containing g. But this is trivially true within the open subset $U \subseteq G$ containing g of Y provided by Claim 2.4. Indeed, a subset $X \subseteq V \cap U$ is open in V if and only if it is G-open in V. □

Remark 2.7. The proof in [Pi1] actually yields a *definable* large subset V of G on which the two topologies coincide.

The following 'finiteness' property holds for groups definable in o-minimal structures. It is known as the Descending Chain Condition, and it will be proved in [El2].

Fact 2.8 (DCC)**.** *Let G be a definable group. Then there is no infinite proper descending chain of definable subgroups of G:*

$$G = G_0 \gneqq G_1 \gneqq G_2 \gneqq \dots$$

3. Semi-linear groups

In this section, we let $\langle M, <, +, 0, \{d\}_{d\in D}\rangle$ be a sufficiently saturated ordered vector space over an ordered division ring D. A definable set in $\mathcal{M}$ is called *semi-linear*, and a definable group is called a *semi-linear group*.

Goal: For every semi-linear group G, define a canonical group homomorphism from G onto some real Lie group. (Rough Pillay's Conjecture for semi-linear groups.)

Example 3.1. Let $\mathcal{M}$ be an elementary extension of $\langle \mathbb{R}, <, +, 0\rangle$, and $(S^1)^{\mathcal{M}}$ be the interpretation of S^1 in $\mathcal{M}$. Then there is a *standard part map* $st : (S^1)^{\mathcal{M}} \to S^1$,

$$st(x) = y \text{ if } \forall n \in \mathbb{N}, |x - y| < \frac{1}{n},$$

where $|\cdot|$ denotes distance in the embedded group.

We are in fact going to show a definable analogue of the following classical result from the theory of Lie groups. (see, for example, [Pon, Theorem 42]).

Fact 3.2. *Every compact, connected, abelian Lie group G is isomorphic to a real torus, that is, to a direct product of copies of the circle topological group S^1.*

Note: The 'algebraic content' of this fact was proved for groups definable in o-minimal expansions of fields in [EdOt]. That is, it was shown there that the k-torsion subgroup $G[k]$ of G is isomorphic to $(\mathbb{Z}/k\mathbb{Z})^n$, where $n = dim(G)$.

The definable versions of the notions of compactness and connectedness are given as follows.

Definition 3.3. Let G be a definable group. Then:

- G is called *definably compact* [Peterzil-Steinhorn (1999)] if for every definable G-continuous $\sigma : (a,b) \subseteq M \to G$ the limit $lim^t_{x\to a}\sigma(x)$ exists (in G).
- G is *definably connected* if it is not the disjoint union of two G-open definable proper subsets.

The definable version of the notion of a torus is more subtle. We proceed with examples of definable groups, which we would like to include in our definition of a 'definable quotient group'.

Example 3.4. Let $\mathcal{M} = \langle \mathbb{R}, <, +, 0\rangle$.

1) $S^1 \cong \langle \mathbb{R}, +\rangle/\mathbb{Z} \cong \langle [0,1), \oplus, 0\rangle$, where

$$x \oplus y = z \Leftrightarrow x + y - z \in \mathbb{Z}.$$

2) $G_2 = S^1 \times S^1 \cong \langle \mathbb{R}^2, +\rangle/\mathbb{Z}^2 \cong \langle [0,1) \times [0,1), \oplus, 0\rangle$, where

$$x \oplus y = z \Leftrightarrow x + y - z \in \mathbb{Z} \times \mathbb{Z}.$$

3) $G_3 = \langle \mathbb{R}^2, +\rangle/\mathbb{Z}^2 \cong \langle [0,1) \times [0,\pi), \oplus, 0\rangle \ncong^{\text{definably}} S^1 \times S^1$, where

$$x \oplus y = z \Leftrightarrow x + y - z \in \mathbb{Z}(1,0) + \mathbb{Z}\left(\frac{1}{2}, \pi\right).$$

4) Let $\mathcal{M}_1 \succ \mathcal{M}$ be saturated.
$G_4 = S^1(\mathcal{M}_1) = \langle [0,1)^{\mathcal{M}_1}, \oplus^{\mathcal{M}_1}, 0\rangle \cong \langle Fin(M_1), +\rangle/\mathbb{Z}$.

Note: the presentation of the above groups as $\langle S, \oplus, 0\rangle$ is definable.

Definition 3.5. Let $U \subseteq M^n$ be a group, and $L \leqslant U$ a subgroup. Then U/L is said to be *a definable quotient group* if there is a definable complete set $S \subseteq U$ of representatives for U/L, such that the induced group structure $\langle S, +_S\rangle$ is definable. In this case, we identify U/L with $\langle S, +_S\rangle$.

Definition 3.6. The abelian subgroup of M^n generated by the elements $v_1, \dots, v_m \in M^n$ is denoted by $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m$. If $v_1, \dots, v_m$ are $\mathbb{Z}$-linearly independent, then the free abelian subgroup $\mathbb{Z}v_1 + \cdots + \mathbb{Z}v_m$ of M^n is called *a lattice of rank m*.

Let $\{X_k : k < \omega\}$ be a collection of definable subsets of M^n. Assume that $U = \bigcup_{k<\omega} X_k$ is equipped with a binary map $\cdot$ so that $\langle U, \cdot\rangle$ is a group. U is called *a $\bigvee$-definable group* if, for all $i, j < \omega$, there is $k < \omega$, such that $X_i \cup X_j \subseteq X_k$ and the restriction of $\cdot$ to $X_i \times X_j$ is a definable function into M^n.

A subset $A \subseteq M^n$ is called *convex* if $\forall x, y \in A, \forall q \in \mathbb{Q} \cap [0,1]$, $qx + (1-q)y \in A$.

Theorem 3.7 (Structure Theorem, [ElSt])**.** *Let $G = \langle G, \oplus, e_G\rangle$ be a definably compact definably connected group definable in $\mathcal{M}$, with $\dim(G) = n$. Then G is definably isomorphic to a definable quotient group U/L, for some convex $\bigvee$-definable subgroup $U \leqslant \langle M^n, +\rangle$, and a lattice $L \leqslant U$ of rank n.*

The main ingredients of the proof the Structure Theorem. We start with recalling basic facts for definability in the linear o-minimal structure $\mathcal{M}$. A *linear* function $f : X \subseteq M^n \to M$ has form

$$f(x_1, \ldots, x_n) = \lambda_1 x_1 + \cdots + \lambda_n x_n + a, \text{ for fixed } \lambda_i \in D \text{ and } a \in M.$$

Linear Cell Decomposition Theorem (v.d. Dries) = Cell Decomposition Theorem + replace definable continuous functions by linear ones.

A set $X \subseteq G$ is *generic* if finitely many $\oplus$-translates of X cover G. Using the definable compactness of G, we can show:

Generic Lemma. *For all definable $X \subseteq G$, either X or $G \setminus X$ is generic.*

For $\vec{\lambda} = (\lambda_1, \ldots, \lambda_n) \in D^n$, $t \in M$, denote $\vec{\lambda}t := (\lambda_1 t, \ldots, \lambda_n t)$. Then:

- G contains a generic *parallelogram* (up to translation):
$$H := \{\vec{\lambda}_1 t_1 + \cdots + \vec{\lambda}_n t_n : -e_i < t_i < e_i\},$$
where $e_1, \ldots, e_n \in M$ are positive, and $\vec{\lambda}_1, \ldots, \vec{\lambda}_n \in D^n$.

We define:
$$U := < H > = \bigcup_{k<\omega} \underbrace{H + \cdots + H}_{k-\text{times}} \leqslant M^n,$$
and
$$\phi : U \ni x_1 + \cdots + x_k \mapsto x_1 \oplus \cdots \oplus x_k \in G,$$
where $x_i \in H$ (and $e_G = 0$). In order to finish the proof of the Structure Theorem, one would have to show that:

(1) $\phi : U \to G$ is a well-defined surjective group homomorphism.
(2) $L := \ker(\phi)$ has rank $n = \dim(G)$. We have $U/L \cong G$.
(3) There is a definable complete set $S \subseteq U$ of representatives for U/L.

□

Sketch of the proof of rough Pillay's Conjecture. Since H is generic in G, by Lemma 1.5, $\dim(H) = n$. It is then not hard to see that every $u \in U$ has a unique form
$$u = \vec{\lambda}_1 u_1 + \cdots + \vec{\lambda}_n u_n, \;\; u_i \in M, \; |u_i| < qe_i, \text{ some } q \in \mathbb{Q}.$$
Let
$$st(u) = \big(st_1(u_1), \ldots, st_n(u_n)\big) \in \mathbb{R}^n,$$
where
$$st_i(u_i) = \sup\{q \in \mathbb{Q} : qe_i < u_i\} \in \mathbb{R}.$$
One can then prove that $st : U \to \mathbb{R}^n$ is a surjective group homomorphism. However, we are aiming to define a standard part map from G (and not U) onto some real Lie group.

Let st_G be the unique map that makes the following diagram commute:

$$\begin{array}{ccc} U & \xrightarrow{\;st\;} & \mathbb{R}^n \\ \big\downarrow{\scriptstyle \phi} & & \big\downarrow{\scriptstyle q} \\ G = U/L & \underset{st_G}{\cdots\cdots\to} & \mathbb{R}^n/st(L) \end{array}$$

One can prove that:

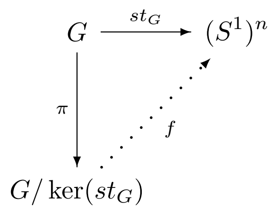
- $st(L) \subseteq \mathbb{R}^n$ is a lattice of rank n. Thus $\mathbb{R}^n/st(L) \cong (S^1)^n$.
- $st_G : G \to (S^1)^n$ is a surjective group homomorphism.

□

An important property that one can show about st_G is:

(1) $$A \subseteq \mathbb{R}^n/st(L) \text{ is closed } \Leftrightarrow st_G^{-1}(A) \text{ is type-definable.}$$

The map $st_G : G \to (S^1)^n$ induces a group isomorphism $f : G/\ker(st_G) \to (S^1)^n$:



By (1), f is an isomorphism between *topological groups* if the quotient $G/\ker(st_G)$ is equipped with the 'logic topology' (Lascar-Pillay 2001):

$A \subseteq G/\ker(st_G)$ is closed in the *logic topology* if $\pi^{-1}(A)$ is type-definable.

We conclude with the complete statement of Pillay's Conjecture. The above argument implies its solution for semi-linear groups. The solution of the conjecture for groups definable in o-minimal expansions of fields can be found in [Pi2] and [HPP], respectively.

Theorem 3.8 (Pillay's Conjecture). *Let $\mathcal{M}$ be a sufficiently saturated o-minimal structure. Let $G = \langle G, \oplus, e_G \rangle$ be a definably compact definably connected group, with $\dim(G) = n$. Then*

(1) *G contains a smallest type-definable normal subgroup G^{00} of bounded index,*
(2) *G/G^{00}, equipped with the logic topology, is a compact connected Lie group,*
(3) *$\dim_{Lie}\left(G/G^{00}\right) = n$.*

(Type-definable: with $< |M|$ many formulas.)
(G^{00} bounded index: $\left|G/G^{00}\right| < |M|$.)

In the above analysis of semi-linear groups, G^{00} is exactly $\ker(st_G)$.

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